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Solution of Implicit Fractional Differential Equation in the Setting of New Perturbed Spaces

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Abstract Distance error between two points, which arises from various causes directly impact measurement accuracy. Although these errors may seem small individually, their cumulative effect can lead to significant discrepancies. Considering this, we introduce the concept of perturbed S-metric spaces and present a noteworthy generalization of fixed points theorems in this context. Also, we prove existence and uniqueness of a solution for a deformable fractional order implicit differential equation.

Keywords Metric space; S-metric space; perturbed mapping; perturbed S-metric space.

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1. Introduction

Banach [3] established the renowned "Banach Contraction Principle" a cornerstone theorem in the realm of fixed-point theory. This principle is celebrated for its extensive applicability, particularly in demonstrating the existence and uniqueness of solutions for various mathematical constructs, including differential equations, integral equations, and fractional integral equations. Given its significance across mathematics and related disciplines, the principle has inspired a multitude of generalizations, expanding its analytical reach.

In 1906, Fréchet made groundbreaking contributions to the theory of metric spaces [7], laying down the foundational concepts that would shape the field. In the years that followed, a host of scholars sought to broaden the definition of metric spaces by relaxing specific conditions and reimagining the metric function itself. This evolution is well-documented, with numerous examples found in the literature (refer to [1, 2, 5, 10, 11, 20, 21, 22, 23, 24, 25, 26, 33, 12]).

A notable advancement in this area occurred in 2012 when Sedghi et al. [33] introduced the concept of S-metric spaces, marking a further generalization of traditional metric spaces. Alongside this introduction, they uncovered several remarkable fixed-point theorems specifically tailored for S-metric spaces, enriching the theoretical landscape and offering new avenues for exploration.an its wide range of application. In 2024, Jleli and Samet

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introduced a new generalization of a metric space, known as a perturbed metric space, where the metric d is not the usual distance metric, but rather an experimental perturbed metric D.

Fractional differential equations (FDEs) have emerged as groundbreaking mathematical tools, revolutionizing the way we model complex phenomena across diverse fields such as engineering, physics, biology, finance, and applied mathematics. Unlike traditional integer-order differential equations, FDEs incorporate derivatives of fractional (non-integer) order. This unique characteristic enables them to capture sophisticated memory effects, hereditary properties, and anomalous diffusion processes that conventional models fail to address adequately.

A pivotal aspect of studying FDEs revolves around establishing the existence and uniqueness of solutions under various boundary or initial conditions. Understanding whether a solution exists, if it is unique, and the conditions under which it can be approximated or computed is crucial for both theoretical exploration and practical application. As a result, a considerable amount of research has been dedicated to creating innovative analytical and numerical methods to tackle these questions. Numerous recent studies, including [4, 9, 15, 27, 28, 29, 30, 31, 32, 6, 13, 16], have delved into the existence, uniqueness, stability, and qualitative behavior of solutions for both ordinary and fractional differential equations across diverse functional frameworks and operator settings. These investigations provide a robust theoretical foundation for applying FDEs to real-world challenges and contribute significantly to the expanding body of literature that highlights the mathematical richness and versatility of fractional calculus.

In [34], Zulfeqarr et al. introduced a novel concept, known as, the deformable fractional derivative. This new approach involves altering the limit technique used in traditional derivatives. It is termed "deformable" because it can continuously transform a function into its derivative. Deformable derivatives can be regarded as fractional-order derivatives. The authors of [18] further explored the properties of this new concept and applied the findings to study a Cauchy problem with a non-local condition:

$$
\begin{aligned}
&D_0^{\omega}(t) = \mathfrak{T}(t, a(t)), t \in (0, k] \\
&a(0) = g(a(t) + a_0,
\end{aligned}
$$

where D_0^{ω} is the deformable derivative of order $\omega \in (0, 1]$ and g is a continuous function on $\mathbb{R}$.

In [19], by using Weissinger's and Krasnosel'skii's fixed point theorems, Mebrat and N'Guerekata studied the existence of solutions for the following problem:

$$
\begin{aligned}
&D_0^{\omega}(t) = \phi(a(t)) + \psi(t, a(t)) + \int_0^t \gamma(t, s, a(s))ds, t \in (0, k] \\
&a(0) = a_0,
\end{aligned}
$$

where $\psi : P \times \mathbb{R} \to \mathbb{R}$ and $\gamma : P \times P \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Motivated by the above results, in this work, we first introduce the notion of a perturbed S-metric space. Then, we define the Banach type contraction in the setting of a perturbed S-metric space and propose some fixed point results for this contraction. As an application, we prove existence and uniqueness of a deformable fractional order implicit differential equation to strengthen our results.

2. Basic Preliminaries

In this section, we define some essential preliminaries.

Definition 2.1. *[14] Let $U \neq \phi$ and $\mathfrak{D}, \mathcal{P} : U \times U \to [0, \infty$ be two mappings such that $\mathfrak{D} - \mathcal{P}(u, v) = \mathfrak{D}(u, v) - \mathcal{P}(u, v)$. Assume that for all $u, v, w \in U$,*

1. $\mathfrak{D} - \mathcal{P}(u, v) \geq 0$;
2. $\mathfrak{D} - \mathcal{P}(u, v) = 0 \iff u = v$;
3. $\mathfrak{D} - \mathcal{P}(u, v) = \mathfrak{D} - \mathcal{P}(v, u)$;
4. $\mathfrak{D} - \mathcal{P}(u, v) \leq \mathfrak{D} - \mathcal{P}(u, w) + \mathfrak{D} - \mathcal{P}(w, v)$.

Then $\mathfrak{D}$ is a perturbed metric on U with respect to the perturbed mapping $\mathcal{P}$. Here, $(U, \mathfrak{D}, \mathcal{P})$ is said to be a perturbed metric space.

Definition 2.2. *[33] Let $U \neq \phi$ be any set and $S : U \times U \times U \to [0, \infty]$ be a function satisfying the following conditions for all $u, v, w, z \in U$:*

1. $S(u, v, w) \geq 0$;
2. $S(u, v, w) = 0 \iff u = v = w$;
3. $S(u, v, w) \leq S(u, u, z) + S(v, v, z) + S(w, w, z)$.

The function S is called an S-metric on X and the pair (U, S) is called an S-metric space.

Definition 2.3. *[33] Let $(U, \mathbb{S})$ be an S-metric space.*

1. *A sequence $\{u_n\}$ in U converges to u, if $\lim_{n\to\infty} S(u_n, u_n, u) = 0$. In this case, we write $u_n \to u$ or $\lim_{n\to\infty} u_n = u$.*
2. *A sequence $\{u_n\}$ in U is called a Cauchy sequence if $\lim_{n,m\to\infty} S(u_n, u_n, u_m) = 0$.*
3. *$(U, \mathbb{S})$ is called complete if every Cauchy sequence converges.*

Definition 2.4. *[8] Let $a(t)$ be a real valued-function on $[0, k]$. The deformable derivative of a of order ω at $t \in (0, k)$ is defined as*

$$D_0^{\omega}(a(t)) = \lim_{n\to\infty} \frac{(1 + \epsilon\beta)a(t + \epsilon\omega) - a(t)}{\epsilon},$$

where $\omega + \beta = 1$, and $\omega \in (0, 1]$. If the limit exists then a is differentiable at t.

If $\omega = 1$ and $\beta = 0$, then a deformable derivative becomes usual derivative.

Definition 2.5. *[8] Let $a(t)$ be a continuous function on $[0, k]$. The deformable integral of a of order ω is defined as*

$$I_0^{\omega}(a(t)) = \frac{1}{\omega} e^{\frac{-\beta}{\omega} t} \int_0^t e^{\frac{\beta}{\omega} s} a(s) ds,$$

where $\omega + \beta = 1$, and $\omega \in (0, 1]$.

If $\omega = 1, \beta = 0$, then a deformable integral becomes the usual Riemann Integral.

Theorem 2.6. *[17] Let $\omega, \omega_1, \omega_2 \in (0, 1]$ be such that $\omega + \beta = 1$ and $\omega_i + \beta_i = 1$, for $i = 1, 2$. Then*

(a) $D^{\omega}(sa + tb) = sD^{\omega}(a) + tD^{\omega}(b)$
(b) $D^{\omega_1} D^{\omega_2} = D^{\omega_2} D^{\omega_1}$
(c) $D^{\omega}(r) = \beta r$ *for constant* r
(d) $D^{\omega}(fg) = (D^{\omega} f)g + \omega f Dg$
(e) $I^{\omega}(sa + tb) = sI^{\omega}(a) + tI^{\omega}(b)$
(f) $I^{\omega_1} I^{\omega_2} = I^{\omega_2} I^{\omega_1}$.

Lemma 2.7. *[34] Let $a(t)$ be a continuous function on $[0,k]$ and $I_0^\omega(a)$ be ω-differentiable on $(0,k)$. Then*

$$D_0^\omega(I^\omega(a))(t) = a(t) \text{ and } I_0^\omega(D_0^\omega(a))(t) = a(t).$$

Lemma 2.8. *[19] Let $\psi \in C(P,\mathbb{R})$ and $0 < \omega \leq 1$, then the initial value problem*

$$D_0^\omega(u(t)) = \psi, (t, u(t), D_0^\omega(u(t))),\ t \in P = (0,k],\ k > 0 \tag{1}$$

$$a(0) + g(a(t)) = +a_0. \tag{2}$$

The system (1-2) is equivalent to the following integral equation

$$u(t) = (u_0 - g(a))e^{\frac{-\beta}{\omega}t} + \frac{1}{\omega}e^{\frac{-\beta}{\omega}t}\int_0^t e^{\frac{\beta}{\omega}s}\phi(s)ds,$$

where $\phi(t) = \psi(t, u(t), D_0^\omega(u(t)))$.

3. Main Results

Definition 3.1. *Let $U \neq \phi$ be any set and $S, \mathcal{P} : U \times U \times U \to [0,\infty]$ be a two mappings with $S - \mathcal{P}(u,v,w) = S(u,v,w) - \mathcal{P}(u,v,w)$. Assume that for all $u,v,w,z \in U$,*

1. $S - \mathcal{P}(u,v,w) \geq 0$;
2. $S - \mathcal{P}(u,v,w) = 0 \iff u = v = w$;
3. $S - \mathcal{P}(u,v,w) \leq S - \mathcal{P}(u,u,z) + S - \mathcal{P}(v,v,z) + S - \mathcal{P}(w,w,z)$.

Then the mapping S is called a Perturbed S-metric with respect to the perturbed mapping $\mathcal{P}$ and $(U,S,\mathcal{P})$is called a Perturbed S-metric space. Also, $\mathcal{S} = S - \mathcal{P}$ is called an exact S-metric .

For $\mathcal{P}(u,v,w) = 0$, every perturbed S-metric space is an S-metric space, but the converse is not true, i.e., every S-metric space is a perturbed S-metric space having $\mathcal{P}(u,v,w) = 0$ (constant).
Now we take an example in which we have an error term, or we call it perturbation, computing the standard S-metric as follows:

Example 3.2. *Consider $\mathcal{K} = [0,1]$. Let $C(\mathcal{K})$ be the set of continuous functions defined on $\mathcal{K}$. Define the mapping $S : C(\mathcal{K}) \times C(\mathcal{K}) \times C(\mathcal{K}) \to [0,\infty)$ such that*

$$S(f,g,h) = \sup_{t\in[0,1]} |f(t) - g(t)| + \sup_{t\in[0,1]} |g(t) - h(t)| + (f(0) + g(0) + h(0))^2.$$

Then $(C(K), S, \mathcal{P})$ is a perturbed S-metric space with respect to the perturbed mapping $\mathcal{P}(f,g,h) = (f(0) + g(0) + h(0))^2$.

Definition 3.3. *Let $(U,S,\mathcal{P})$ be perturbed S-metric space. For $r > 0$ and $u \in U$, we define open and closed ball with radius r and centre u respectively*

$$B(u,r) = \{\mathbb{S}(u,v,v) < r : v \in U\}$$

$$B[u,r] = \{\mathbb{S}(u,v,v) \leq r : v \in U\}.$$

Example 3.4. *Let $U = \mathbb{R}$. Define the mapping $S : U \times U \times U \to [0,\infty)$ such that $S = \frac{|u+v-2w|+|v-w|}{2} + u^2v^4w^6$. Then (U,S,P) is a perturbed S-metric with respect to the perturbed mapping $P(u,v,w) = u^2v^4w^6$. The open ball with* *centre 0 and radius* 1 *is*

$$\begin{aligned} B[0,1] &= \{v \in \mathbb{R} : \mathbb{S}(v,v,o) < 1\} \\ &= \{v \in \mathbb{R} : \frac{3|v|}{2} < 1\} \\ &= (-\frac{2}{3}, \frac{2}{3}). \end{aligned}$$

The closed ball with centre 1 *and radius* 1 *is*

$$\begin{aligned} B(1,1) &= \{v \in U : \mathbb{S}(v,v,1) \leq 1\} \\ &= \{v \in \mathbb{R} : \frac{3|v-1|}{2} \leq 1\} \\ &= \{v \in \mathbb{R} : |v-1| \leq \frac{2}{3}\} \\ &= \left[\frac{-1}{3}, \frac{5}{2}\right]. \end{aligned}$$

Definition 3.5. *Let $(U, S, \mathcal{P})$ be perturbed S-metric space and $\{u_n\}$ be a sequence in U.*
(i) A subset A of U is said to be a perturbed S-open if for every $u \in A$, there exists $r > 0$ such that $B_{\mathbb{S}}(x,r) \subset A$, i.e., if A is open in an S- metric space.
(ii) A subset A of U is said to be perturbed S-bounded if A is bounded in S- metric space.
(iii) $\{u_n\}$ is perturbed convergent in $(U, S, \mathcal{P})$ if $\{u_n\}$ is convergent in the metric space $(U; \mathcal{S})$, where $\mathbb{S} = S - \mathcal{P}$ is the exact S-metric .
(iv) $\{u_n\}$ is perturbed Cauchy in $(U, S, \mathcal{P})$ if $\{u_n\}$ is Cauchy in the S-metric space $(U; \mathbb{S})$.
(v) We say that $(U; S, \mathcal{P})$ is a complete perturbed S-metric space, if $(U; \mathbb{S})$ is a complete S-metric space.
(vi) We say that $T : U \to U$ is a perturbed continuous mapping, if T is continuous with respect to the exact S-metric.

Lemma 3.6. *Let $(U, S, \mathcal{P})$ be a perturbed S-metric space. Then we have $\mathbb{S}(v,v,u) = \mathbb{S}(u,u,v)$.*

Proof
Since $\mathbb{S}$ =S$-\mathcal{P}$ is an exact S-metric . Following the proof of Lemma 2.5 in [33], we get the desired result. □

Now, we shall establish the Banach's fixed point result in the setting of perturbed S-metric spaces. Since in an S-metric space, every contraction map is continuous, but in the case of a perturbed metric space, the contraction map need not be continuous.

Theorem 3.7. *Let $(U, S, \mathcal{P})$ be a complete perturbed S- metric space and $T : U \to U$ be a continuous map satisfying*

$$S(fu, fu, fv) \leq \theta S(u,u,v) \tag{3}$$

for $u, v \in U$, and $\theta \in [0,1)$. Then f has a fixed point.

Proof
Let $u_0 \in U$ be arbitrary, define picard sequence $u_n = fu_{n-1}$ for all $n \geq 1$. For $\theta \in (0,1)$, from (4) we have

$$\begin{aligned} S(u_2,u_2,u_1) &= S(fu_1, fu_1, fu_0) \\ &\leq \theta S(u_1,u_1,u_0). \end{aligned}$$

Similarly, for $u_3 = fu_2$, we have

$$S(u_3,u_3,u_2) \leq \theta\, S(u_2,u_2,u_1).$$

Continuing in this manner, for $u_{n+1} = fu_n$, we have

$$\begin{aligned} S(u_{n+1}, u_{n+1}, u_n) &\leq \theta S(u_n, u_n, u_{n-1}) \\ &\leq \theta^2 S(u_{n-1}, u_{n-1}, u_{n-2}) \\ &\leq . \\ &\leq . \\ &\leq \theta^n S(u_1, u_1, u_0). \end{aligned}$$

Claim: the sequence $\{u_n\}$ is perturbed Cauchy. It is sufficient to prove that it is Cauchy in $(U, \mathbb{S})$. Now, for $n \geq 1$, we have

$$\begin{aligned} \mathbb{S}(u_{n+1}, u_{n+1}, u_n) &= S(u_{n+1}, u_{n+1}, u_n) - \mathcal{P}(u_{n+1}, u_{n+1}, u_n) \\ &\leq \theta^n S(u_1, u_1, u_0) - \mathcal{P}(u_{n+1}, u_{n+1}, u_n). \end{aligned}$$

Since $\mathcal{P}(u_m, u_m, u_n) \geq 0$, letting $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{S}(u_{n+1}, u_{n+1}, u_n) = 0.$$

Thus, the sequence $\{u_n\}$ is Cauchy in the S-metric space, which implies it is perturbed Cauchy in $(U, S, \mathcal{P})$. Since $(U, S, \mathcal{P})$ is a complete perturbed S-metric space, there exists $u^* \in U$ such that $\lim_{n\to\infty} u_n = u^*$. Since f is continuous as well, $u^* = fu^*$, which implies u^* is a fixed point of f.

Uniqueness: Let v^* be another fixed point of f. Then for $\theta \in [0, 1)$

$$\begin{aligned} S(fu^*, fu^*, fv^*) &= S(u^*, u^*, v^*) \\ &\leq \theta S(u^*, u^*, v^*), \end{aligned}$$

which holds if $\theta = 0$, i.e., $u^* = v^*$. Hence, the fixed point is unique. □

Example 3.8. *Let $U = [0, 1]$, and define the functions $S, \mathcal{P} : [0, 1]^3 \to [0, \infty)$ such that*

$$S(u, v, w) = |u - v| + |v - w| + u^2 v^2 w^2$$

for all $u, v, w \in U$. Then S is a perturbed S-metric on U with respect to the perturbed mapping $\mathcal{P} : [0, 1]^3 \to [0, \infty)$ defined by $\mathcal{P}(u, v, w) = u^2 v^2 w^2$. Then the triplet $(U, S, \mathcal{P})$ is a perturbed S-metric space. Let $f : U \to U$ be defined by

$$f(u) = \frac{sinu}{2}.$$

Then

$$\begin{aligned} S(fu, fu, fv) &= \frac{1}{2}|sinu - sinv| + \frac{1}{2^6} sin^4 u sin^2 v \\ &\leq \frac{1}{2}|u - v| + \frac{1}{2} u^4 v^2 \\ &\leq \theta |u - v| \end{aligned}$$

for $\theta \in (\frac{1}{2}, 1) \subset [0, 1)$. Thus, f satisfies all the postulates of Theorem 3.7. Hence, f has a unique fixed point in U, which is 0.

Corollary 3.9. *Let $(U, \mathbb{S})$ be a complete S-metric space and $T : U \to U$ be map satisfying*

$$\mathbb{S}(fu, fu, fv) \leq \theta \mathbb{S}(u, u, v) \tag{4}$$

for $u, v \in U$, where $\theta \in [0, 1)$. Then f has a fixed point.

Proof

Taking $\mathcal{P}(u, v, w) = 0$, following the proof of Theorem 3.7 we get the desired result. □

4. Application

Let $U = C(\mathcal{K}, \mathbb{R})$ be the set of all continuous functions from $\mathcal{K} = [0, P]$ to $\mathbb{R}$. Define the perturbed S-metric such that

$$S(u, v, w) = \sup_{t\in[0,1]} |u(t) - v(t)| + \sup_{t\in[0,1]} |v(t) - w(t)| + \frac{|u(0) - 2v(0) + w(0)|}{2}$$

with perturbed mapping $\mathcal{P}(u, v, w) = \frac{|u(0)-2v(0)+w(0)|}{2}$ for all $u, v, w \in U$. Then $(U, S, \mathcal{P})$ is a perturbed S-metric space. Define a deformable fractional order implicit differential equation such that

$$D_0^{\omega}(u(t)) = \psi(t, u(t), D_0^{\omega}(u(t)), \ t \in \mathcal{K} = (0, \theta] \tag{5}$$

$$u(0) = u_0, \tag{6}$$

where $0 < \omega < 1$, D_0^{ω} is the deformable fractional derivative defined in [34].
From Lemma 2.8, problem (5-9) is equivalent to

$$u(t) = u_0 e^{\frac{-\beta}{\omega}t} + \frac{1}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \phi(s) ds$$

where $\phi(t) \in C(\mathcal{K}, \mathbb{R})$ with $\phi(t) = \psi(t, u(t), D_0^{\omega}(t))$.

Theorem 4.1. *Assume that there exist constants $0 \le \gamma_1, \gamma_2 < 1$ such that*

$$|\psi(t, u(t), \phi(t)) - \psi'(t, v(t), \phi'(t)| \le \gamma_1 \beta |u - v| + \gamma_2 |\phi(t) - \phi'(t)|, \tag{7}$$

for all $t \in [0, P]$, where $\frac{\gamma_1}{(1-\gamma_2)} \le \frac{\theta}{2}$ for some $\theta \in [\frac{1}{2}, 2)$. Then the initial value problem (5-6) has a unique solution.

Proof
To prove the existence of a solution, it is sufficient to find a fixed point for the self-map $f : U \to U$ defined by

$$fu = u_0 e^{\frac{-\beta}{\omega}t} + \frac{1}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \phi(s) ds.$$

Now, for $u, v \in C(P)$, we have

$$\begin{aligned} S(fu, fu, fv) &= \sup_{t\in P} \frac{1}{\omega} |e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \phi(s) ds - e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \phi'(s) ds| + \frac{1}{2}|u_0 - v_0| \\ &\le \sup_{t\in P} \frac{1}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} |\phi(s) - \phi'(s)| + \frac{1}{2}|u_0 - v_0|. \end{aligned}$$

From inequality(7), we have

$$\begin{aligned} S(fu, fu, fv) &\le \sup_{t\in P} \frac{1}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} (\gamma_1 \beta |u - v| + \gamma_2 |\phi(s) - \phi'(s)|) ds + \frac{1}{2}|u_0 - v_0| \\ &\le \sup_{t\in P} \frac{\beta}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \left(\frac{\gamma_1}{1 - \gamma_2} |u - v| \right) ds + \frac{1}{2}|u_0 - v_0| \\ &\le \sup_{t\in P} \frac{\beta}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} \left(\frac{\theta}{2} |u - v| \right) ds + \frac{1}{2}|u_0 - v_0| \\ &\le \frac{\theta}{2} \sup_{t\in P} |u - v| \frac{\beta}{\omega} e^{\frac{-\beta}{\omega}t} \int_0^t e^{\frac{\beta}{\omega}s} ds + \frac{1}{2}|u_0 - v_0| \\ &= \frac{\theta}{2} \sup_{t\in P} |u - v| + \frac{1}{2}|u_0 - v_0| \\ &\le \theta \sup_{t\in P} |u - v| + \frac{\theta}{2}|u(0) - v(0)| \\ &= \theta S(u, u, v). \end{aligned}$$

We see f satisfies all the postulates of Theorem 3.7. Hence, f has a unique fixed point. Therefore the problem (5-6) has a unique solution. □

Example 4.2. *Consider the initial value problem*

$$D_0^{\frac{1}{2}}(u(t)) = \frac{1}{30(1+|u(t)|)} + \frac{1}{90(1+|D_0^{\frac{1}{2}}(u(t))|)},\ t \in [0,1] \tag{8}$$

$$u(0) = 0. \tag{9}$$

First, we check the existence and uniqueness of the above problem by Theorem 4.1.
Here,

$$\psi(t, u(t), \phi(t) = \frac{1}{30(1+|u(t)|)} + \frac{1}{90(1+|\phi(t)|)}.$$

Therefore,

$$\begin{aligned}|\psi(t,u(t),\phi(t)) - \psi(t,u(t),\phi'(t)| &= \left|\frac{1}{30(1+|u(t)|)} + \frac{1}{90(1+|\phi(t)|)} - \frac{1}{30(1+|u(t)|)} + \frac{1}{90(1+|\phi'(t)|)}\right| \\ &\leq \frac{1}{30}|u(t)-v(t)| + \frac{1}{90}|\phi(t)-\phi'(t)| \\ &\leq \sup_{t\in P}\frac{1}{30}|u(t)-v(t)| + \frac{1}{90}|\phi(t)-\phi'(t)| \\ &= \frac{1}{30}||u(t)-v(t)||_\infty + \frac{1}{90}|\phi(t)-\phi'(t)|\end{aligned}$$

Since $\beta = \frac{1}{2}$, we have $\gamma_1 = \frac{1}{15}$ and $\gamma_2 = \frac{1}{90}$.
Therefore, the problem outlined above meets all the conditions of Theorem 4.1, indicating that a unique solution exists for this problem.

Next, we demonstrate numerically that the problem has a unique solution in the interval $[0,1]$. To achieve this, we employ the Picard iteration scheme. Let $y(t) := D_0^{1/2}u(t)$. Then the equation becomes

$$y(t) = \frac{1}{30(1+|u(t)|)} + \frac{1}{90(1+|y(t)|)}.$$

Starting with $u_0(t) = 0,\ y_0(t) = 0$, the Picard iteration is

$$\begin{aligned}y_{n+1}(t) &= \frac{1}{30(1+|u_n(t)|)} + \frac{1}{90(1+|y_n(t)|)}, \\ u_{n+1}(t) &= \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}\, y_{n+1}(s)\,\mathrm{d}s.\end{aligned}$$

t	u_1	y_1	u_2	y_2	u_3	y_3	u_4	y_4	u_5	y_5
0.0000	10.245405	0.044444	3.135664	0.013601	4.384975	0.019022	3.940482	0.017094	4.073628	0.017671
0.0100	10.249134	0.044444	3.136578	0.013601	4.386163	0.019019	3.941606	0.017091	4.074761	0.017669
0.1002	10.259846	0.044444	3.139206	0.013598	4.389575	0.019015	3.944833	0.017088	4.078012	0.017664
0.2504	10.269046	0.044444	3.141463	0.013595	4.392502	0.019010	3.947606	0.017085	4.080806	0.017661
0.5008	10.279430	0.044444	3.144010	0.013593	4.395808	0.019006	3.950736	0.017082	4.083958	0.017658
0.7496	10.287352	0.044444	3.145954	0.013590	4.398330	0.019001	3.953123	0.017079	4.086363	0.017654
1.0000	10.294079	0.044444	3.147604	0.013589	4.400470	0.018998	3.955150	0.017076	4.088404	0.017652

Table 1. Picard iterations for $u(t)$ and $y(t) = D^{1/2}u(t)$ at selected t values (up to 5 iterations).

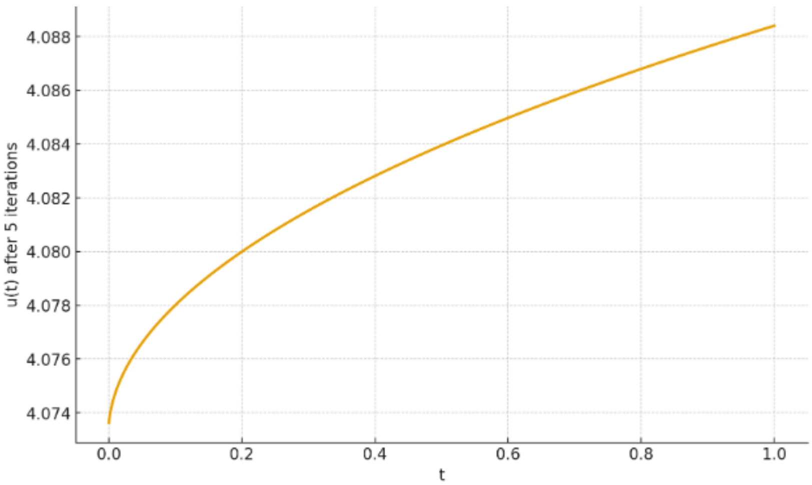

Figure 1. Numerical solution of the deformable fractional order $\omega = \frac{1}{2}$ implicit differential equation

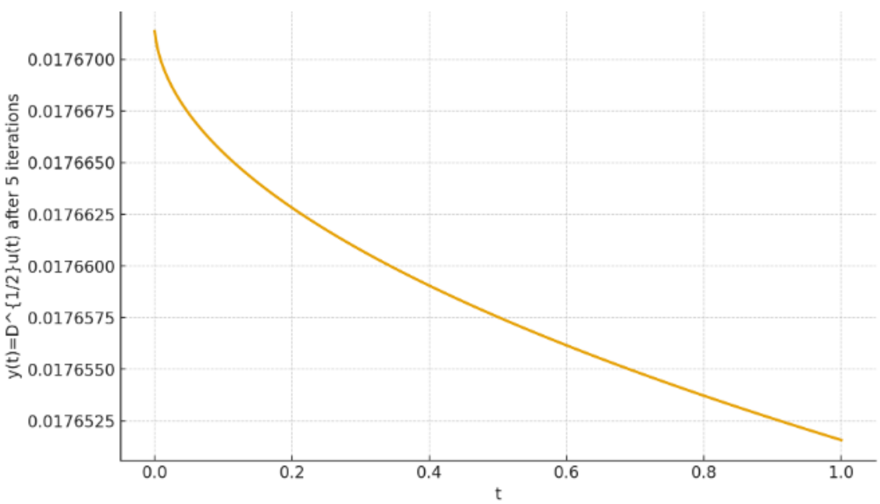

Figure 2. Numerical solution of the deformable fractional order $\omega = \frac{1}{2}$ implicit differential equation

The Picard iteration converges rapidly to a consistent pair $(u(t), y(t))$*. After 3 iterations, the solution is essentially stabilized:*

$$y(t) \approx 0.01765$$

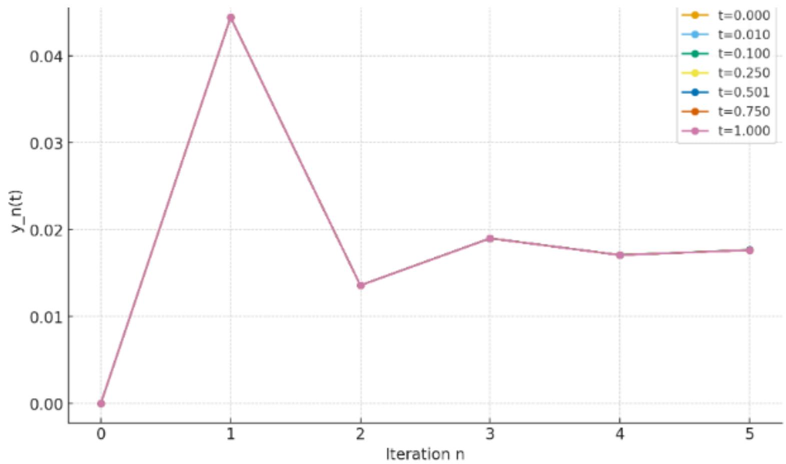

Figure 3. Numerical solution of the deformable fractional order $\omega = \frac{1}{2}$ implicit differential equation

nearly constant over [0, 1]*,*

$$u(1) \approx 4.0884.$$

5. Conclusion

In this article, we first proved fixed point theorems for Banach contraction in the framework of perturbed S-metric spaces, which depend on the experimental distance measuring function. Also, we provided examples and an application related to deformable fractional order differential equations to vindicate our results. We showed that a unique solution exists numerically by solving a deformable fractional order implicit differential equation.

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Conflict of interest

The authors assert that there are no conflicts of interest to disclose.

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