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Different Methods to Estimate Reliability for Exponentiated Mukherjee–Islam Distribution under Type-II Censoring

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Abstract Many reliability and survival datasets are effectively bounded above because of administrative follow-up, warranty limits, or planned termination times. The Exponentiated Mukherjee–Islam distribution (EMID) is a finite-range lifetime model on $(0, \theta)$ with flexible skewness and hazard-rate shapes. This paper studies parameter estimation and reliability inference for EMID under Type-II censoring using maximum likelihood estimation (MLE) and maximum product of spacings (MPS). Since the model depends on the product of the shape parameters αp, results are reported mainly for αp and the scale parameter θ.
An extensive Monte Carlo simulation study with $N = 1000$ replications compares MLE and MPS across sample sizes $n \in \{30, 50, 100, 300\}$, censoring ratios $C \in \{0.0, 0.1, 0.2, 0.3\}$, and several parameter settings, using RMSE and relative efficiency. Overall, RMSE decreases as n increases, while heavier censoring mainly inflates error for θ. Under moderate to high censoring, MPS is generally more stable for θ and yields less variable reliability estimates, whereas under light censoring MLE remains competitive for αp. The proposed analysis is also illustrated using a lung cancer survival dataset from the `survival` package, where graphical diagnostics and goodness-of-fit tests support the adequacy of the EMID fit under different censoring levels.

Keywords Exponentiated Mukherjee–Islam distribution, Type-II censoring, maximum likelihood, maximum product of spacings, reliability.

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1. Introduction

The Exponentiated Mukherjee–Islam distribution was first introduced by Aafa Rather and Subramanian [1]. It belongs to a family of distributions based on the Mukherjee and Islam distribution, which was originally introduced in 1983 [2]. It is a finite-range distribution, which is one of its key properties. In this work, the EMID is parameterized by two shape parameters (α, p) and one scale parameter θ. Researchers have studied its properties and associated results in detail.

Let us consider the Mukherjee–Islam distribution with the probability density function (pdf) that is given by

$$f(t) = \frac{p}{\theta^p} t^{p-1}, \qquad 0 < t < \theta,\ p, \theta > 0, \tag{1}$$

with cumulative distribution function (cdf)

$$F(t) = \left(\frac{t}{\theta}\right)^p, \tag{2}$$

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where p is the shape parameter and θ is the scale parameter. Let a random variable T be said to have an exponentiated distribution if its distribution function is defined as follows:

$$F_{\alpha}(t) = [F(t)]^{\alpha}, \qquad \alpha > 0, \tag{3}$$

where $F(t)$ is the baseline distribution function. By substituting (2) into (3), we obtain the distribution function of the Exponentiated Mukherjee–Islam distribution [3].

In reliability and lifetime data analysis, complete observation of all failure times is often not possible because of time limits, cost, or experimental constraints. This situation leads to censored data. Among the common censoring schemes, Type-II censoring is frequently used in life-testing experiments, where the test is stopped after a fixed number of failures. The statistical aspects of Type-II censored data and their practical consequences have been discussed by several authors; see, for example, Habibi-Rad et al. [4] and Zheng and Park [5]. Estimation problems under Type-II censoring have therefore been considered in many reliability studies. Balakrishnan and Kateri [6] examined maximum likelihood estimation for Weibull models based on complete and censored data and showed how censoring affects estimation accuracy. Dey et al. [7] studied inference for the two-parameter Rayleigh distribution under progressively Type-II censored samples and evaluated estimator performance for different censoring levels. Reliability analysis under more complicated Type-II censoring schemes has also been investigated. For example, Eliwa and Ahmed [8] considered accelerated life tests with progressive first failure Type-II censoring and used EM and MCMC algorithms for estimation. In addition, Al-Rassam et al. [9] compared different estimation methods for reliability-related functions in bounded-support lifetime models. Despite these contributions, direct comparisons between maximum likelihood estimation and maximum product of spacings under Type-II censoring for finite-range distributions are still limited, especially with respect to numerical stability, parameter identifiability, and reliability estimation.

Motivation and contributions. Bounded-support lifetime models are often required when failure times are observed within a fixed time limit, such as a warranty period, inspection horizon, or administrative follow-up. In such cases, finite-range distributions are more appropriate than heavy-tailed models, since the data cannot exceed a known upper bound. Moreover, many life-testing experiments are terminated after observing a pre-specified number of failures, which leads to Type-II censoring. This censoring scheme reduces the amount of information available in the upper tail of the distribution and may cause instability in likelihood-based estimation, particularly for the scale parameter and for reliability estimates at later times. For this reason, alternative estimation approaches are of practical interest. In this paper, we study parameter estimation for the Exponentiated Mukherjee–Islam distribution under Type-II censoring using maximum likelihood estimation (MLE) and maximum product of spacings (MPS), with particular emphasis on the identifiable parameter αp and the reliability function. A Monte Carlo simulation study is carried out for different sample sizes and censoring levels to compare the finite-sample performance of the estimators using RMSE and relative efficiency, and to examine numerical convergence. The proposed methods are also illustrated using a real survival dataset, including graphical diagnostics, goodness-of-fit assessment, and reliability estimation with confidence intervals. Based on the simulation and data analysis results, we discuss situations in which MLE or MPS may be preferable, especially for small sample sizes or moderate to high censoring levels.

The paper is organized as follows. Section 2 introduces the Exponentiated Mukherjee–Islam distribution (EMID), its reliability and hazard functions, and the estimation methods under complete and Type-II censored data. Section 3 presents the simulation design and applies the proposed methods to real survival data and examines model fit and reliability estimation. Section 4 Discusses the numerical the simulation and the application findings. And finally, section 5 concludes the paper with a summary of the main results and remarks.

2. Material and methods

This section presents the EMID model, its reliability and hazard functions, and parameter estimation under complete and Type-II censored samples using maximum likelihood (ML) and maximum product of spacings (MPS). The resulting estimators form the basis for the simulation study in section 3.

2.1. *Exponentiated Mukherjee–Islam Distribution*

The cumulative distribution function (CDF) of the EMID is given by

$$F(t) = \left[\left(\frac{t}{\theta} \right)^{p} \right]^{\alpha} = \left(\frac{t}{\theta} \right)^{\alpha p}, \qquad p, \theta, \alpha > 0, \tag{4}$$

where α and p are shape parameters, and θ is a scale parameter.

From (4), the parameters α and p enter the model only through their product αp. Therefore, α and p are not separately identifiable from the distribution of T. In this work, we report and compare estimation results mainly in terms of the product of the shape parameters αp (together with the scale parameter θ), which is identifiable and fully determines the PDF, CDF, reliability and hazard functions.

The corresponding probability density function (PDF) is

$$f(t) = \frac{\alpha p}{\theta^{\alpha p}} \, t^{\alpha p - 1}, \qquad 0 < t < \theta, \tag{5}$$

and the reliability (survival) and hazard rate functions are defined, respectively, as

$$R(t) = 1 - \left(\frac{t}{\theta} \right)^{\alpha p}, \tag{6}$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\alpha p \, t^{\alpha p - 1}}{\theta^{\alpha p} - t^{\alpha p}}. \tag{7}$$

The EMID is particularly useful for modeling lifetime or reliability data that exhibit skewness and bounded support $(0, \theta)$ [1, 2]. Its additional shape parameter α provides greater flexibility than the base Mukherjee–Islam or Weibull distributions. Overall, EMID is suitable for data with bounded support and flexible skewness/hazard shapes [3].

Figures 1–3 illustrate how varying the shape parameter α affects the EMID when $p = 1$ and $\theta = 5.7$. Figure 1 shows that as α increases, the PDF becomes more concentrated toward larger values of t. Figure 2 indicates that the CDF becomes more concave for larger α, meaning probability accumulates more slowly at small t. Finally, Figure 3 shows that the reliability function $R(t)$ declines more gradually as α increases, implying systems with higher α values exhibit longer expected lifetimes.

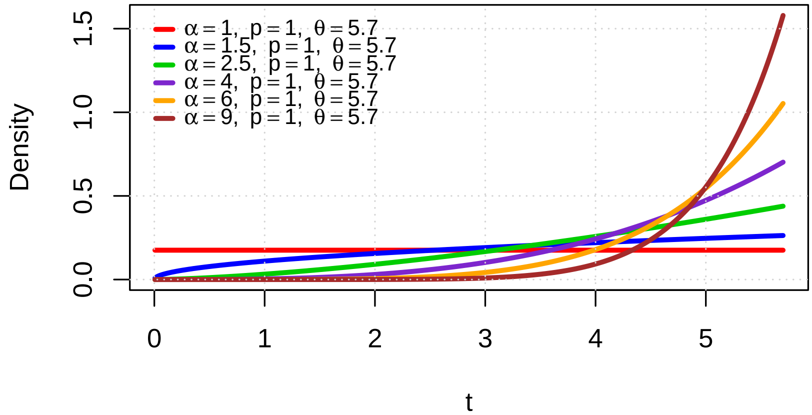

Figure 1. Theoretical probability density functions (PDFs) of the EMID for different values of α ($p = 1$, $\theta = 5.7$).

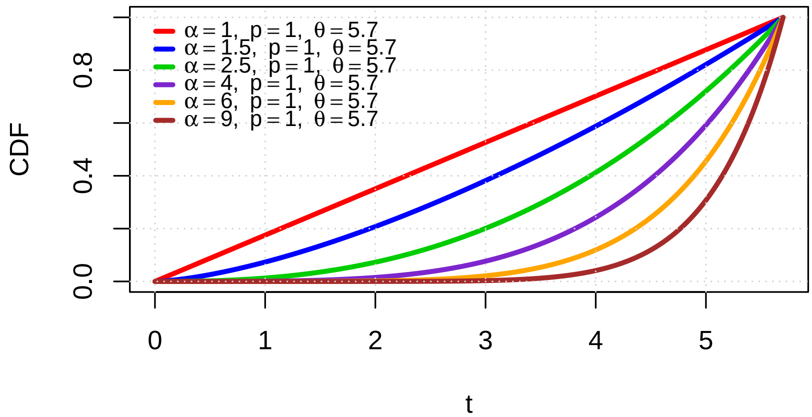

Figure 2. Theoretical cumulative distribution functions (CDFs) of the EMID for different values of α ($p = 1$, $\theta = 5.7$).

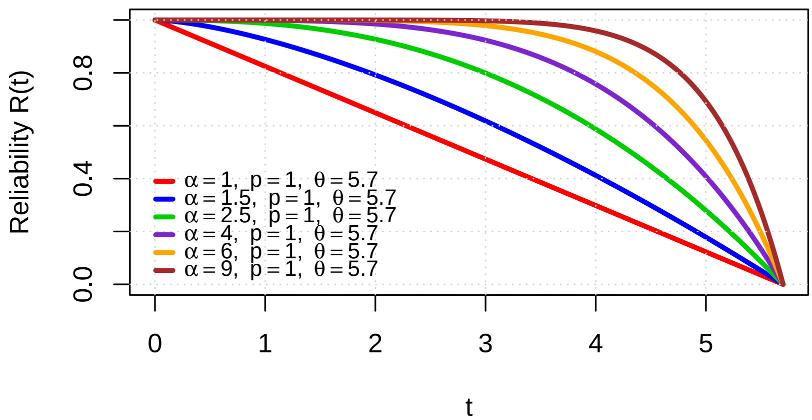

Figure 3. Theoretical reliability functions $R(t)$ of the EMID for different values of α ($p = 1$, $\theta = 5.7$).

2.2. Inference on the EMID Parameters

Let $t_{(1)} < t_{(2)} < \cdots < t_{(r)}$ denote the ordered failure times from a life test of n units drawn from the EMID with parameters α, p, and θ, where $r < n$. This setup corresponds to a Type-II censoring scheme, in which the experiment terminates after observing r failures and the remaining $(n - r)$ units are right-censored [5, 10].

Both the MLE and MPS estimation methods are derived under this censoring structure, and the corresponding reliability function is estimated for comparison.

2.2.1. Maximum Likelihood Estimation (MLE) The likelihood function of EMID under Type-II censoring is given by

$$L(t; \alpha, p, \theta) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(t_{(i)}) \, [R(t_{(r)})]^{(n-r)}, \tag{8}$$

where $f(t)$ and $R(t) = 1 - F(t)$ are the probability density and reliability functions of EMID, respectively [11, 12].

Substituting equations (4) and (5) yields

$$L(t; \alpha, p, \theta) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left[\frac{\alpha p}{\theta^{\alpha p}} \, t_{(i)}^{\alpha p - 1} \right] \left[1 - \left(\frac{t_{(r)}}{\theta} \right)^{\alpha p} \right]^{(n-r)}. \tag{9}$$

Taking the natural logarithm gives

$$\begin{aligned} \ln L(t; \alpha, p, \theta) = \ln \frac{n!}{(n-r)!} + r \ln \alpha + r \ln p - r\alpha p \ln \theta \\ + (\alpha p - 1) \sum_{i=1}^{r} \ln t_{(i)} + (n-r) \ln \left[1 - \left(\frac{t_{(r)}}{\theta} \right)^{\alpha p} \right]. \end{aligned} \tag{10}$$

The score equations are obtained by differentiating (10) with respect to each parameter:

For α:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{r}{\alpha} - rp \ln \theta + p \sum_{i=1}^{r} \ln t_{(i)} - (n-r) \frac{p \left(\frac{t_{(r)}}{\theta} \right)^{\alpha p} \ln(t_{(r)}/\theta)}{1 - \left(\frac{t_{(r)}}{\theta} \right)^{\alpha p}} = 0, \tag{11}$$

For p:

$$\frac{\partial \ln L}{\partial p} = \frac{r}{p} - r\alpha \ln\theta + \alpha \sum_{i=1}^{r} \ln t_{(i)} - (n-r) \frac{\alpha \left(\frac{t_{(r)}}{\theta}\right)^{\alpha p} \ln(t_{(r)}/\theta)}{1 - \left(\frac{t_{(r)}}{\theta}\right)^{\alpha p}} = 0, \tag{12}$$

For θ:

$$\frac{\partial \ln L}{\partial \theta} = -\frac{r\alpha p}{\theta} + \frac{\alpha p(n-r)}{\theta} \frac{(t_{(r)}/\theta)^{\alpha p}}{1 - (t_{(r)}/\theta)^{\alpha p}} = 0. \tag{13}$$

The maximum likelihood estimators $\hat{\alpha}_{\text{ML}}$, $\hat{p}_{\text{ML}}$, and $\hat{\theta}_{\text{ML}}$ are obtained by simultaneously solving the nonlinear equations (11)–(13). Since these equations have no closed-form solutions, they are solved numerically using an iterative method such as Newton–Raphson. The invariance property of MLE allows the reliability function to be estimated directly by substituting the parameter estimates into equation (6).

2.2.2. Maximum Product of Spacings Estimation (MPS) In statistics, the *maximum product of spacings* (MPS) method provides an alternative to the MLE for estimating parameters of univariate models. It is particularly useful when the likelihood function is ill-behaved or unbounded, leading to numerical instability or inconsistent estimators. MPS estimators are known to be consistent, asymptotically efficient, and often more robust than MLEs under general conditions [13, 14].

Let $t_{(1)}, \ldots, t_{(r)}$ denote the ordered observed failure times from $\text{EMID}(\alpha, p, \theta)$.

Complete Sample Case The MPS estimators are obtained by maximizing the logarithmic spacing function

$$S(\alpha, p, \theta) = \sum_{j=1}^{n+1} \ln\left[F(t_{(j)}; \alpha, p, \theta) - F(t_{(j-1)}; \alpha, p, \theta)\right], \tag{14}$$

where $F(t; \alpha, p, \theta)$ is the CDF of EMID, with $F(t_{(0)}; \alpha, p, \theta) = 0$ and $F(t_{(n+1)}; \alpha, p, \theta) = 1$.

Substituting the EMID CDF yields

$$S(\alpha, p, \theta) = \sum_{j=1}^{n+1} \ln\left[\left(\frac{t_{(j)}}{\theta}\right)^{\alpha p} - \left(\frac{t_{(j-1)}}{\theta}\right)^{\alpha p}\right]. \tag{15}$$

Type-II Censoring Case For Type-II censored samples, where only the first r failure times $t_{(1)} < \cdots < t_{(r)}$ are observed and the remaining $(n-r)$ items are censored, the modified log–spacing function (Ng *et al.*, following Cheng and Amin [15]) is

$$S_{\text{censored}}(\alpha, p, \theta) = \sum_{j=1}^{r+1} \ln\left[F(t_{(j)}; \alpha, p, \theta) - F(t_{(j-1)}; \alpha, p, \theta)\right] + (n-r)\ln\left[1 - F(t_{(r)}; \alpha, p, \theta)\right], \tag{16}$$

where $F(t_{(0)}) = 0$ and $F(t_{(r+1)}) = 1$, by definition of the support endpoints.

Substituting the EMID CDF gives

$$S_{\text{censored}}(\alpha, p, \theta) = \sum_{j=1}^{r+1} \ln\left[\left(\frac{t_{(j)}}{\theta}\right)^{\alpha p} - \left(\frac{t_{(j-1)}}{\theta}\right)^{\alpha p}\right] + (n-r)\ln\left[1 - \left(\frac{t_{(r)}}{\theta}\right)^{\alpha p}\right]. \tag{17}$$

Maximizing (17) with respect to α, p, and θ yields the MPS estimators $\hat{\alpha}_{\text{MPS}}$, $\hat{p}_{\text{MPS}}$, and $\hat{\theta}_{\text{MPS}}$. The corresponding estimate of the reliability function is then obtained by substituting these estimates into equation (6).

3. Results

To make a direct comparison between the MLE and MPS methods, a set of EMID scenarios was considered based on the identifiable product of the shape parameters $k = \alpha p$. In the simulation study, the values $k \in \{0.12,\ 0.8,\ 2.4,\ 7.2\}$ were used, while the scale parameter was fixed at $\theta = 5.7$. These values were selected in order to represent different distribution shapes and different reliability behaviors. For each scenario, the sample size was taken as $n \in \{30, 50, 100, 300\}$, and Type-II censoring was applied with censoring ratios $C \in \{0, 0.1, 0.2, 0.3\}$, where $C = 1 - r/n$.

In each Monte Carlo replicate, the parameters $(\alpha p, \theta)$ were estimated using the MLE and MPS methods. The performance of the estimators was evaluated using the root mean square error (RMSE). Smaller values of RMSE indicate better estimation performance. In addition, the relative efficiency was also reported in terms of mean squared error (MSE), which is directly related to RMSE.

3.1. Simulation Study

A Monte Carlo simulation study was carried out to compare the efficiency of the ML and MPS estimators for the parameters of the EMID. Since the parameters α and p appear in the model only through their product αp, the simulation results are mainly summarized for the product of the shape parameters αp and the scale parameter θ. For each simulation setting, $N = 1000$ Monte Carlo replications were generated.

The comparison between the estimators is based on the RMSE criterion, which is defined as

$$\text{RMSE}(\psi) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\psi}_i - \psi)^2}, \quad \psi \in \{\alpha p, \theta\}, \tag{18}$$

where N denotes the number of Monte Carlo replications.

Moreover, the relative efficiency of the MPS estimator with respect to the ML estimator is defined in terms of MSE as

$$\text{RE}(\psi) = \left(\frac{\text{RMSE}_{\text{ML}}(\psi)}{\text{RMSE}_{\text{MPS}}(\psi)}\right)^2, \tag{19}$$

where values greater than one indicate that the MPS estimator is more efficient, values less than one favor the ML estimator, and values close to one indicate similar performance.

All simulation experiments were performed using a Monte Carlo framework with $N = 1000$ replications for each combination of sample size and censoring level. In each replication, random samples were generated from the EMID model using the inverse transform method. Type-II censoring was then applied by keeping only the first r ordered observations.

The unknown parameters were estimated numerically for both the MLE and MPS methods using the `optim` function in `R` with the `L-BFGS-B` optimization algorithm. To avoid numerical problems, fixed initial values $(\alpha p)^{(0)} = 1$ and $\theta^{(0)} = \max(t) + 1$ were used in all simulations. In addition, parameter constraints were imposed to guarantee $\alpha p > 0$ and $\theta > \max(t)$ during the optimization process. An estimation run was regarded as successful only if the algorithm converged and all estimated parameter values were finite.

The convergence behavior of both methods was carefully monitored over all replications in order to evaluate numerical stability. Out of the 1000 simulated samples, the MLE converged successfully in 993 cases, corresponding to a non-convergence rate of 0.7%. For the MPS estimator, convergence was achieved in 963 cases, giving a non-convergence rate of 3.7%. When valid estimates from both methods were required simultaneously, 958 replications were retained, which leads to an overall exclusion rate of 4.2%. Replications with non-convergent solutions or non-finite estimates were excluded from the computation of RMSE, and hence of RE. Overall, both estimation methods show acceptable numerical stability under the considered simulation settings, although the MPS method appears slightly more sensitive to heavy censoring. Non-convergence was mainly observed for small n with $C = 0.3$, where the objective surface becomes flatter in θ.

The numerical results of the simulation study for different parameter values, sample sizes, and censoring ratios are reported in Tables 1–4.

Further, to support the numerical results in Tables 1–4, we also plot the estimated reliability functions. Figure 4 shows the MLE- and MPS-based reliability curves for two censoring cases: no censoring ($C = 0$) and heavy censoring ($C = 0.3$). We show these two cases because they represent the two extremes, while the intermediate censoring levels ($C = 0.1$ and $C = 0.2$) gave similar patterns and are omitted for shortness.

Table 1. Simulation results for the EMID under Type-II censoring with parameters $\alpha p = 0.3 \times 0.4 = 0.12$ and $\theta = 5.7$, showing estimated values, RMSEs, and REs of MLE and MPS methods across different sample sizes (n) and censoring ratios (C).

n	Method	$C = 0.0$						$C = 0.1$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	0.1276	0.0260		4.6178	1.4762		0.1248	0.0252		5.5910	2.0798	
	MPS	0.1440	0.0308	0.713	5.7667	1.3244	1.242	0.1410	0.0314	0.644	6.8189	3.2362	0.413
50	MLE	0.1255	0.0197		5.0076	0.9884		0.1225	0.0185		5.9361	1.5782	
	MPS	0.1440	0.0293	0.452	5.8032	0.7936	1.551	0.1412	0.0284	0.425	6.4866	1.9148	0.679
100	MLE	0.1228	0.0129		5.3189	0.5608		0.1210	0.0124		5.9865	1.1410	
	MPS	0.1462	0.0293	0.194	5.8098	0.2425	5.348	0.1409	0.0286	0.188	6.3505	1.7004	0.450
300	MLE	0.1207	0.0069		5.5657	0.1952		0.1200	0.0070		5.9845	0.6884	
	MPS	0.1430	0.0241	0.082	5.6970	0.1677	1.355	0.1428	0.0238	0.087	5.7654	0.4185	2.712
n	Method	$C = 0.2$						$C = 0.3$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	0.1254	0.0259		6.9140	5.5737		0.1225	0.0245		8.7637	9.8280	
	MPS	0.1455	0.0341	0.577	6.8189	4.0715	1.874	0.1412	0.0316	0.602	8.3912	7.6769	1.637
50	MLE	0.1212	0.0197		6.7236	3.4233		0.1216	0.0189		7.7416	6.7232	
	MPS	0.1461	0.0312	0.399	5.8808	1.7950	3.628	0.1426	0.0289	0.428	6.4099	2.8142	5.707
100	MLE	0.1202	0.0124		6.5818	2.3353		0.1201	0.0128		7.1455	3.8757	
	MPS	0.1478	0.0290	0.183	6.2689	1.3819	2.853	0.1475	0.0276	0.215	8.1012	4.4017	0.775
300	MLE	0.1194	0.0071		6.2238	1.1820		0.1191	0.0071		6.4582	1.7346	
	MPS	0.1432	0.0242	0.086	5.7012	0.4236	7.792	0.1433	0.0243	0.085	5.6680	0.4782	13.158

Table 2. Simulation results for the EMID under Type-II censoring with parameters $\alpha p = 2 \times 0.4 = 0.8$ and $\theta = 5.7$, showing estimated values, RMSEs, and REs of MLE and MPS methods across different sample sizes (n) and censoring ratios (C).

n	Method	$C = 0.0$						$C = 0.1$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	0.8544	0.1752		5.6630	0.2152		0.8412	0.1693		5.8104	0.3383	
	MPS	0.7790	0.1533	1.1430	5.9157	0.3115	0.6910	0.7723	0.1559	1.0860	6.0501	0.5563	0.6080
50	MLE	0.8307	0.1283		5.6715	0.1371		0.8189	0.1275		5.7995	0.2555	
	MPS	0.7817	0.1182	1.0850	5.8203	0.1844	0.7440	0.7749	0.1229	1.0370	5.9341	0.3860	0.6620
100	MLE	0.8187	0.0853		5.6857	0.0718		0.8047	0.0853		5.7907	0.1780	
	MPS	0.7917	0.0810	1.0530	5.7583	0.0921	0.7800	0.7801	0.0857	0.9950	5.8557	0.2458	0.7240
300	MLE	0.8043	0.0479		5.6955	0.0237		0.7981	0.0476		5.7547	0.1024	
	MPS	0.7939	0.0477	1.0040	5.7194	0.0305	0.7770	0.7884	0.0486	0.9790	5.7751	0.1241	0.8250
n	Method	$C = 0.2$						$C = 0.3$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	0.8404	0.1664		5.8974	0.4807		0.8205	0.1598		6.0044	0.6827	
	MPS	0.7715	0.1557	1.0690	6.1896	0.8029	0.5990	0.7486	0.1596	1.0010	6.4161	1.1673	0.5850
50	MLE	0.8066	0.1214		5.8691	0.3733		0.8069	0.1203		5.9855	0.5746	
	MPS	0.7617	0.1232	0.9850	6.0409	0.5625	0.6640	0.7579	0.1241	0.9690	6.2391	0.8653	0.6640
100	MLE	0.7986	0.0839		5.8450	0.2732		0.7981	0.0853		5.9049	0.3881	
	MPS	0.7730	0.0871	0.9630	5.9330	0.3697	0.7390	0.7711	0.0897	0.9510	6.0224	0.5258	0.7380
300	MLE	0.7969	0.0466		5.7773	0.1492		0.7958	0.0519		5.8210	0.2240	
	MPS	0.7868	0.0482	0.9670	5.8052	0.1796	0.8310	0.7849	0.0540	0.9610	5.8611	0.2693	0.8320

Table 3. Simulation results for the EMID under Type-II censoring with parameters $\alpha p = 2 \times 1.2 = 2.4$ and $\theta = 5.7$, showing estimated values, RMSEs, and REs of MLE and MPS methods across different sample sizes (n) and censoring ratios (C).

n	Method	$C = 0.0$						$C = 0.1$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	2.5827	0.5298		5.8115	0.1379		2.5336	0.5024		5.8674	0.2010	
	MPS	2.3563	0.4586	1.1550	5.8956	0.2128	0.6480	2.3255	0.4577	1.0980	5.9461	0.2856	0.7040
50	MLE	2.5145	0.3855		5.7649	0.0816		2.4359	0.3683		5.8065	0.1311	
	MPS	2.3660	0.3491	1.1040	5.8145	0.1251	0.6520	2.3048	0.3634	1.0140	5.8501	0.1793	0.7310
100	MLE	2.4647	0.2608		5.7337	0.0409		2.4149	0.2467		5.7633	0.0798	
	MPS	2.3870	0.2445	1.0670	5.7579	0.0625	0.6540	2.3426	0.2486	0.9920	5.7832	0.1028	0.7760
300	MLE	2.4146	0.1443		5.7111	0.0136		2.4008	0.1394		5.7303	0.0415	
	MPS	2.3792	0.1436	1.0050	5.7192	0.0207	0.6570	2.3721	0.1413	0.9870	5.7367	0.0490	0.8470
n	Method	$C = 0.2$						$C = 0.3$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	2.4913	0.4773		5.8892	0.2416		2.4587	0.4764		5.9094	0.2800	
	MPS	2.2814	0.4560	1.0470	5.9843	0.3506	0.6890	2.2436	0.4735	1.0060	6.0305	0.4283	0.6540
50	MLE	2.4165	0.3671		5.8341	0.1765		2.4298	0.3686		5.8531	0.2098	
	MPS	2.2824	0.3733	0.9830	5.8890	0.2412	0.7320	2.2890	0.3751	0.9830	5.9243	0.2969	0.7070
100	MLE	2.4033	0.2516		5.7815	0.1118		2.3894	0.2564		5.8023	0.1444	
	MPS	2.3267	0.2580	0.9750	5.8085	0.1434	0.7800	2.3068	0.2692	0.9520	5.8411	0.1894	0.7620
300	MLE	2.3836	0.1404		5.7403	0.0586		2.3898	0.1447		5.7488	0.0763	
	MPS	2.3530	0.1468	0.9560	5.7495	0.0694	0.8440	2.3570	0.1509	0.9590	5.7617	0.0911	0.8370

Table 4. Simulation results for the EMID under Type-II censoring with parameters $\alpha p = 6 \times 1.2 = 7.2$ and $\theta = 5.7$, showing estimated values, RMSEs, and REs of MLE and MPS methods across different sample sizes (n) and censoring ratios (C).

n	Method	$C = 0.0$						$C = 0.1$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	7.6685	1.5314		5.8632	0.1653		7.6101	1.5673		5.8810	0.1847	
	MPS	6.9858	1.3350	1.147	5.8919	0.1937	0.853	6.9820	1.4243	1.100	5.9068	0.2122	0.870
50	MLE	7.5585	1.1691		5.7985	0.0997		7.3394	1.0733		5.8124	0.1153	
	MPS	7.0846	1.0449	1.119	5.8150	0.1161	0.859	6.9438	1.0472	1.025	5.8271	0.1313	0.878
100	MLE	7.3628	0.7515		5.7490	0.0496		7.2545	0.7699		5.7596	0.0618	
	MPS	7.1298	0.7365	1.020	5.7573	0.0579	0.857	7.0330	0.7679	1.003	5.7664	0.0694	0.890
300	MLE	7.2592	0.4252		5.7162	0.0164		7.1889	0.4263		5.7222	0.0239	
	MPS	7.1665	0.4182	1.017	5.7190	0.0191	0.859	7.1046	0.4336	0.984	5.7243	0.0264	0.905
n	Method	$C = 0.2$						$C = 0.3$					
		αp			θ			αp			θ		
		Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE	Est	RMSE	RE
30	MLE	7.4808	1.5237		5.8907	0.1969		7.4129	1.4905		5.8999	0.2105	
	MPS	6.8509	1.4435	1.056	5.9226	0.2327	0.846	6.7555	1.4785	1.008	5.9409	0.2581	0.816
50	MLE	7.2885	1.1264		5.8227	0.1283		7.2818	1.1459		5.8263	0.1346	
	MPS	6.8798	1.1338	0.994	5.8415	0.1497	0.857	6.8604	1.1643	0.984	5.8494	0.1620	0.831
100	MLE	7.2144	0.7476		5.7638	0.0684		7.1436	0.7474		5.7695	0.0770	
	MPS	6.9833	0.7660	0.976	5.7729	0.0788	0.868	6.9017	0.7975	0.937	5.7815	0.0913	0.843
300	MLE	7.1734	0.4300		5.7260	0.0294		7.1322	0.4356		5.7293	0.0354	
	MPS	7.0811	0.4445	0.968	5.7291	0.0331	0.888	7.0370	0.4601	0.947	5.7334	0.0404	0.876

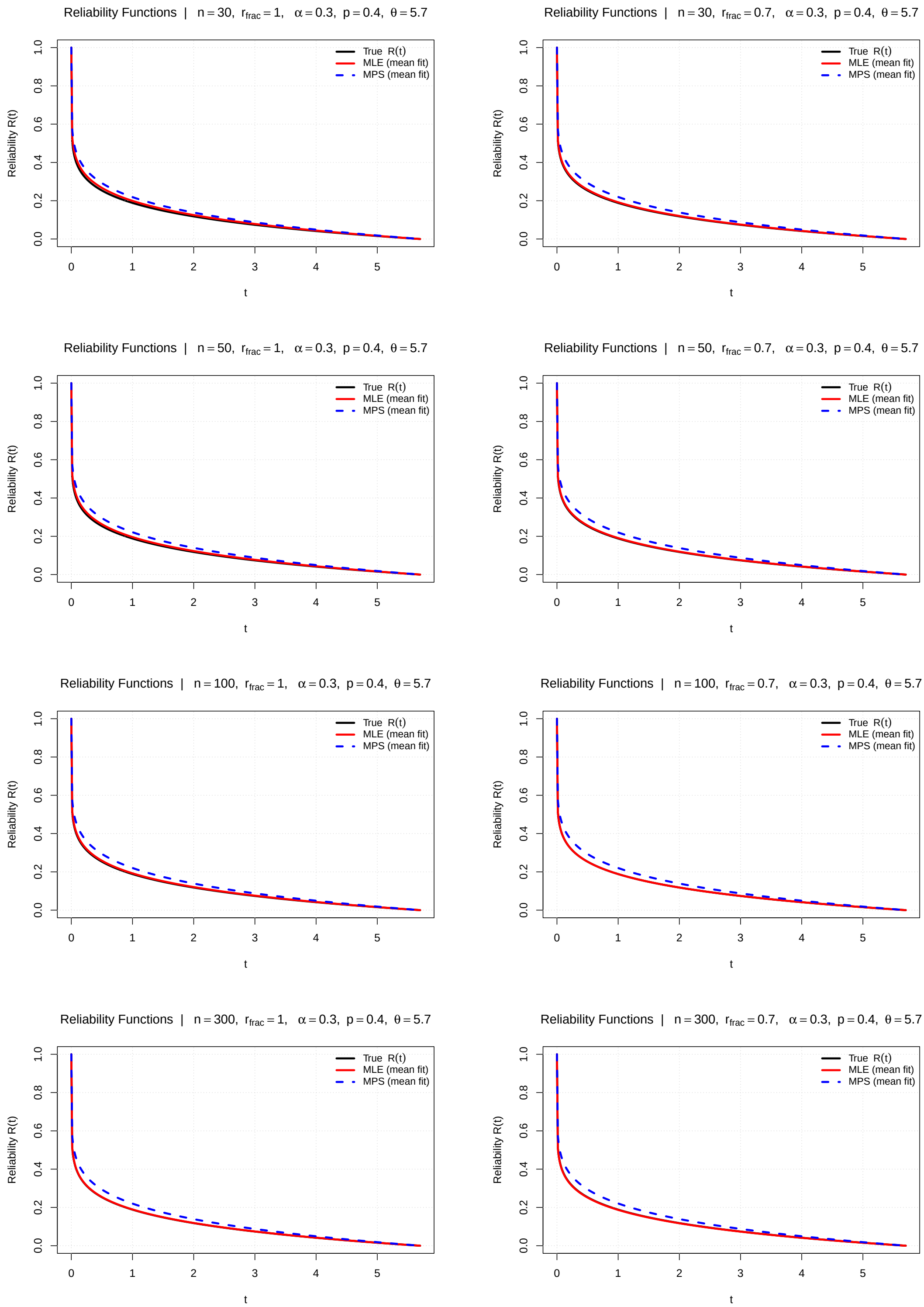

Figure 4. Estimated reliability functions for different sample sizes ($n = 30, 50, 100, 300$) and censoring ratios $C \in \{0.0, 0.3\}$ with parameters $\alpha = 0.3$, $p = 0.4$, and $\theta = 5.7$.

3.2. Application to Lung Cancer Survival Data

We used the lung cancer survival data available in the `survival` package of the `R` software [16]. The lung cancer data originate from a Mayo Clinic study reported in [17]. A preliminary inspection of the data indicates that the empirical distribution of the survival times is compatible with the general shape characteristics of the EMID. Although the original dataset contains additional covariates as well as a right-censoring indicator, the present analysis focuses exclusively on the survival times. The main objective is to evaluate the suitability of the EMID as a marginal distribution and to examine the behavior of parameter estimation and reliability functions under controlled Type-II censoring schemes.

In particular, the first 40 observed survival times exhibit distributional features that are well aligned with the EMID family, making them appropriate for illustrating EMID-based parameter estimation, efficiency comparison, and reliability analysis. This subset was selected solely for illustrative purposes, and the selection was not based on survival outcomes or covariate information.

The original lung cancer dataset was not collected under a Type-II censoring design, mainly due to the limited availability of real datasets that follow such a censoring mechanism and are compatible with the EMID model. To address this limitation, the behavior of the EMID parameter estimators was investigated under controlled conditions by considering several values for the number of observed failures, namely $r = n$, $r = 0.9n$, $r = 0.8n$, and $r = 0.7n$. These choices correspond to censoring ratios $C \in \{0.0, 0.1, 0.2, 0.3\}$. This approach allows a systematic investigation of the EMID model under different censoring levels while preserving the realistic nature of the observed survival data.

For each censoring level, the EMID model parameters were estimated using both the MLE and the MPS methods. Model adequacy was examined using several graphical diagnostic tools, including comparisons between empirical and fitted cumulative distribution functions, histograms with superimposed probability density functions, reliability functions, and probability–probability (P–P) plots. In addition, the Kolmogorov–Smirnov and Anderson–Darling goodness-of-fit tests were applied for formal assessment. To illustrate the practical interpretation of the fitted EMID model, reliability estimates were computed at selected time points together with bootstrap-based confidence intervals.

Table 5. Summary statistics of lung cancer survival times (days) for the selected sample ($n = 40$).

Minimum	Q_1	Median	Mean	Q_3	Maximum
12.0	121.0	366.0	401.1	615.8	1022.0

Table 5 presents the basic descriptive statistics of the selected survival times. The data exhibit a wide range and a moderate degree of right skewness, as indicated by the difference between the mean and the median. Such characteristics are typical for survival data and are consistent with the flexible shape properties provided by the EMID model.

Table 6. Parameter estimates, RMSE, and relative efficiency of the MLE and MPS estimators for the EMID model under different Type-II censoring schemes.

C	Method	$\widehat{\alpha p}$	RMSE($\widehat{\alpha p}$)	RE($\widehat{\alpha p}$)	$\widehat{\theta}$	RMSE($\widehat{\theta}$)	RE($\widehat{\theta}$)
0.0	MLE	0.7437	0.1314		1022.0000	47.2281	
	MPS	0.6974	0.1128	1.3572	1058.8330	37.9105	1.5515
0.1	MLE	0.7933	0.1280		929.6107	82.3987	
	MPS	0.7385	0.1524	0.7054	970.7510	75.4107	1.1938
0.2	MLE	0.8418	0.1797		852.5139	97.2897	
	MPS	0.7766	0.1468	1.4987	899.8639	116.7849	0.6941
0.3	MLE	0.8290	0.1957		871.8303	129.4933	
	MPS	0.7582	0.1557	1.5801	937.6236	171.3262	0.5711

The results reported in Table 6 indicate that the performance of both MLE and MPS estimators depends on the parameter being estimated and the level of censoring. In particular, the MPS method outperforms the MLE method in several cases, as shown by smaller RMSE values and relative efficiency measures greater than one, especially for the scale parameter under heavier censoring. A relative efficiency value greater than one indicates superior performance relative to the competing estimator under the MSE criterion. In this study, the relative efficiency is computed using Eq. (19). These findings are consistent with the robustness properties commonly associated with spacing-based estimation methods [18].

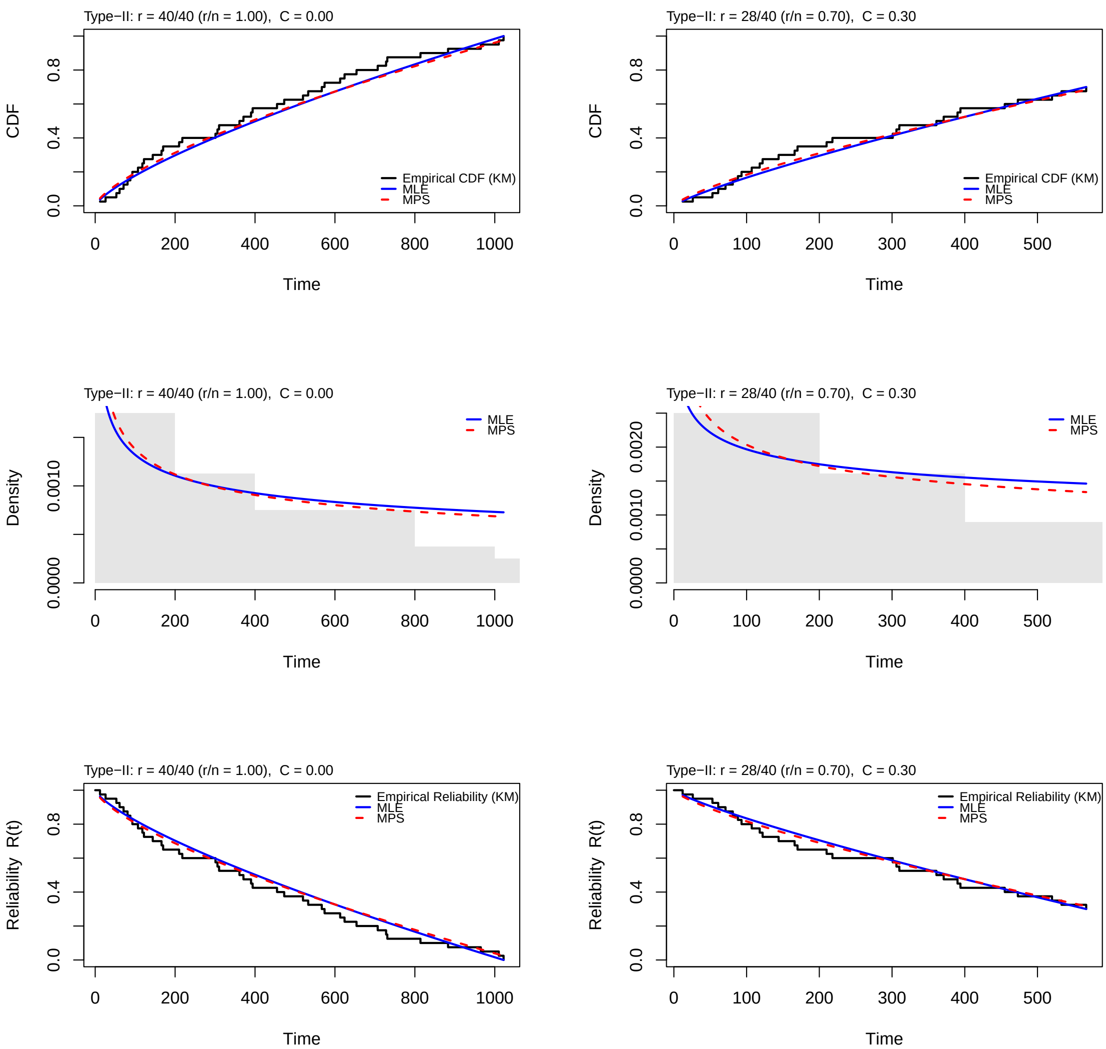

Figure 5. Empirical and EMID fitted functions for the real data under Type-II censoring. The first row shows the empirical and fitted CDFs, the second row shows histograms with the fitted PDFs, and the third row presents the empirical and fitted reliability functions. Results are displayed for two censoring ratios, $C = 0.0$ (left column) and $C = 0.7$ (right column).

All plots (Figures 5 and 6) were constructed in a censoring-aware manner to ensure valid comparison between the empirical and fitted models. The cumulative distribution function (CDF) and the reliability function were estimated using the Kaplan–Meier (KM) approach based on the constructed Type-II censored samples, where r

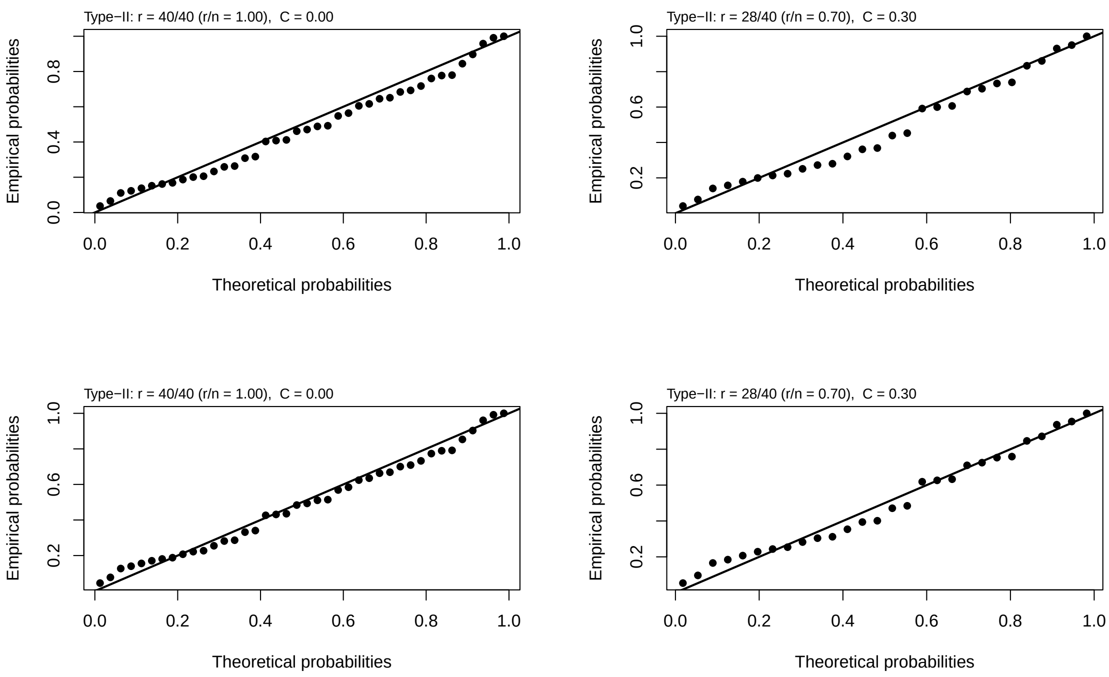

Figure 6. Probability–probability plots for the EMID model under Type-II censoring. The first row corresponds to the MLE and the second row to the MPS. Results are shown for two censoring ratios, $C = 0.0$ (left column) and $C = 0.7$ (right column).

failures were observed from a fixed sample size n. Under this censoring scheme, the KM estimator of the survival function remains strictly positive at the censoring time t_r, and therefore the corresponding empirical CDF, defined as $\widehat{F}(t) = 1 - \widehat{S}(t)$, does not reach one within the observable time range. This behavior correctly reflects the presence of right censoring and represents the unconditional distribution of the lifetimes over the interval $[0, t_r]$ [19].

The P–P plots were constructed using censoring-adjusted CDF values. Under Type-II censoring, the observed failure times follow the conditional distribution given that the failure occurs before the censoring time t_r. Accordingly, for each observed failure time t_i, the transformed values $U_i = F(t_i)/F(t_r)$ were computed, where $F(\cdot)$ denotes the cumulative distribution function of the EMID model. When the fitted model is appropriate, these transformed values follow a $\mathrm{Uniform}(0, 1)$ distribution [18]. Separate P–P plots were constructed for the MLE and MPS estimators to visually assess and compare their goodness-of-fit performance under the imposed Type-II censoring scheme. For graphical illustration, results are presented for the uncensored case ($C = 0$) and for a heavy Type-II censoring scheme ($C = 0.3$), representing the two extreme scenarios. Intermediate censoring levels exhibited similar qualitative behavior and are therefore omitted from the plots for brevity.

Table 7. Bootstrap Kolmogorov–Smirnov and Anderson–Darling goodness-of-fit p-values for the EMID model under Type-II censoring.

Method	Test	$C = 0.0$	$C = 0.1$	$C = 0.2$	$C = 0.3$
MLE	KS	0.519	0.537	0.398	0.321
	AD	0.463	0.761	0.767	0.650
MPS	KS	0.677	0.846	0.735	0.636
	AD	0.714	0.935	0.914	0.870

To illustrate the practical implications of the fitted EMID model, reliability estimates were computed at selected clinically meaningful time points. These estimates were compared with the nonparametric Kaplan–Meier estimator and were accompanied by 95% parametric bootstrap confidence intervals.

Table 8. Estimated reliability $\widehat{R}(t)$ of the EMID model with Kaplan–Meier comparison and 95% bootstrap confidence intervals under Type-II censoring

C	Time	KM	MLE			MPS		
			$\widehat{R}(t)$	L	U	$\widehat{R}(t)$	L	U
0.0	100	0.8000	0.8225	0.7207	0.9124	0.8071	0.6839	0.8880
	200	0.6500	0.7027	0.5883	0.8180	0.6872	0.5567	0.7857
	365	0.5000	0.5350	0.4259	0.6571	0.5242	0.4072	0.6260
	600	0.2750	0.3271	0.2360	0.4193	0.3271	0.2412	0.4114
	900	0.0750	0.0902	0.0000	0.1130	0.1072	0.0440	0.1476
0.1	100	0.8000	0.8295	0.7293	0.9215	0.8133	0.6905	0.8934
	200	0.6500	0.7045	0.5905	0.8249	0.6886	0.5584	0.7904
	365	0.5000	0.5237	0.4096	0.6435	0.5144	0.3966	0.6155
	600	0.2750	0.2934	0.1753	0.3815	0.2990	0.1947	0.3844
0.2	100	0.8000	0.8354	0.7366	0.9253	0.8184	0.6968	0.8974
	200	0.6500	0.7049	0.5878	0.8229	0.6890	0.5561	0.7906
	365	0.5000	0.5104	0.3788	0.6241	0.5038	0.3707	0.6038
	600	0.2750	0.2560	0.0279	0.3439	0.2700	0.0879	0.3621
0.3	100	0.8000	0.8339	0.7353	0.9255	0.8168	0.6962	0.8966
	200	0.6500	0.7049	0.5851	0.8247	0.6901	0.5599	0.7900
	365	0.5000	0.5141	0.3715	0.6292	0.5110	0.3758	0.6144

Note: KM denotes the Kaplan–Meier estimator. L and U represent the lower and upper 95% parametric bootstrap confidence limits for the EMID reliability function under Type-II censoring.

4. Discussion

This section discusses the main results from the simulation study and the real data application. The simulation results give controlled insight into the behavior of the estimators under different sample sizes and censoring levels, while the application shows how these results appear in a real survival analysis setting. Taken together, the findings provide a clear view of the performance of the EMID model and the estimation methods under Type-II censoring.

4.1. Discussion of Simulation Results

Tables 1–4 summarize the simulation results for different values of n and censoring levels. Overall, RMSE values decrease as the sample size increases, indicating improved estimation accuracy for both αp and θ. Type-II censoring has a stronger impact in small samples, where RMSE values increase noticeably under moderate and heavy censoring. When censoring is light, MLE often performs slightly better for αp, while under higher censoring MPS shows more stable behavior, especially for θ. These numerical patterns are consistent across the considered settings and are further illustrated by the reliability curves in Figure 4.

The higher sensitivity of θ to Type-II censoring is related to its dependence on upper-tail information. Under Type-II censoring, observations beyond the rth failure are not observed, which limits information near the upper endpoint. As a result, the objective function becomes relatively flat with respect to θ. MLE is more affected by this problem because it relies strongly on extreme observations, whereas MPS is based on distribution function spacings and uses information from the whole sample. This explains the more robust behavior of MPS for θ under heavier censoring.

The estimated reliability curves support these numerical findings. As shown in Figure 4, both MLE- and MPS-based estimates are close to the true reliability curve when the sample size is moderate or large. For small samples with heavy censoring, MPS curves appear smoother, while MLE curves show larger fluctuations. This visual behavior is consistent with the RMSE results and indicates an advantage of spacing-based estimation when failure information is limited.

4.2. *Discussion of Real Data Application*

The conclusions drawn from the simulation study are largely supported by the real data application using lung cancer survival times. This analysis allows the EMID model and the estimation procedures to be examined under realistic conditions, where censoring and data structure are not controlled by design.

Overall, the EMID model provides an adequate fit to the observed survival data across the considered censoring levels. The bootstrap Kolmogorov–Smirnov and Anderson–Darling tests produce relatively large p-values, indicating no strong evidence against the fitted model. This suggests that EMID is capable of describing the right-skewed pattern observed within the available follow-up period.

Although survival times are not theoretically bounded, in practice the observation window is limited by administrative follow-up. In the lung cancer dataset, the largest observed survival time corresponds to the end of follow-up rather than a biological limit. In this sense, θ can be viewed as an effective upper bound induced by the study design, which makes the EMID model a reasonable approximation for the observed data.

The comparison between MLE and MPS in the real data analysis follows the same general trend observed in the simulation study. When censoring is light, MLE performs well and gives competitive estimates. As censoring increases, the variability of the MLE estimates becomes more apparent, especially for θ, while MPS remains relatively stable.

The reliability analysis provides further practical insight. At early and intermediate time points, where sufficient failure information is available, reliability estimates from both methods are close to the Kaplan–Meier estimator. At later times, and particularly under heavier censoring, uncertainty increases due to limited information in the tail. In such cases, differences between the methods become more visible and long-term reliability estimates should be interpreted with caution.

The P–P plots also support the adequacy of the fitted model. The transformed probabilities follow the reference line reasonably well, especially for the MPS estimator under moderate and high censoring. Overall, the real data application confirms the main findings of the simulation study and supports the use of the EMID model for censored survival data.

From a practical perspective, some simple guidance can be drawn from these results. When the sample size is small and censoring is moderate to high, MPS appears to be a safer option for estimating θ due to its better numerical stability. When censoring is low or data are nearly complete, MLE remains competitive and may provide slightly better performance for αp. For reliability curves under heavy censoring, differences between the methods mainly appear in the tail region, and interpretation should focus more on early and middle follow-up times.

5. Conclusion

This study examined parameter estimation and reliability analysis for the EMID under Type-II censoring. Two estimation methods were considered, namely maximum likelihood estimation (MLE) and maximum product of spacings (MPS), with inference reported for the identifiable shape product αp and the scale parameter θ. The behavior of the estimators was investigated using an extensive Monte Carlo simulation study and illustrated with a real survival dataset.

The simulation results show clear and consistent patterns. For both methods, estimation accuracy improves as the sample size increases, which is reflected by decreasing RMSE values. Type-II censoring has a stronger effect in small samples, and this effect is most pronounced for the scale parameter θ, since information from the upper tail is lost. When censoring is absent or light, MLE often provides slightly better estimation for αp. As censoring becomes moderate or heavy, MPS shows more stable behavior, particularly for θ, and produces reliability estimates with less variability. These findings highlight the practical advantage of spacing-based estimation when the likelihood surface becomes flat or unstable under censoring.

The real data application supports the simulation findings. The EMID model provides an adequate fit to the lung cancer survival data based on graphical diagnostics and goodness-of-fit tests. Under low censoring, MLE and MPS lead to similar parameter and reliability estimates. As the censoring level increases, MLE estimates, especially for θ, become more variable, while MPS remains relatively stable. EMID-based reliability estimates are close to the Kaplan–Meier estimator at early and intermediate times, whereas uncertainty increases at later times due to limited tail information.

From a practical viewpoint, the results suggest that MPS is a reliable alternative to MLE when sample sizes are small or censoring is moderate to high, particularly for estimating the scale parameter and tail reliability. When censoring is low and data are nearly complete, MLE remains competitive and may be preferred for estimating αp. In all cases, reliability estimates at late times under heavy censoring should be interpreted with caution.

Several directions for future work are possible. First, the performance of EMID can be compared with other bounded-support or commonly used lifetime distributions under the same censoring settings. Second, alternative estimation approaches, such as Bayesian methods, penalized likelihood, or other robust estimators, may be explored to further improve stability for θ. Finally, extending the model to other censoring schemes or regression settings, and applying it to additional real datasets, would help clarify the range of applications where EMID is most useful.

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