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A Hybrid Approach for Solving Fractional Caputo Partial Differential Equations with Convergence Analysis

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Abstract In this research, fractional Caputo partial differential equations are addressed using the q-Homotopy analysis method merged with the Sawi transform method through the construction of a novel algorithm. This approach combines the Sawi transform with the q-Homotopy analysis method to demonstrate how complex fractional differential equations can be solved analytically in a straightforward manner. The proposed algorithm illustrates the effectiveness of applying the Sawi transform in conjunction with the q-Homotopy method to overcome the challenges associated with handling nonlinear terms numerically. Several examples are provided to verify the accuracy and efficiency of the proposed approach. The results indicate that the method converges to the exact solutions when suitable parameters are chosen. Therefore, the proposed method proves to be a robust and flexible algorithm for solving nonlinear fractional partial differential equations.

Keywords Homotopy analysis method; Caputo derivative; Fractional differential equations

AMS 2010 subject classifications 26A33, 35R11, 65M70

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1. Introduction

Fractional calculus has received huge interests in recent decades [1, 2, 3] because of its numerous applications in different fields of science and engineering. Different kinds of fractional order differential equations have been studied [4, 5, 6] to explain some complex processes and systems in many scientific fields, such as viscoelastic materials, mathematical physics, manufacturing innovation, process innovation and image processing. Authors have established many definitions of fractional derivatives such as Caputo, Baleanu, Atangana and others. Many complex mathematical models nowadays depend on fractional partial differential equations (FPDEs), especially those containing Caputo derivatives, because they enable the simulation of memory and the preservation of the inheritance property of materials and processes more easily [7, 8], also Caputo derivative is easy to handle and need less computations than other fractional derivatives. These equations can be solved using different kinds of numerical and analytical methods, but we need to consider optimizing the accuracy and increasing the convergence [9, 10].

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Recent studies highlight the expanding role of applied mathematical modeling and intelligent computational methods in addressing complex real-world problems. Mohammad et al. [11] demonstrated that integrating artificial intelligence with fuzzy systems improves educational equity by enabling adaptive decision-making under uncertainty. Meanwhile, Al-Sawaie et al. [12] employed a production-function modeling approach to estimate potential economic output, emphasizing the importance of quantitative analytical tools in macroeconomic analysis. These contributions illustrate the growing interdisciplinary impact of mathematical and computational techniques in supporting data-driven solutions across education and economic systems.

There are some complex problems that arise when solving fractional differential equations, especially that contain nonlinear terms, that complicates getting solutions for them analytically and numerically [13]. The homotopy analysis method showed its effectiveness in establishing approximate solutions for linear and nonlinear differential equations, as seen in [14, 15]. Mathematicians have recently worked on merging the homology analysis method (HAM) with other analytical methods, such as the Adomian decomposition method and the variational iteration method, to reduce the restrictions when handling fractional differential problems [16, 17]. The q-homotopy analysis method (q-HAM) has got great attention of researchers due to its effectiveness in improving these methods, providing more flexibility to get the solutions by interring some auxiliary parameters to be fixed to the solution to determine the best convergence region [18, 19].

The Sawi integral transform was first established to handle the difficulty and shown when solving fractional derivatives in FPDEs [20]. It provides some advantages over the existing transforms because of its simplicity when handling differential equations, and to find the inversion formula, also its duality to Laplace transform gives it more strength. Moreover, the simplicity of applying Sawi transform to the fractional Caputo derivative gives it more attention by researchers [21, 22]. We study the q-Homotopy analysis method combined with Sawi transform to obtain (q-HASTM). This is a new hybrid method for solving FPDEs in the sense of Caputo derivatives [23]. This technique employs the characteristics of Sawi transform to reduce the difficulty of Caputo fractional differential equations and expand the solutions utilizing homotopy analysis theory [24, 25]. Due to the properties of Sawi transform, we choose it to be the key of the presented algorithm in solving the equations [26]. The main contribution of this paper is to construct a new algorithm for solving nonlinear fractional partial differential equations, the method is simple and accurate, it presents the solution in a form of infinite series that converges rapidly to the exact solution in comparison to other numerical methods [27, 28, 29].

This article presents the properties of Sawi integral transform and the main related theorems, its application to Caputo fractional derivative is presented. We define the q-HAM method and shows its applicability to handle nonlinear problems and the effects of memory to FPDEs [26]. Moreover, this article introduces some numerical examples of nonhomogeneous nonlinear FPDEs that are solved by q-HASTM, which proves its applicability and accuracy such as in similar problems [30, 31].

2. Basic Concepts of Sawi Transform

In this section, we present some properties, existence conditions, linearity, the inverse, the convolution theorem, and the derivative properties of the Sawi transform [20, 25].

Definition 2.1
Assume that $z(t)$ is a function given with the domain $[0,\infty)$. Then, the Sawi integral transform of the function $z(t)$, denoted by $\mathcal{S}[z(t)]$, is defined as

$$\mathcal{S}[z(t)] = Z(u) = \frac{1}{u^2}\int_0^\infty z(t)e^{-\frac{t}{u}}\,dt, \qquad \Re(u) > 0. \tag{1}$$

The inverse Sawi transformation is provided as

$$\mathcal{S}^{-1}[Z(u)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{u^2}e^{\frac{t}{u}}Z(u)\right)du = z(t), \qquad c\in\mathbb{R}. \tag{2}$$

Theorem 2.1
Given the function $z(t)$ to be continuous for $t > 0$ and satisfies the property

$$|\mathcal{S}[z(t)]| \leq \mathcal{K} e^{\mathscr{b} t}, \quad \text{for some } \mathcal{K} \text{ and } \mathscr{b} > 0. \tag{3}$$

Suppose that $\mathcal{S}[z(t)] = Z(u)$ and $\mathcal{S}[f(t)] = F(u)$ and $a, j \in \mathbb{R}$, then we get the following characteristic:

$$\begin{aligned}
\mathcal{S}[a\, z(t) + j f(t)] &= a\, \mathcal{S}[z(t)] + j\, \mathcal{S}[f(t)], \\
\mathcal{S}^{-1}[a\, Z(u) + j F(u)] &= a\, \mathcal{S}^{-1}[Z(u)] + j\, \mathcal{S}^{-1}[F(u)], \\
\mathcal{S}[t^a] &= u^{a-1}\Gamma(a+1), \\
\mathcal{S}[e^{at}] &= \frac{1}{u(1-au)}, \\
\mathcal{S}[\cos(at)] &= \frac{1}{u(1+a^2u^2)}, \\
\mathcal{S}[\sin(at)] &= \frac{a}{1+a^2u^2}, \\
\mathcal{S}[\cosh(at)] &= \frac{1}{u(1-a^2u^2)}, \\
\mathcal{S}[\sinh(at)] &= \frac{a}{1-a^2u^2}, \\
\mathcal{S}\left[\frac{d^n z(t)}{dt^n}\right] &= \frac{Z(u)}{u^n} - \sum_{j=0}^{n-1} \frac{z^{(j)}(0)}{u^{n-j+1}}, \qquad n \in \mathbb{N}.
\end{aligned}$$

Theorem 2.2
Let $\mathcal{S}[z(t)] = Z(u)$. Then for the Heaviside function $H(t)$, we get

$$\mathcal{S}[z(t-j)H(t-j)] = e^{-\frac{j}{u}} Z(u), \tag{4}$$

where $H(t)$ is defined as

$$H(t-j) = \begin{cases} 1, & t > j, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3
If $\mathcal{S}[z(t)] = Z(u)$ and $\mathcal{S}[h(t)] = H(u)$, then

$$\mathcal{S}[z(t) * h(t)] = u^2 Z(u) H(u), \tag{5}$$

where the convolution is defined as

$$z(t) * h(t) = \int_0^t z(\tau)\, h(t-\tau)\, d\tau.$$

3. Fractional Calculus and Sawi Transform

In this part of the study, we present the needed definitions and properties related to fractional calculus to be used in this research.

Definition 3.1
The Caputo fractional derivative of a function $z(t)$ of order v is defined as

$$D_t^v z(t) = \frac{1}{\Gamma(r-v)} \int_0^t (t-q)^{r-v-1} z^{(r)}(q)\, dq, \tag{6}$$

where $r-1 < v \leq r$, $r \in \mathbb{N}$ and $z(t)$ is a continuous function on $[0, \infty)$.

Theorem 3.1
If $Z(u)$ is the Sawi transform of the function $z(t)$, then the Sawi transform of the fractional derivative of $z(t)$ in the Caputo sense can be expressed as

$$\mathcal{S}\left[D_t^v z(t)\right] = \frac{1}{u^v}\left(Z(u) - \sum_{k=0}^{v-1}\frac{z^{(k)}(0)}{u^{1-k}}\right), \qquad r-1 < v \le r. \tag{7}$$

Proof. Using the idea of the convolution property, one can obtain

$$\int_0^t (t-\tau)^{r-v-1} z^{(r)}(q)\, dq = t^{r-v-1} * z^{(r)}(t).$$

Therefore,

$$\begin{aligned}\mathcal{S}\left[D_t^v z(t)\right] &= \frac{1}{u^2}\int_0^\infty \left(\frac{1}{\Gamma(r-v)}\int_0^t (t-q)^{r-v-1} z^{(r)}(q)\, dq\right) e^{-t/u} dt\\ &= \frac{1}{\Gamma(r-v)}\mathcal{S}\left[t^{r-v-1} * z^{(r)}(t)\right]\\ &= \frac{1}{\Gamma(r-v)}\left(u^2 \mathcal{S}[t^{r-v-1}]\mathcal{S}[z^{(r)}(t)]\right).\end{aligned}$$

Also, from the Sawi transform identities, one can obtain

$$\begin{aligned}\mathcal{S}\left[D_t^v z(t)\right] &= \frac{u^2}{\Gamma(r-v)}\left(u^{r-v-2}\Gamma(r-v)\left(\frac{Z(u)}{u^r} - \sum_{i=0}^{r-1}\frac{z^{(i)}(0)}{u^{r-i+1}}\right)\right)\\ &= \frac{Z(u)}{u^v} - \sum_{i=0}^{r-1}\frac{z^{(i)}(0)}{u^{v-i+1}}.\end{aligned}$$

Thus,

$$\mathcal{S}\left[D_t^v z(t)\right] = \frac{1}{u^v}\left(Z(u) - \sum_{i=0}^{r-1}\frac{z^{(i)}(0)}{u^{1-i}}\right), \qquad r-1 < v \le r. \tag{8}$$

Corollary 3.1
If the Sawi transform of the function of two variables $z(\rho,\vartheta)$ with respect to ϑ is given by $Z(\rho,u)$ and $0 < v \le 1$, then

$$\mathcal{S}\left[D_\vartheta^v z(\rho,\vartheta)\right] = \frac{1}{u^v}\left(Z(\rho,u) - \frac{1}{u}z(\rho,0)\right) = \frac{1}{u^v}\left(\mathcal{S}[z(\rho,\vartheta)] - \frac{1}{u}z(\rho,0)\right). \tag{9}$$

3.1. Analysis of the Method (q-HASTM)

The fundamental idea of q-homotopy analysis merged with the Sawi transform is presented here for solving fractional Caputo partial differential equations. To simplify the method, we follow the steps of q-HASTM.

Let us begin with the differential equation of nth order

$$D_\vartheta^v z(\rho,\vartheta) = \varphi(\rho,\vartheta) + L(z(\rho,\vartheta)) + N(z(\rho,\vartheta)), \quad \vartheta > 0,\ n-1 < v < n,\ n \in \mathbb{N}, \tag{10}$$

subject to the conditions

$$\frac{d^w z(\rho,0)}{d\vartheta^w} = \varphi_w(\rho), \qquad w = 0,1,\ldots,n-1. \tag{11}$$

Here, L and N are linear and nonlinear differential operators, $D_\vartheta^v z(\rho,\vartheta)$ denotes the Caputo fractional derivative, $z(\rho,\vartheta)$ is the unknown function, and $\varphi(\rho,\vartheta)$ is a known function.

Applying the Sawi transform to Eq.(10), we obtain

$$\mathcal{S}\left[D_{\vartheta}^{v} z(\rho,\vartheta)\right]=\mathcal{S}\left[\varphi(\rho,\vartheta)+L(z(\rho,\vartheta))+N(z(\rho,\vartheta))\right]. \tag{12}$$

Simplifying, we get

$$\mathcal{S}[z(\rho,\vartheta)]=u^{v}\left(\sum_{i=0}^{v-1}\frac{z^{(i)}(\rho,0)}{u^{v-i+1}}+\mathcal{S}\left[\varphi(\rho,\vartheta)+L(z(\rho,\vartheta))+N(z(\rho,\vartheta))\right]\right). \tag{13}$$

The nonlinear operator is defined as

$$\begin{aligned}\mathcal{N}[\mathcal{G}(\rho,\vartheta;q)]&=\mathcal{S}[\mathcal{G}(\rho,\vartheta;q)]\\&\quad-u^{v}\left(\sum_{i=0}^{v-1}\frac{\mathcal{G}^{(i)}(\rho,\vartheta;q)(0)}{u^{v-i+1}}+\mathcal{S}\left[\varphi(\rho,\vartheta)+L(\mathcal{G})+N(\mathcal{G})\right]\right),\end{aligned} \tag{14}$$

where $\mathcal{G}(\rho,\vartheta;q)$ is a function of ρ, ϑ, $q\in[0,1/m]$, and $m\geq 1$ is the homotopy parameter.

The homotopy equation is defined as

$$(1-\tau q)\mathcal{S}\left[\mathcal{G}(\rho,\vartheta;q)-z_0(\rho,\vartheta)\right]=hq\,\mathcal{H}[\mathcal{G}(\rho,\vartheta;q)], \tag{15}$$

where $h\neq 0$ is an auxiliary parameter.

For $q=0$ and $q=\frac{1}{\tau}$, we have

$$\mathcal{G}(\rho,\vartheta;0)=z_0(\rho,\vartheta),\qquad \mathcal{G}\left(\rho,\vartheta;\frac{1}{\tau}\right)=z(\rho,\vartheta). \tag{16}$$

Using Taylor expansion,

$$\mathcal{G}(\rho,\vartheta;q)=z_0(\rho,\vartheta)+\sum_{m=1}^{\infty}z_m(\rho,\vartheta)q^m, \tag{17}$$

where

$$z_m(\rho,\vartheta)=\frac{1}{m!}\left.\frac{\partial^m\mathcal{G}(\rho,\vartheta;q)}{\partial q^m}\right|_{q=0},\quad m=0,1,2,\ldots \tag{18}$$

Thus,

$$z(\rho,\vartheta)=z_0(\rho,\vartheta)+\sum_{m=1}^{\infty}\frac{z_m(\rho,\vartheta)}{\tau^m}. \tag{19}$$

Differentiating the zero-order deformation equation m times with respect to q, setting $q=0$, and dividing by $m!$, yields

$$\mathcal{S}\left[z_m(\rho,\vartheta)-\kappa_m z_{m-1}(\rho,\vartheta)\right]=h\,\mathcal{H}(\rho,\vartheta)\mathcal{R}[\vec{z}_{m-1}(\rho,\vartheta)]. \tag{20}$$

The vector form is defined as

$$\vec{z}_m(\rho,\vartheta)=(z_0,z_1,z_2,\ldots,z_m). \tag{21}$$

Applying the inverse Sawi transform on Eq. (**??**), implies

$$z_m(\rho,\vartheta)=\kappa_m z_{m-1}(\rho,\vartheta)+h\,\mathcal{S}^{-1}\left[\mathcal{H}(\rho,\vartheta)\mathcal{R}[\vec{z}_{m-1}]\right], \tag{22}$$

where

$$\begin{aligned}\mathcal{R}[\vec{z}_{m-1}]&=\mathcal{S}[z_{m-1}(\rho,\vartheta)]-\left(1-\frac{\kappa_m}{n}\right)\frac{\varphi_0(\rho)}{u}\\&\quad-u^{v}\mathcal{S}\left[\left(1-\frac{\kappa_m}{n}\right)\varphi(\rho,\vartheta)+L(z_{m-1})+N(z_{m-1})\right],\end{aligned} \tag{23}$$

and

$$\kappa_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases}$$

Utilizing the Eq.s (22) and (23), we can obtain the value of . Finally, the q-HASTM series solution can expressed to be

$$z(\rho,\vartheta) = \sum_{m=0}^{\infty} z_m(\rho,\vartheta). \tag{24}$$

4. Applications

Some examples are discussed and solved in this section to prove the simplicity and effectiveness of the presented q-HASTM.

Example 4.1
Consider the following fractional linear equation with constant coefficients:

$$\frac{\partial^v z(\rho,\vartheta)}{\partial \vartheta^v} + \mu \frac{\partial z(\rho,\vartheta)}{\partial \rho} = 0, \quad 0 < v < 1, \tag{25}$$

subject to the conditions

$$z(\rho, 0) = \rho^2. \tag{26}$$

Applying Sawi transform on Eq. (25), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right] = \frac{1}{u} z(\rho,0) - u^v \mathcal{S}\left[\mu \frac{\partial z(\rho,\vartheta)}{\partial \rho}\right]. \tag{27}$$

Using the initial condition (26), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right] = \frac{\rho^2}{u} - \mu u^v \mathcal{S}\left[\frac{\partial z(\rho,\vartheta)}{\partial \rho}\right]. \tag{28}$$

Applying q-HASTM on Eq.(28), we get

$$\mathcal{N}\left[\mathcal{G}(\rho,\vartheta;q)\right] = \mathcal{S}\left[\mathcal{G}(\rho,\vartheta;q)\right] - \frac{\rho^2}{u} + \mu u^v \mathcal{S}\left[\frac{\partial \mathcal{G}(\rho,\vartheta;q)}{\partial \rho}\right], \tag{29}$$

and we have

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = \mathcal{S}\left[z_{m-1}(\rho,\vartheta)\right] - \left(1 - \frac{\kappa_m}{n}\right)\frac{\rho^2}{u} + \mu u^v \mathcal{S}\left[\frac{\partial z_{m-1}(\rho,\vartheta)}{\partial \rho}\right]. \tag{30}$$

Hense, we can define the mth-order deformed functions as

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = h^{-1}\mathcal{S}\left[\vec{z}_m(\rho,\vartheta) - \kappa_m z_{m-1}(\rho,\vartheta)\right]. \tag{31}$$

Operating inverse Sawi transform to Eq.(31) to obtain

$$z_m(\rho,\vartheta) = \kappa_m z_{m-1}(\rho,\vartheta) + h\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{m-1}\right)\right].$$

Note that, the first few terms of $z_m(\rho,\vartheta)$ is given by.

$$z_0(\rho,\vartheta) = z(\rho,0) = \rho^2,$$

we can find the first two iterations as

$$\begin{aligned} z_1(\rho,\vartheta) &= \kappa_1 z_0(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_0\right)\right] \\ &= h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_0(\rho,\vartheta)\right] - \frac{\rho^2}{u} + \mu u^v \mathcal{S}\left[\frac{\partial z_0(\rho,\vartheta)}{\partial \rho}\right]\right] \\ &= h\,\mathcal{S}^{-1}\left[\frac{\rho^2}{u} - \frac{\rho^2}{u} + \mu u^v \mathcal{S}[2\rho]\right] \\ &= 2\mu h\rho\,\mathcal{S}^{-1}\left[u^{v-1}\right] = \frac{2\mu h\rho\,\vartheta^v}{\Gamma(v+1)}\,, \end{aligned}$$

$$\begin{aligned} z_2(\rho,\vartheta) &= \kappa_2 z_1(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_1\right)\right] \\ &= \frac{2n\mu h\rho\,\vartheta^v}{\Gamma(v+1)} + h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_1(\rho,\vartheta)\right] + \mu u^v \mathcal{S}\left[\frac{\partial z_1(\rho,\vartheta)}{\partial \rho}\right]\right] \\ &= \frac{2\mu h(h+n)\rho\,\vartheta^v}{\Gamma(v+1)} + \frac{2\mu^2 h^2\,\vartheta^{2v}}{\Gamma(2v+1)}\,. \end{aligned}$$

In the same way, one can obtain

$$z_3(\rho,\vartheta) = 2\mu h(h+n)^2 \frac{\rho\,\vartheta^v}{\Gamma(v+1)} + 4(\mu h)^2(h+n)\frac{\vartheta^{2v}}{\Gamma(2v+1)}.$$

$$\vdots$$

We can find the rest of terms of $z_m(\rho,\vartheta)$, $m \geq 4$, by a similar procedure. Hens we can define the approximate analytic solution as

$$z(\rho,\vartheta) = z_0(\rho,\vartheta) + \sum_{m=1}^{\infty} \frac{z_m(\rho,\vartheta)}{n^m}.$$

Figure 1 illustrates the plot of the function $z(\rho,\vartheta)$ for different values of ϑ (0.5, 0.7, 0.9, and 1). The function is evaluated over a range of ρ values from 0 to 2. Each line represents a different value of ϑ, demonstrating how the function varies with ρ and ϑ. For $h=-1,\ n=1$ and $v=1$, the approximate solution converges to the exact one.

$$z(\rho,\vartheta) = \rho^2 - 2\mu\rho\vartheta + \mu^2\vartheta^2.$$

Figure 2 is the plot comparing both functions $z(\rho,\vartheta)$. The blue line represents the exact solution, while the red dashed line corresponds to the approximate solution. Both functions are plotted for $\vartheta = 1$ and over the range of ρ from 0 to 2, showing their distinct behaviors.

Example 4.2
Consider the initial value problem:

$$\frac{\partial^v z(\rho,\vartheta)}{\partial \vartheta^v} + \rho\frac{\partial z(\rho,\vartheta)}{\partial \rho} = 0, \quad 0 < v < 1, \tag{32}$$

subject to the conditions

$$z(\rho,0) = \rho^2. \tag{33}$$

Applying Sawi transform on Eq. (32), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right] = \frac{1}{u} z(\rho,0) - u^v \mathcal{S}\left[\rho\frac{\partial z(\rho,\vartheta)}{\partial \rho}\right]. \tag{34}$$

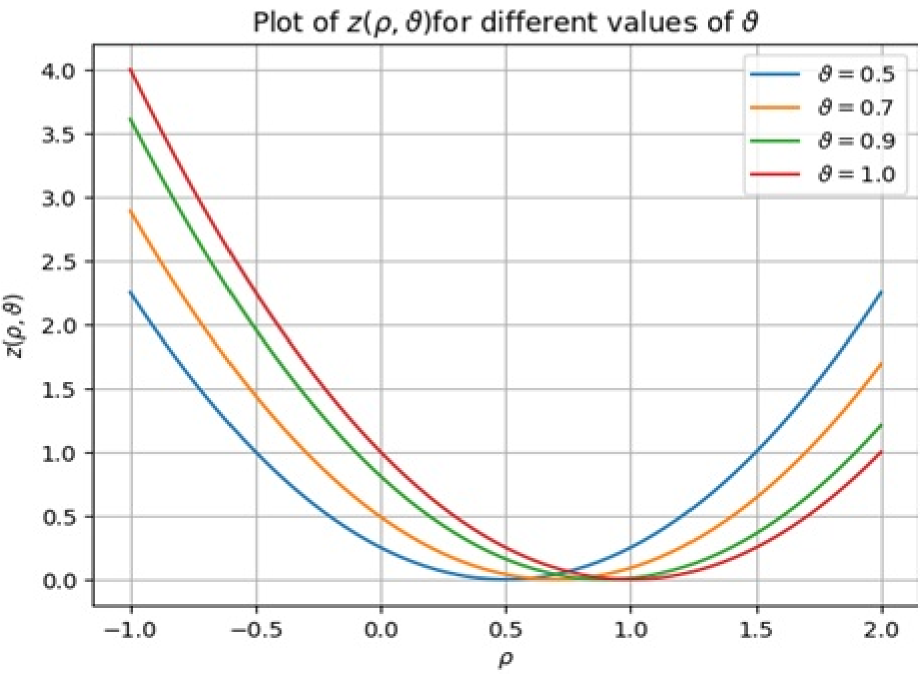

Figure 1. Plot of the function $z\left(\rho,\vartheta\right)$ for different values of $\vartheta(0.5, 0.7, 0.9, and 1)$.

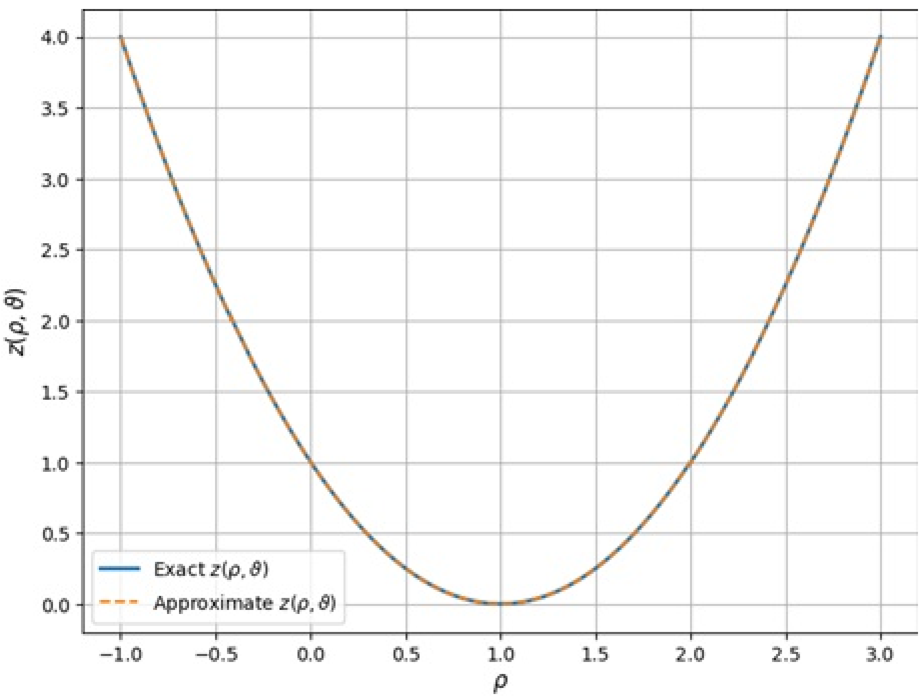

Figure 2. Comparison between the exact and approximate solutions.

Using the initial condition (33), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right]=\frac{\rho^2}{u}-u^v\mathcal{S}\left[\rho\frac{\partial z(\rho,\vartheta)}{\partial\rho}\right]. \tag{35}$$

Applying q-HASTM on Eq.(35), we get

$$\mathcal{N}\left[\mathcal{G}(\rho,\vartheta;q)\right] = \mathcal{S}\left[\mathcal{G}(\rho,\vartheta;q)\right] - \frac{\rho^2}{u} + u^v \mathcal{S}\left[\rho \frac{\partial \mathcal{G}(\rho,\vartheta;q)}{\partial \rho}\right], \tag{36}$$

and we have

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = \mathcal{S}\left[z_{m-1}(\rho,\vartheta)\right] - \left(1 - \frac{\kappa_m}{n}\right)\frac{\rho^2}{u} + u^v \mathcal{S}\left[\rho \frac{\partial z_{m-1}(\rho,\vartheta)}{\partial \rho}\right]. \tag{37}$$

Thus, we define the mth-order equations:

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = h^{-1}\mathcal{S}\left[\vec{z}_m(\rho,\vartheta) - \kappa_m z_{m-1}(\rho,\vartheta)\right]. \tag{38}$$

Running inverse Sawi transform to Eq.(38), to get

$$z_m(\rho,\vartheta) = \kappa_m z_{m-1}(\rho,\vartheta) + h\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{m-1}\right)\right]. \tag{39}$$

Note that, the first few terms of $z_m(\rho,\vartheta)$ is given by

$$z_0(\rho,\vartheta) = z(\rho,0) = \rho^2,$$

$$\begin{aligned}
z_1(\rho,\vartheta) &= \kappa_1 z_0(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_0\right)\right] \\
&= h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_0(\rho,\vartheta)\right] - \frac{\rho^2}{u} + u^v \mathcal{S}\left[\rho \frac{\partial z_0(\rho,\vartheta)}{\partial \rho}\right]\right] \\
&= h\,\mathcal{S}^{-1}\left[\frac{\rho^2}{u} - \frac{\rho^2}{u} + u^v \mathcal{S}\left[2\rho^2\right]\right] \\
&= 2h\rho^2\,\mathcal{S}^{-1}\left[u^{v-1}\right] = \frac{2h\rho^2\,\vartheta^v}{\Gamma(v+1)}.
\end{aligned}$$

In the same way, we get

$$z_2(\rho,\vartheta) = 2h(h+n)\frac{\rho^2\,\vartheta^v}{\Gamma(v+1)} + 4h^2\frac{(\rho\vartheta^v)^2}{\Gamma(2v+1)},$$

$$z_3(\rho,\vartheta) = 2h(h+n)^2\frac{\rho^2\,\vartheta^v}{\Gamma(v+1)} + 8h^2(h+n)\frac{(\rho\vartheta^v)^2}{\Gamma(2v+1)} + 8h^3\frac{\rho^2\vartheta^{3v}}{\Gamma(3v+1)}.$$

$$\vdots$$

Similarly, we get the components of $z_m(\rho,\vartheta)$, $m \geq 4$.
Thus, the expanded solution is given as

$$z(\rho,\vartheta) = z_0(\rho,\vartheta) + \sum_{m=1}^{\infty}\frac{z_m(\rho,\vartheta)}{n^m}.$$

Figure 3 below, is the plot of the function $z(\rho,\vartheta)$ for different values of ϑ (0.5, 0.7, 0.9, and 1). The graph shows how the function behaves as ρ changes, with decreasing values of $z(\rho,\vartheta)$ as ϑ increases. For $h=-1,\ n=1$ and $v=1$ then obviously, the series solution is converging to the exact one.

$$z(\rho,\vartheta) = \rho^2 - 2\rho^2\vartheta + \rho^2\frac{(2\vartheta)^2}{2!} - \rho^2\frac{(2\vartheta)^3}{3!} + \ldots = \rho^2 e^{-2\vartheta}.$$

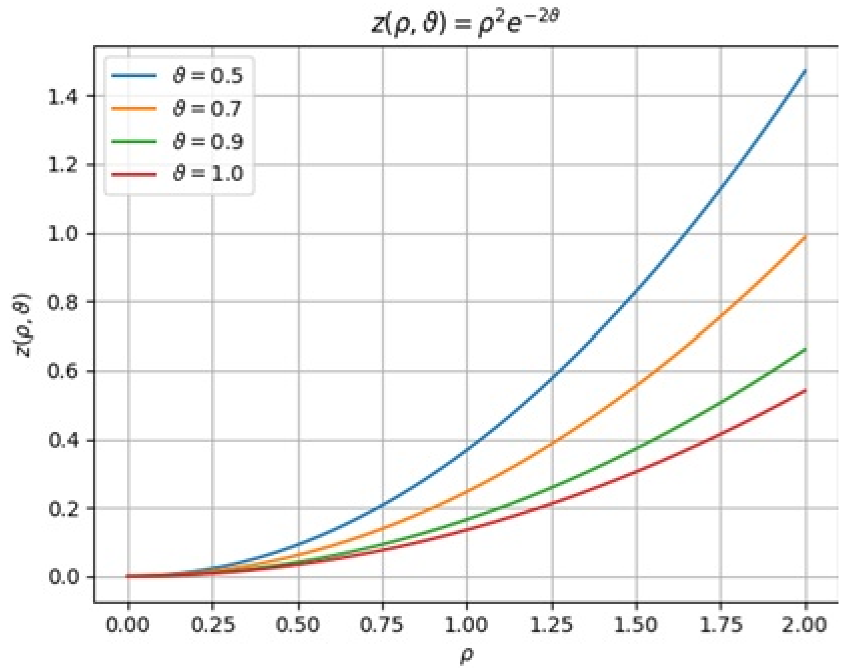

Figure 3. Plot of the function $z(\rho, \vartheta)$ for different values of $\vartheta(0.5, 0.7, 0.9, and 1)$.

Example 4.3
Consider the initial value problem:

$$\frac{\partial^v z(\rho, \vartheta)}{\partial \vartheta^v} + \frac{\partial z(\rho, \vartheta)}{\partial \rho} = \rho, \quad 0 < v < 1, \tag{40}$$

subject to the conditions

$$z(\rho, 0) = e^\rho. \tag{41}$$

Running Sawi transform on Eq. (40), we obtain

$$\mathcal{S}\left[z(\rho, \vartheta)\right] = \frac{1}{u} z(\rho, 0) - u^v \mathcal{S}\left[\frac{\partial z(\rho, \vartheta)}{\partial \rho} - \rho\right]. \tag{42}$$

Using the initial condition (41), we obtain

$$\mathcal{S}\left[z(\rho, \vartheta)\right] = \frac{e^\rho}{u} - u^v \mathcal{S}\left[\frac{\partial z(\rho, \vartheta)}{\partial \rho} - \rho\right]. \tag{43}$$

Applying q-HASTM on Eq.(43), we get

$$\mathcal{N}\left[\mathcal{G}(\rho, \vartheta; q)\right] = \mathcal{S}\left[\mathcal{G}(\rho, \vartheta; q)\right] - \frac{e^\rho}{u} + u^v \mathcal{S}\left[\frac{\partial \mathcal{G}(\rho, \vartheta; q)}{\partial \rho} - \rho\right], \tag{44}$$

and we have

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = \mathcal{S}\left[z_{m-1}(\rho, \vartheta)\right] - \left(1 - \frac{\kappa_m}{n}\right) \frac{e^\rho}{u} + u^v \mathcal{S}\left[\frac{\partial z_{m-1}(\rho, \vartheta)}{\partial \rho} - \left(1 - \frac{\kappa_m}{n}\right) \rho\right]. \tag{45}$$

Hens we define mth-order functions of the form:

$$\mathcal{R}\left(\vec{z}_{m-1}\right) = h^{-1} \mathcal{S}\left[\vec{z}_m(\rho, \vartheta) - \kappa_m z_{m-1}(\rho, \vartheta)\right]. \tag{46}$$

Applying inverse Sawi transform to Eq.(46) to obtain

$$z_m(\rho,\vartheta) = \kappa_m z_{m-1}(\rho,\vartheta) + h\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{m-1}\right)\right]. \tag{47}$$

Note that, the first few terms of $z_m(\rho,\vartheta)$ is given by

$$z_0(\rho,\vartheta) = z(\rho,0) = e^{\rho},$$

$$\begin{aligned} z_1(\rho,\vartheta) &= \kappa_1 z_0(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_0\right)\right] \\ &= h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_0(\rho,\vartheta)\right] - \frac{e^{\rho}}{u} + u^v\mathcal{S}\left[\frac{\partial z_0(\rho,\vartheta)}{\partial\rho} - \rho\right]\right] = h\frac{\left(e^{\rho}-\rho\right)\vartheta^v}{\Gamma(v+1)}. \end{aligned}$$

To get the value of the second approximation

$$\begin{aligned} z_2(\rho,\vartheta) &= \kappa_2 z_1(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_1\right)\right] \\ &= nh\frac{\left(e^{\rho}-\rho\right)\vartheta^v}{\Gamma(v+1)} + h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_1(\rho,\vartheta)\right] + u^v\mathcal{S}\left[\frac{\partial z_1(\rho,\vartheta)}{\partial\rho}\right]\right] \\ &= n = h(h+n)\frac{\left(e^{\rho}-\rho\right)\vartheta^v}{\Gamma(v+1)} + h^2\frac{\left(e^{\rho}-1\right)\vartheta^{2v}}{\Gamma(2v+1)}. \end{aligned}$$

Repeating the same procedure, we get $z_3(\rho,\vartheta)$ as

$$\begin{aligned} z_3(\rho,\vartheta) &= \kappa_3 z_2(\rho,\vartheta) + h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_2\right)\right] \\ &= nh(h+n)\frac{\left(e^{\rho}-\rho\right)\vartheta^v}{\Gamma(v+1)} + nh^2\frac{\left(e^{\rho}-1\right)\vartheta^{2v}}{\Gamma(2v+1)} \\ &\quad + h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_2(\rho,\vartheta)\right] + u^v\mathcal{S}\left[\frac{\partial z_2(\rho,\vartheta)}{\partial\rho}\right]\right] \\ &= h(h+n)^2\frac{\left(e^{\rho}-\rho\right)\vartheta^v}{\Gamma(v+1)} + 2h^2(n+h)\frac{\left(e^{\rho}-1\right)\vartheta^{2v}}{\Gamma(2v+1)} + h^3\frac{\left(e^{\rho}-1\right)\vartheta^{3v}}{\Gamma(3v+1)}. \end{aligned}$$

$$\vdots$$

Thus, similarly we can compute the rest terms of $z_m(\rho,\vartheta)$, $m \geq 4$.

The expanded solution can be defined

$$z(\rho,\vartheta) = z_0(\rho,\vartheta) + \sum_{m=1}^{\infty}\frac{z_m(\rho,\vartheta)}{n^m}.$$

Putting $h=-1,\ n=1$ and $v=1$, we obtain the series solution converging to the exact one:

$$z(\rho,\vartheta) = \vartheta\left(\rho - \frac{1}{2}\vartheta\right) + e^{\rho-\vartheta}.$$

Example 4.4

Consider the nonhomogeneous problem:

$$\frac{\partial^v z(\rho,\vartheta)}{\partial\vartheta^v} = \frac{\partial^2 z(\rho,\vartheta)}{\partial\rho^2} - \left(1+4\rho^2\right)z(\rho,\vartheta),\ \ 0<v<1, \tag{48}$$

subject to the conditions

$$z(\rho,0) = e^{\rho^2}. \tag{49}$$

Applying Sawi transform on Eq. (48), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right]=\frac{1}{u}z(\rho,0)+u^{v}\mathcal{S}\left[\frac{\partial^{2}z(\rho,\vartheta)}{\partial\rho^{2}}-\left(1+4\rho^{2}\right)z(\rho,\vartheta)\right]. \tag{50}$$

Using the initial condition (49), we obtain

$$\mathcal{S}\left[z(\rho,\vartheta)\right]=\frac{e^{\rho^{2}}}{u}+u^{v}\mathcal{S}\left[\frac{\partial^{2}z(\rho,\vartheta)}{\partial\rho^{2}}-\left(1+4\rho^{2}\right)z(\rho,\vartheta)\right]. \tag{51}$$

Applying q-HASTM on Eq.(51), we get

$$\mathcal{N}\left[\mathcal{G}(\rho,\vartheta;q)\right]=\mathcal{S}\left[\mathcal{G}(\rho,\vartheta;q)\right]-\frac{e^{\rho^{2}}}{u}-u^{v}\mathcal{S}\left[\frac{\partial^{2}\mathcal{G}(\rho,\vartheta;q)}{\partial\rho^{2}}-\left(1+4\rho^{2}\right)\mathcal{G}(\rho,\vartheta;q)\right], \tag{52}$$

and we have

$$\mathcal{R}\left(\vec{z}_{m-1}\right)=\mathcal{S}\left[z_{m-1}(\rho,\vartheta)\right]-\left(1-\frac{\kappa_{m}}{n}\right)\frac{e^{\rho^{2}}}{u}-u^{v}\mathcal{S}\left[\frac{\partial^{2}z_{m-1}(\rho,\vartheta)}{\partial\rho^{2}}-\left(1+4\rho^{2}\right)z_{m-1}(\rho,\vartheta)\right]. \tag{53}$$

Moreover, we define the mth-order deformation functions:

$$\mathcal{R}\left(\vec{z}_{m-1}\right)=h^{-1}\mathcal{S}\left[\vec{z}_{m}(\rho,\vartheta)-\kappa_{m}z_{m-1}(\rho,\vartheta)\right]. \tag{54}$$

Running inverse Sawi transform to Eqs.(54) to obtain

$$z_{m}(\rho,\vartheta)=\kappa_{m}z_{m-1}(\rho,\vartheta)+h\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{m-1}\right)\right]. \tag{55}$$

Note that, the first few terms of $z_{m}(\rho,\vartheta)$ is given by

$$z_{0}(\rho,\vartheta)=z(\rho,0)=e^{\rho^{2}}.$$

$$\begin{aligned}
z_{1}(\rho,\vartheta)&=\kappa_{0}z_{0}(\rho,\vartheta)+h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{0}\right)\right]\\
&=h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_{0}(\rho,\vartheta)\right]-\frac{e^{\rho^{2}}}{u}-u^{v}\mathcal{S}\left[\frac{\partial^{2}z_{0}(\rho,\vartheta)}{\partial^{2}\rho}-\left(1+4\rho^{2}\right)z_{0}(\rho,\vartheta)\right]\right]\\
&=-e^{\rho^{2}}h\,\mathcal{S}^{-1}\left[u^{v-1}\right]=\frac{-he^{\rho^{2}}\vartheta^{v}}{\Gamma(v+1)}.
\end{aligned}$$

The second and third iterations can be obtained similarly,

$$z_{2}(\rho,\vartheta)=\kappa_{2}z_{1}(\rho,\vartheta)+h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{1}\right)\right]=\frac{-h(n+h)e^{\rho^{2}}\vartheta^{v}}{\Gamma(v+1)}+\frac{h^{2}e^{\rho^{2}}\vartheta^{2v}}{\Gamma(2v+1)},$$

$$\begin{aligned}
z_{3}(\rho,\vartheta)&=\kappa_{3}z_{2}(\rho,\vartheta)+h\,\mathcal{S}^{-1}\left[\mathcal{R}\left(\vec{z}_{2}\right)\right]\\
&=-\frac{nh(n+h)e^{\rho^{2}}\vartheta^{v}}{\Gamma(v+1)}+\frac{nh^{2}e^{\rho^{2}}\vartheta^{2v}}{\Gamma(2v+1)}\\
&\quad+h\,\mathcal{S}^{-1}\left[\mathcal{S}\left[z_{2}(\rho,\vartheta)\right]-u^{v}\mathcal{S}\left[\frac{\partial^{2}z_{2}(\rho,\vartheta)}{\partial^{2}\rho}-(1+4\rho^{2})z_{2}(\rho,\vartheta)\right]\right]\\
&=-\frac{h(n+h)^{2}e^{\rho^{2}}\vartheta^{v}}{\Gamma(v+1)}+\frac{2h^{2}(n+h)e^{\rho^{2}}\vartheta^{2v}}{\Gamma(2v+1)}-\frac{h^{3}e^{\rho^{2}}\vartheta^{3v}}{\Gamma(3v+1)}.
\end{aligned}$$

$$\vdots$$

The rest terms of $z_m(\rho, \vartheta)$, $m \geq 4$, are obtained similarly.
The expanded solution is defined to be

$$z(\rho, \vartheta) = z_0(\rho, \vartheta) + \sum_{m=1}^{\infty} \frac{z_m(\rho, \vartheta)}{n^m}.$$

For $h = -1,\ n = 1$ and $v = 1$, we get the series solution converging to the exact one

$$z(\rho, \vartheta) = e^{\rho^2 + \vartheta}.$$

5. Conclusion

In this paper, we successfully demonstrated the application of the q-Homotopy Analysis of Sawi Transform Method (q-HASTM) to solve fractional Caputo partial differential equations. By integrating the Sawi transform with the homotopy analysis technique, we have provided a robust method capable of deriving approximate analytical solutions to complex fractional equations. The q-HASTM showed flexibility and accuracy, especially in handling nonlinear and non-homogeneous equations, validating its potential as a powerful tool for future research in fractional calculus. The results obtained through various examples confirmed the convergence of the method to exact solutions with appropriate parameter settings. Furthermore, the proposed method offers valuable insights for solving problems in applied mathematics, physics, and engineering, where fractional derivatives play a significant role [32, 33, 34]. The potential for further improvement of q-HASTM, such as extending its applications to multi-dimensional fractional problems or incorporating other types of fractional derivatives, remains promising. Future studies could also explore the combination of q-HASTM with other analytical methods to expand its usability in broader contexts [35, 36, 37].

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There is no data used in this article.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

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