STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–9.
Published online in International Academic Press (www.IAPress.org)

Boundary Optimal Control of Infinite Order Linear Elliptic Systems Under Pointwise Control Constraints

Basima Abd ElHakim*, Samira El-Tamimy, Ghada E. Mostafa

Department of Mathematics, Faculty of Sciences, Al-Azhar University [Girl's Branch], Nasr City, Cairo, Egypt

Abstract This paper presents a rigorous analysis of an optimal boundary control problem governed by a linear elliptic equation of infinite order subject to pointwise control constraints. Such problems arise naturally in various applications but remain insufficiently studied due to the analytical difficulties associated with infinite-order operators and control constraints. The main objective of this work is to establish the Well-posedness of the state equation and derive optimality conditions for the associated control problem. Under assumptions on the system coefficient and admissible control set, we prove the existence and uniqueness of the weak solution to the state equation. Under pointwise control constraints on the boundary, we demonstrate the existence of an optimal control using convexity and compactness arguments that are adapted to the infinite order setting. By deriving the associated adjoint system, we formulate first order necessary optimality conditions in the form of a variational inequality involving the boundary adjoint variable. Furthermore, we discuss optimality conditions under coercivity assumptions on the infinite order operator. The results presented in this paper extend several known results for finite order elliptic systems to the infinite order framework, thereby filling an important gap in the existing literature on boundary optimal control.

Keywords Optimal control, boundary control, linear elliptic equation, necessary optimality conditions.

AMS 2010 subject classifications 46E35, 35J25, 35F05, 35D30, 34K06.

DOI: 10.19139/soic-2310-5070-3172

1. Introduction

Optimal control problems governed by partial differential equations have attracted considerable attention due to their wide range of applications in science and engineering. Boundary control problems for elliptic systems play a central role in many physical and engineering models where the control acts on the boundary of the spatial domain. Over the past decades, extensive research has been devoted to finite order elliptic systems, leading to well-established theoretical frameworks concerning existence, uniqueness and optimality conditions.

In contrast, elliptic systems of infinite order have received significantly less attention, despite their relevance in advanced models of continuum mechanics, signal processing and control theory. The analysis of such systems presents substantial mathematical challenges, mainly due to the nonstandard structure of infinite-order differential operators and the lack of compactness properties commonly used in finite-order setting. These difficulties become even more pronounced when pointwise control constraints are imposed.

Infinite order operators provide a powerful framework for describing multi-scale phenomena, long-range interactions and regularizing mechanisms not captured by standard second-or fourth-order elliptic equations. They naturally arise in anomalous diffusion, material science and quantum mechanics.

Several authors have investigated boundary optimal control problems for elliptic equations and systems of finite order under various types of constraints. However the extension of these results to infinite-order elliptic systems

*Correspondence to: Basima Abd ElHakim (Email: BasemaAbdElhakim27082@azhar.edu.eg). Department of Mathematics, Faculty of Sciences, Al-Azhar University [Girl's Branch], Nasr City, Cairo, Egypt.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

remains largely unexplored. In particular, the literature lacks comprehensive studies addressing the existence of optimal control and the derivation of necessary optimality conditions in the presence of pointwise control constraints for infinite-order systems.

The purpose of this paper is to fill this gap by providing a rigorous theoretical analysis of a boundary optimal control problem governed by linear elliptic systems of infinite order.

The main objective of this study is to establish a rigorous analytical framework that ensures the well-posedness of the state equations, characterizes the mapping from boundary controls to system states and derives first-order optimality conditions for the constrained control problems by using [1, 2].

For linear elliptic control problems of infinite order with pointwise control constraints were established in the paper by [11, 23, 26] and constraints on the state were investigated by [21, 24] , for a semilinear problems of infinite order with finite dimension, this obtained by [12, 20].

The paper which a near connection to our work we refer to [16, 17, 18, 28]. The existence of the Lagrange multipliers was discussed by [13, 25, 27] for elliptic case.

In Dubinskii [4, 5] studied the Cauchy Dirichlet problem

$$L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma}u) = h(x), \qquad x \in \Omega$$
$$D^{|\omega|}u(x)|_{\partial\Omega} = 0, \qquad |\omega| = 0, 1, 2, \cdots$$

infinite order Sobolev spaces

$$W^{\infty}\{a_{\alpha}, P_{\alpha}\}(\Omega) = \{u(x) \in C_0^{\infty}(\Omega) : P(x) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha}u\|_{P_{\alpha}}^{P_{\alpha}} < \infty\},$$

where $a_{\alpha} \geq 0$ and $P_{\alpha} \geq 1$ are numerical sequences and established of $W^{\alpha}\{a_{\alpha}, P_{\alpha}\}$ and boundary value problem above is investigated where $\Omega \subset R^N$.

Gali. et al. [6] presented a set of inequalities defining on optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [7].

Gali. et al. [8, 9, 10] presented some control problems generated by both elliptic and hyperbolic linear operator of infinite order with finite number of variables. El-Zahaby et al. [15, 19, 22] obtained the optimal control of problems governed by variational inequalities of an infinite order with finite domain.

We define the Sobolev space $W^{\infty}\{a_{\alpha}, 2\}$ of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of R^n, $n \geq 1$ as follows

$$W^{\infty}\{a_{\alpha}, 2\} = \{\phi \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_{\alpha}\|D^{\alpha}\phi\|_2^2 < \infty\},$$

where $a_{\alpha} \geq 0$ is a numerical sequence and $\|.\|_2$ is the canonical norm with space $L^2(R^n)$ all functions are assumed to be the real value on

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index for differentiation $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$.

The duality paring of the space $W^{\infty}\{a_{\alpha}, 2\}$ and $W^{-\infty}\{a_{\alpha}, 2\}$ is postulated by the formula

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_{\alpha} \int_{R^n} \psi_{\alpha}(x) D^{\alpha}\phi(x) dx,$$

where $\phi \in W^{\infty}\{a_\alpha, 2\}, \psi \in W^{-\infty}\{a_\alpha, 2\}$.

From above, $W^{\infty}\{a_\alpha, 2\}$ is everywhere dense in $L^2(R^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ dense the topological dual space with respect to $L^2(R^n)$, so we have the following chain

$$W^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}.$$

Analogous to the above chain, we have

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\},$$

where $W_0^{\infty}\{a_\alpha, 2\}$ is the set of all function of $W^{\infty}\{a_\alpha, 2\}$ which vanish on the boundary Γ of R^n, i.e.,

$$W_0^{\infty}\{a_\alpha, 2\} =$$
$$\{\phi(x) \in C_0^{\infty}(R^n) : \|\phi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty, \;\; D^{|\omega|}\phi|_\Gamma = 0, \;\; |\omega| = 0, 1, \cdots\}.$$

The main contributions of this paper are threefold:

1. We develop a new variational formulation for infinite-order elliptic boundary control, proving existence, uniqueness and regularity of solutions driven solely by boundary inputs.
2. We establish the existence of an optimal control under pointwise constraints on the control, relying on convexity and compactness arguments tailored to the infinite-order setting.
3. We derive a complete optimality system, including the adjoint equation, boundary trace characterization and variational inequality describing the optimal control. Additionally, we formulate first-order necessary optimality conditions using the Lagrange principle and adjoint system techniques. .

The novelty of this work lies in extending classical boundary optimal control results from finite-order elliptic systems to the infinite-order case under pointwise control constraints. The results presented here contribute new theoretical insights to the field of optimization and control of partial differential equations and provide a foundation for future studies on nonlinear systems and numerical approximations.

The paper is structured as follows: In section one, we introduce for functional spaces of infinity order with finite dimension. The existence of optimal control is given in section two. In section three, we derive optimality conditions for boundary control problem of infinite order with pointwise control constraints. In section four, we formulate Lagrange principle. Finally, section five concludes the paper and outlines possible directions for future research.

2. Existence of Optimal Control

Assumption 2.1. Let $\Omega \subset R^N$ be a bounded domain with Lipschitz-Continuous boundary Γ and suppose that $\lambda \geq 0$, $y_\Omega \in L^2(\Omega)$, $y_\Gamma \in L^2(\Gamma)$, $\beta \in L^2(\Gamma)$ with $\beta(x) \geq 0$ for almost all $x \in \Gamma$ and $u_a, u_b \in L^2(\Gamma)$ with $u_a(x) \leq u_b(x)$.

We consider the following linear elliptic control problem with boundary control

$$(P) \begin{cases} \min J(y,u) = \dfrac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \dfrac{\lambda}{2}\|u\|_{L^2(\Gamma)} \\ \text{subject to} \\ \qquad Ay = 0 \qquad\qquad \text{in} \quad \Omega \\ \qquad y^{|\omega|}|_\Gamma = \beta(u - y) \qquad \text{on} \quad \Gamma, \quad |\omega| = 0, 1, 2, \cdots \\ \text{and} \\ \qquad u_a(x) \leq u(x) \leq u_b(x), \end{cases}$$

where A is infinite order operator with finite dimension of the form

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y, \qquad a_\alpha > 0.$$

This operator is bounded, self adjoint mapping $W_0^\infty\{a_\alpha, 2\}$ onto $W_0^\infty\{a_\alpha, 2\}$. We define the set of admissible control U_{ad} by:

$$U_{ad} = \{u \in L^2(\Gamma) : u_a(x) \le u(x) \le u_b(x) \quad \text{for almost every} \quad x \in \Gamma\}.$$

U_{ad} is non-empty, closed, convex and bounded in $L^2(\Gamma)$.

For the existence of a unique solution to problem (P), we suppose

$$\int_\Gamma (\beta(x))^2 \; ds(x) > 0$$

Definition 2.1. We call $\bar{u} \in U_{ad}$ an optimal control for (P) if $J(y(\bar{u}), \bar{u}) \le J(y(u), u)$ for all $u \in U_{ad}$. The function $y = y(\bar{u})$ is said to be the (associated) optimal state, we say that $\bar{u} \in U_{ad}$ is a local optimal solution for (P) if there exist an $\epsilon > 0$ so that

$$J(y(\bar{u}), \bar{u}) \le J(y(u), u) \qquad \text{for all} \quad u \in U_{ad} \quad \text{with} \quad \|u - \bar{u}\|_{L^2(\Gamma)} \le \epsilon.$$

The operator $G : L^2(\Gamma) \to W^\infty\{a_\alpha, 2\}$, $u \mapsto y(u)$ is continuous.

We interpret G as a continuous linear operator mapping $L^2(\Gamma)$ into $L^2(\Omega)$ that is we take $S = E_Y G$ and $S : L^2(\Gamma) \to L^2(\Omega)$, $u \mapsto y(u)$.

We transformed the control problem into so-called reduced cost functional in term of pure control u:

$$f(u) = \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Gamma)}. \tag{2.1}$$

Theorem 2.1. *With Assumption 2.1 holding there exist a unique weak solution $y \in W^\infty\{a_\alpha, 2\}$ for every $u \in L^2(\Gamma)$. i.e.*

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega a_\alpha (D^\alpha y)(x)(D^\alpha v)(x) \; dx + \int_\Gamma \beta y v \; ds = \int_\Gamma \beta u v \; ds \tag{2.2}$$

Furthermore

$$\|y\|_{W^\infty\{a_\alpha,2\}} \le c\|u\|_{L^2(\Gamma)},$$

for a constant c depend on $\beta \in L^\infty(\Gamma)$.

Proof

In order to apply the Lax-Milgram Lemma, we put $V = W^\infty\{a_\alpha, 2\}$ and define the functional $F(v)$ and the bilinear form a, respectively, by

$$F(v) = \int_\Gamma \beta u v ds$$

$$\begin{aligned} a[y, v] &= \sum_{|\alpha|=0}^{\infty} ((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), v(x))_{L^2(R^n)} \\ &= \sum_{|\alpha|=1}^{\infty} ((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), v(x))_{L^2(R^n)} \\ &\quad + (q(x)y(x), v(x))_{L^2(R^n)} + (\beta(x)y(x), v(x))_{L^2(\Gamma)} \\ &= \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha v(x))_{L^2(R^n)} \\ &\quad + (q(x)y(x), v(x))_{L^2(R^n)} + (\beta(x)y(x), v(x))_{L^2(\Gamma)} \end{aligned}$$

where $q(x), \beta(x)$ are real valued functions. To this end, We have to verify that the bilinear form a is bounded and V-elliptic First, one easily derives

$$
\begin{aligned}
|a[y, v]| = | \sum_{|\alpha|=1}^{\infty} & (a_\alpha D^\alpha y(x), D^\alpha v(x))_{L^2(R^n)} \\
& + (q(x)y(x), v(x))_{L^2(R^n)} + (\beta(x)y(x), v(x))_{L^2(\Gamma)} \\
\leq \alpha_0 & \|y\|_{W^\infty\{a_\alpha,2\}} \|v\|_{W^\infty\{a_\alpha,2\}}
\end{aligned}
$$

i.e the boundedness of a. Indeed, this is an immediate consequence of the estimates

$$
\begin{aligned}
| \int_\Omega q(x) y v(x) dx| &\leq \|q\|_{L^\infty(\Omega)} \|y\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \|q\|_{L^\infty(\Omega)} \|y\|_{W^\infty\{a_\alpha,2\}} \|v\|_{W^\infty\{a_\alpha,2\}}
\end{aligned}
$$

$$
\begin{aligned}
| \int_\Gamma \beta y v ds| &\leq \|\beta\|_{L^\infty(\Gamma)} \|y\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\
&\leq \|\beta\|_{L^\infty(\Gamma)} c_0 \|y\|_{W^\infty\{a_\alpha,2\}} \|v\|_{W^\infty\{a_\alpha,2\}}
\end{aligned}
$$

where the trace has been used for the latter estimate. To show the V-ellipticity, we argue as follows. In view of assumptions, we have $q \neq 0$ in $L^\infty(\Omega)$ or $\beta \neq 0$ in $L^\infty(\Gamma)$. In case where $\beta \neq 0$ there exist a measurable set $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| > 0$ and $\delta > 0$ such that $\beta(x) \geq \delta$ for all $x \in \Gamma_1$. For $q \neq 0$, we have $q(x) \geq \delta$ for all $x \in \Omega$ we find that

$$
\begin{aligned}
a[y, y] = & \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} \\
& + (q(x)y(x), y(x))_{L^2(R^n)} + (\beta(x)y(x), y(x))_{L^2(\Gamma)} \\
\geq & (\sum_{|\alpha|=1}^{\infty} a_\alpha D^\alpha y(x), D^\alpha y(x))_{L^2(R^n)} + \delta (y(x), y(x))_{L^2(\Gamma_1)} \\
\geq & \frac{\min\{1, \delta\}}{c(\Gamma_1)} \|y\|_{W^\infty_{\{a_\alpha,2\}}}
\end{aligned}
$$

Consequently, the assumptions of Lax-Milgram lemma are satisfied. In addition, employing the trace theorem once more, We can conclude as follows:

$$
\begin{aligned}
|F(v)| = | \int_\Gamma & \beta u v ds| \\
& \leq \|\beta\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\
& \leq \|\beta\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} c_0 \|v\|_{W^\infty\{a_\alpha,2\}} \\
& \leq c \|u\|_{L^2(\Gamma)} \|v\|_{W^\infty\{a_\alpha,2\}}
\end{aligned}
$$

But this means that $\|F\|_{W^{-\infty}\{a_\alpha,2\}} \leq c\|u\|_{L^2(\Gamma)}$ and the asserted estimate for $\|y\|_{W^\infty\{a_\alpha,2\}}$ then follows from the Lax-Milgram Lemma. This cocludes the proof. □

Theorem 2.2. *Let $U_{ad} \subset L^2(\Gamma)$ be nonempty, bounded, closed convex and $y_\Omega \in L^2(\Omega)$, $\lambda \geq 0$. The mapping $S : L^2(\Gamma) \to L^2(\Omega)$ is assumed to be a linear and continuous operator. Then there exist an optimal control $\bar{u}$ solving*

$$
\min_{u \in U_{ad}} f(u) = \frac{1}{2} \|su - y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \|u\|^2_{L^2(\Gamma)}
$$

If $\lambda > 0$ holds or if S is injective, then $\bar{u}$ is uniquely determined.

Proof
Since $f(u) \geq 0$ holds, the infimum

$$j = \inf_{u \in U_{ad}} f(u) \qquad \text{exists.}$$

By Assumption, $U_{ad} \neq 0$. Thus, there is a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} f(u_n) = j$.

The set U_{ad} is bounded and closed. From the convexity of U_{ad} we infer that U_{ad} is weakly sequentially compact. Thus, there exist a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ and an element $\bar{u} \in U_{ad}$ satisfying $u_{n_k} \to \bar{u}$ for $k \to \infty$.

Since S is continuous, f is continuous from the convexity of J we infer

$$f(\bar{u}) \leq \lim_{k\to\infty} f(u_{nk}) = j$$

recall that j is the infimum of all function value $f(u), u \in U_{ad}$. From $\bar{u} \in U_{ad}$ we have $f(\bar{u}) \geq j$.

Thus, $f(\bar{u}) = j$ and $\bar{u}$ is an optimal control.

This proof is standard (Lions [14]), it is enough to note also that, even if $\lambda > 0$, this implies that J is strictly convex and there exist an unique optimal solution. □

3. First Order Optimality Conditions

We derive first order optimality conditions.

Lemma 3.1. *Let U be a real Banach spaces, $\mathcal{U} \subset U = L^2(\Gamma)$ be open, $C \subset \mathcal{U}$ be a convex and $f : \mathcal{U} \to \mathbb{R}$ be a function which is Gateaux differentiable in $\mathcal{U}$. Suppose that $\bar{u} \in C$ is a solution to*

$$\min_{u\in C} f(u) = \frac{1}{2}\|Su - z_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|_{L^2(\Gamma)}. \tag{3.1}$$

Then the following variational inequality holds

$$f'(\bar{u})(u - \bar{u}) \geq 0 \qquad \textit{for all} \qquad u \in C \tag{3.2}$$

If $\bar{u} \in c$ solves (3.2) and f is convex, then $\bar{u}$ is a solution to (3.1).

Next, we apply the previous Lemma to (3.1).

Theorem 3.2. *Let $U_{ad} \subset U = L^2(\Gamma)$ be a nonempty, convex and $z_d \in L^2(\Omega)$, $\lambda > 0$ be given. Furthermore, assume that $S \in L(L^2(\Omega), L^2(\Omega))$. Then $\bar{u} \in U_{ad}$ solve (3.1) if the variational inequality*

$$(S\bar{u} - z_d, y - \bar{y})_{L^2(\Omega)} + \lambda(\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0, \qquad \textit{for all} \quad u \in U_{ad} \tag{3.3}$$

Proof
The gradient of f is given by

$$f'(u) = S^*(S\bar{u} - z_d) + \lambda\bar{u}$$

then from (3.2) we have (3.3).

The variational inequality (3.3) can be expressed as

$$(S\bar{u} - z_d, Su - S\bar{u})_{L^2(\Omega)} + \lambda(\bar{u}, u - \bar{u})_{L^2(\Gamma)} \geq 0, \qquad \text{for all} \quad u \in U_{ad}.$$

□

Definition 3.1. The weak solution $p \in W_0^{\infty}\{a_\alpha, 2\}$ of the adjoint equation

$$\begin{aligned} &Ap = \bar{y} - z_d && \text{in} \quad \Omega \\ &p^{|\omega|}|_\Gamma + \beta p = 0 && \text{on} \quad \Gamma, \quad |\omega| = 0, 1, 2, \cdots \end{aligned} \tag{3.4}$$

with $\bar{y} = s\bar{u}$ is called the associated adjoint or dual state, let $z_d \in L^2(\Omega)$.

Furthermore, $\bar{y} \in W_0^{\infty}\{a_\alpha, 2\} \to L^2(\Omega)$.

Thus, $z_d - \bar{y}$ belong to $L^2(\Omega)$. By Lax-Milgram Lemma there exist a unique state $p \in W_0^{\infty}\{a_\alpha, 2\}$ satisfying (3.4).

Let y be the weak solution to (2.2).

Now, choosing p as a test function in the weak formulation of (2.1), we obtain

$$\sum_{|\alpha|=1}^{\infty} \int_\Omega a_\alpha (D^\alpha y)(x)(D^\alpha p)(x)\ dx + \int_\Omega q(x)y(x)p(x)\ dx + \int_\Gamma \beta y(x)p(x)\ ds$$
$$= \int_\Gamma \beta u(x) up(x)\ ds$$

on the other-hand, for p we obtain the test function $y \in W_0^{\infty}\{a_\alpha, 2\}$ that

$$\sum_{|\alpha|=1}^{\infty} \int_\Omega a_\alpha (D^\alpha p)(x)(D^\alpha y)(x)\ dx + \int_\Omega q(x)p(x)y(x)\ dx + \int_\Gamma \beta p(x)y(x)\ ds$$
$$= \int_\Omega (\bar{y} - z_d)\ dx$$

since the left hand sides are equal the assertion immediately follows and we find that

$$\int_\Omega (\bar{y} - z_d)y = \int_\Gamma \beta p(x)u\ ds$$

substitute in (3.3), we obtain

$$\int_\Gamma (\lambda\bar{u} + \beta p)(u - \bar{u})\ ds \geq 0 \qquad \forall \quad u \in U_{ad} \tag{3.5}$$

Theorem 3.3. *Let $\bar{u}$ denote an optimal control for (P) and let $\bar{y}$ denote the associated state. Then the adjoint equation (3.4) has a unique weak solution p that satisfies the variational inequality*

$$\int_\Gamma (\lambda\bar{u} + \beta p)(u - \bar{u})\ ds \geq 0 \qquad \forall \quad u \in U_{ad} \tag{3.6}$$

Conversely, every control $\bar{u} \in U_{ad}$ which together with its associated state $\bar{y} = y(\bar{u}) = S\bar{u}$ and the solution p to (3.4) satisfies the variational inequality (3.6) is optimal solution to (P) .

In this case, we obtain that

$$\bar{u}(x) = \begin{cases} u_a(x) & \text{if} \quad \beta(x)p(x) + \lambda\bar{u}(x) > 0 \\ \in [u_a(x), u_b(x)] & \text{if} \quad \beta(x)p(x) + \lambda\bar{u}(x) = 0 \\ u_b(x) & \text{if} \quad \beta(x)p(x) + \lambda\bar{u}(x) < 0 \end{cases} \tag{3.7}$$

and the weak minimum principle becomes

$$\min_{u_a(x) \leq v \leq u_b(x)} \{(\beta(x)p(x) + \lambda\bar{u}(x))v\} = \{(\beta(x)p(x) + \lambda\bar{u}(x))\bar{u}(x)\}$$

for almost every $x \in \Gamma$.

In addition, we have the following result.

Theorem 3.4. *(Minimum Principle)*
Suppose that $\bar{u}$ is an optimal control for (P) and let p denote the associated adjoint state. Then, for almost every $x \in \Gamma$, the minimum

$$\min_{u_a(x)\leq v\leq u_b(x)} \{\beta(x)p(x)v + \frac{\lambda}{2}v^2\}$$

is attained at $v = \bar{u}(x)$.

Hence, for $\lambda > 0$ we have for almost every $x \in \Gamma$ the projection formula

$$\bar{u}(x) = P[u_a(x), u_b(x)]\{\frac{-1}{\lambda}\beta(x)p(x)\} \tag{3.8}$$

Conversely, a control $\bar{u} \in U_{ad}$ is optimal if it satisfies, together with the associated adjoint state p, the projection formula (3.8). Summarizing, a control u is optimal for (P) if and only if u satisfies together with y and p.

The following first-order necessary optimality system

$$\begin{array}{lll} Ay = 0 & \text{in} \quad \Omega & Ap = \bar{y} - z_d \\ y^{|\omega|}|_\Gamma = \beta(u-y), & |\omega| = 0,1,2,\cdots & p^{|\omega|}|_\Gamma + \beta p = 0 \\ u \in U_{ad} & & \\ (\lambda\bar{u} + \beta p, v - \bar{u})_{L^2(\Gamma)} \geq 0 & & \forall \quad v \in U_{ad} \end{array}$$

4. The Formal Lagrange Principle

In this section, we derive the optimality conditions by utilizing the Lagrange functional.

We treat all Lagrange multipliers as function in $L^2(\Gamma)$ without any proof. Therefore, we call this procedure "formal". But the main goal of this section is to explain the use strategy which can be also applied to much more complex problems.

The Lagrangian function for problem (P) is defined by

$$\mathcal{L}(y,u,p) = J(y,u) + \int_\Omega Ayp_1 \; dx + \int_\Gamma (y^{|\omega|}|_\Gamma - \beta(u-y))p_2 \; ds$$

where $p_1 : \Omega \to \mathbb{R}$, and, $p_2 : \Gamma \to \mathbb{R}$ are the Lagrange multipliers associated with the partial differential equation and bounded condition, respectively.

We set $p = (p_1, p_2)$. From the Lagrange principle we conclude that $(\bar{y}, \bar{u})$ together with the Lagrange multipliers p_1, p_2 satisfies the first-order necessary optimality conditions for

$$\min_{(y,u)} \mathcal{L}(y,u,p), \qquad u \in U_{ad}$$

Since y is now formally unconstrained, the derivative of $\mathcal{L}$ with respect to y has to vanish that is

$$D_y\mathcal{L}(\bar{y},\bar{u},p)h = 0 \qquad \text{for all} \qquad y \in W^\infty\{a_\alpha, 2\}$$

is equivalent with the weak form (3.4).

Moreover, from the box constraints for u we deduce the variational inequality

$$D_u\mathcal{L}(\bar{y},\bar{u},p)(u-\bar{u}) \geq 0 \qquad \text{for all} \qquad u \in U_{ad}$$

is equivalent with the variational inequality (3.6).

The proof can be found in [3].

5. Conclusion

This study examined a boundary optimal control problem governed by an infinite-order linear elliptic system under pointwise control constraints. By establishing an appropriate functional framework for infinite-order operators, we proved the well-posedness of the state equation and derived the necessary optimality conditions using variational methods and the Lagrange multiplier approach. The analysis showed that, despite the additional complexities introduced by the infinite-order operator and the presence of pointwise constraints, the control problem remains mathematically tractable and admits a unique optimal solution. The results extend classical elliptic optimal control theory to a more general and nonlocal setting. Future work may focus on semilinear extensions, numerical approximation schemes, and further applications of infinite-order models.

REFERENCES

1. E. Casas and F. Tröltzsch, *Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAM Journal on Optimization, vol. 13, no. 2, pp. 406–431, 2002.
2. C. Meyer and F. Tröltzsch, *On an elliptic optimal control problem with pointwise mixed control-state constraints*, Recent Advances in Optimization, Springer, pp. 187–204, 2006.
3. F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*, American Mathematical Society, vol. 112, 2010.
4. Ju. A. Dubinskiĭ, *Sobolev spaces of infinite order and the behavior of solutions of some boundary value problems with unbounded increase of the order of the equation*, Mathematics of the USSR-Sbornik, vol. 27, no. 2, pp. 143–160, 1975.
5. Yu. A. Dubinskii, *Some imbedding theorems for Sobolev spaces of infinite order*, Doklady Akademii Nauk, vol. 242, no. 6, pp. 1241–1244, 1978.
6. I. M. Gali and H. A. El-Saify, *Optimal control of a system governed by a self-adjoint elliptic operator with an infinite number of variables*, Proc. Int. Conf. on Functional-Differential Systems and Related Topics II, Warsaw, pp. 126–133, 1981.
7. I. M. Gali, H. A. El-Saify, and S. A. El-Zahabi, *Optimal control of a system governed by a hyperbolic operator*, Ordinary and Partial Differential Equations, Springer, pp. 157–163, 1985.
8. I. M. Gali and H. A. El-Saify, *Distributed control of a system governed by Dirichlet and Neumann problems for a self-adjoint elliptic operator with an infinite number of variables*, Journal of Optimization Theory and Applications, vol. 39, pp. 293–298, 1983.
9. I. M. Gali, H. A. El-Saify, and S. A. El-Zahabi, *Optimal control of a system governed by a hyperbolic operator*, Ordinary and Partial Differential Equations, Springer, pp. 157–163, 1985.
10. I. M. Gali and S. A. El-Zahaby, *Optimal control of a system governed by variational inequalities for infinite order elliptic operator*, Kyungpook Mathematical Journal, vol. 31, no. 2, pp. 193–200, 1991.
11. S. A. El-Zahaby and S. El-Tamimy, *Optimal control problems of systems governed by elliptic operator of infinite order with pointwise control constraint*, Journal of Mathematics and Statistical Science, pp. 199–212, 2015.
12. S. A. El-Zahaby and S. El-Tamimy, *Second order optimality conditions for a semilinear elliptic control problem of infinite order with pointwise control constraints*, Journal of Mathematics and Statistical Science, vol. 5, pp. 263–273, 2019.
13. A. Rösch and F. Tröltzsch, *Existence of regular Lagrange multipliers for elliptic optimal control problems with pointwise control-state constraints*, SIAM Journal on Control and Optimization, vol. 45, no. 2, pp. 548–564, 2006.
14. J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, vol. 170, 1971.
15. S. A. El-Zahaby, *Optimal control of systems governed by infinite order operator*, POT, vol. 151, 2000.
16. D. M. Abdou, H. M. Rezk, A. S. Zaghrout, and S. H. Saker, *Some new dynamic inequalities involving the dynamic Hardy operator with kernels*, Al-Azhar Bulletin of Science, vol. 35, no. 2, 2024.
17. H. M. Serag et al., *Distributed control for cooperative elliptic systems under conjugation conditions*, Al-Azhar Bulletin of Science, vol. 29, no. 2-B, pp. 1–10, 2018.
18. G. A. M. Esmaeil, A. A. S. Zaghrout, and A. A. El-Deeb, *Dynamic Ulam stability behavior for first-order nonlinear equations on time scales*, Al-Azhar Bulletin of Science, vol. 36, no. 1, 2025.
19. S. A. El-Zahaby and I. M. El-Zoulati, *Optimal control of problems governed by obstacle type for infinite order*, Journal of Progressive Research in Mathematics, vol. 5, no. 5, pp. 656–663, 2015.
20. S. A. El-Zahaby and S. El-Tamimy, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations of infinite order*, Applied Mathematics, vol. 18, no. 5, pp. 1075–1080, 2024.
21. S. El-Tamimy, *Linear elliptic control problems of infinite order with pointwise state constraints*, Applied Mathematics, vol. 16, no. 6, pp. 1005–1008, 2022.
22. S. A. El-Zahaby and Gh. H. Mostafa, *Optimal control of variational inequalities of infinite order*, Journal of the Mathematical Egyptian Society, vol. 13, pp. 99–106, 2005.
23. M. Hinze, *A variational discretization concept in control constrained optimization: the linear-quadratic case*, Computational Optimization and Applications, vol. 30, no. 1, pp. 45–61, 2005.
24. M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE Constraints*, Springer, vol. 23, 2008.
25. K. Ito and K. Kunisch, *Lagrange Multiplier Approach to Variational Problems and Applications*, SIAM, 2008.
26. E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM Journal on Control and Optimization, vol. 24, pp. 1309–1322, 1986.

27. V. Karl and D. Wachsmuth, *An augmented Lagrange method for elliptic state constrained optimal control problems*, Computational Optimization and Applications, vol. 69, no. 3, pp. 857–880, 2018.
28. W. Gong and Z. Tan, *A new finite element method for elliptic optimal control problems with pointwise state constraints in energy spaces*, Journal of Scientific Computing, vol. 102, no. 1, 2025.