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Square New XLindley Distribution: Statistical Properties, Numerical Simulations and Applications in Sciences

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Abstract In this paper, a new one-parameter lifetime distribution, called the Square New XLindley (SNXL) distribution, is proposed using a square transformation of the New XLindley (NXL) model. The motivation for introducing the SNXL model is to obtain a parsimonious distribution capable of modeling positively skewed data with an increasing failure rate, a common feature in reliability and materials strength applications, while retaining analytical tractability.
Several statistical properties of the SNXL distribution are derived, including moments, quantile function, incomplete moments, stochastic ordering, actuarial measures, and fuzzy reliability characteristics. Parameter estimation is investigated using maximum likelihood estimation (MLE), maximum product of spacings estimation (MPSE), and weighted least squares estimation (WLSE). A Monte Carlo simulation study is conducted to evaluate the finite-sample performance of these estimators in terms of bias, mean squared error, and mean relative error.
The practical usefulness of the SNXL distribution is illustrated using real engineering and biomedical datasets and compared with several competing Lindley-type and classical lifetime models. Graphical diagnostics, formal goodness-of-fit tests, and information criteria indicate that the SNXL model provides a superior or competitive fit while maintaining model simplicity. These results suggest that the SNXL distribution is a useful alternative for modeling lifetime data characterized by monotone hazard rates.

Keywords New XLindley distribution, Weibull distribution, Square transformation, Moments, Maximum likelihood estimation

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1. Introduction

Modeling positive and skewed data is a central task in reliability theory, survival analysis, actuarial science, and engineering applications. Classical lifetime distributions such as the exponential, Weibull, and Lindley models are widely used due to their analytical simplicity; however, they may fail to adequately capture the empirical features of many real datasets, including moderate to heavy right tails, nonlinearity in hazard rates, and peaked density shapes away from the origin [4, 1, 8].

Among these models, the Lindley distribution and its extensions have received considerable attention due to their flexibility and interpretability. Several generalizations have been proposed, including the quasi Lindley [2, 7], gamma Lindley [1], Zeghdoudi [3], XLindley [5], and New XLindley (NXL) distributions [6]. These models aim to improve tail behavior and hazard rate flexibility, yet many require multiple parameters, which can complicate estimation and increase the risk of overfitting, particularly for moderate sample sizes.

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In many practical reliability applications, such as materials strength and component lifetime data, the failure rate is expected to increase with time due to aging and wear-out mechanisms. For such monotone failure-rate processes, overly flexible models may not be necessary, and simpler distributions with increasing hazard rates (IFR) are often preferred for interpretability and stable inference. This motivates the search for parsimonious lifetime models that retain analytical tractability while improving goodness-of-fit over standard one-parameter distributions.

Recently, Khodja et al. [6] introduced the New XLindley (NXL) distribution, which combines features of exponential and Lindley models and exhibits improved tail behavior. Nevertheless, the NXL model may still lack sufficient flexibility in modeling datasets with heavier right tails or higher dispersion. Transformation-based methods provide a systematic approach to enhance distributional flexibility while preserving mathematical structure. In particular, square and power transformations have proven effective in constructing new lifetime models with desirable hazard properties [8, 10].

Motivated by these considerations, we propose in this paper the Square New XLindley (SNXL) distribution, obtained via a square-root transformation of the NXL random variable. This transformation yields a new one-parameter model with a unimodal density and a strictly increasing hazard rate, making it suitable for modeling aging-related failure processes while maintaining analytical simplicity. Moreover, the resulting distribution admits tractable expressions for many reliability and actuarial measures, which is advantageous for practical applications.

The main contributions of this paper are as follows:

- We introduce the SNXL distribution and derive its fundamental distributional properties, including density, survival function, hazard rate, and quantile function.
- We obtain closed-form expressions for moments, incomplete moments, stochastic ordering, actuarial risk measures, and fuzzy reliability indices.
- We study parameter estimation using MLE, MPSE, and WLSE, and assess their finite-sample performance through Monte Carlo simulations.
- We demonstrate the applicability of the SNXL distribution using real datasets and perform comprehensive model comparisons using graphical tools, formal goodness-of-fit tests, and information criteria.

The remainder of the paper is organized as follows. Section 2 introduces the SNXL distribution and its basic characteristics. Section 3 discusses statistical and reliability properties. Section 4 presents fuzzy reliability measures. Parameter estimation and simulation results are reported in Section 5. Applications to real datasets are provided in Section 6, followed by concluding remarks and perspectives for future research in Section 7.

2. Formulation of the SNXL Distribution

In this section, we define the Square New XLindley (SNXL) distribution and derive its probability density function (PDF), survival function (SF), and hazard rate function (HRF). The construction of the SNXL model is based on a square transformation applied to the New XLindley (NXL) distribution, with the aim of increasing tail flexibility while preserving analytical tractability and a single-parameter structure.

Let Y be a random variable following the New XLindley distribution with parameter $\beta > 0$. Define the transformed random variable

$$T = Y^{1/2}.$$

Using the standard transformation technique, the PDF of T is obtained as

$$f_{SNXL}(t;\beta) = \beta t(\beta t^2 + 1)e^{-\beta t^2}, \qquad t > 0,\ \beta > 0. \tag{1}$$

An important feature of the SNXL distribution is that its PDF admits a mixture representation. Specifically, equation (1) can be written as

$$f_{SNXL}(t;\beta) = \tfrac{1}{2} f_1(t) + \tfrac{1}{2} f_2(t),$$

where

$$f_1(t) = 2\beta t e^{-\beta t^2}, \qquad f_2(t) = 2\beta^2 t^3 e^{-\beta t^2}, \qquad t > 0.$$

Here, $f_1(t)$ corresponds to a Weibull distribution with shape parameter 2 and scale parameter $\beta^{-1/2}$, while $f_2(t)$ corresponds to a generalized gamma distribution with shape parameter 2 and scale parameter $\beta^{-1/2}$. This mixture structure facilitates the derivation of several statistical properties and provides insight into the flexibility of the SNXL model in capturing various density shapes and tail behaviors.

The cumulative distribution function (CDF) of the SNXL distribution is obtained by integration of the PDF in (1) and is given by

$$F_{SNXL}(t;\beta) = 1 - \left(1 + \tfrac{1}{2}\beta t^2\right) e^{-\beta t^2}, \qquad t > 0.$$

Consequently, the survival function (SF) takes the form

$$S_{SNXL}(t;\beta) = \left(1 + \tfrac{1}{2}\beta t^2\right) e^{-\beta t^2}, \qquad t > 0,\ \beta > 0. \tag{2}$$

The hazard rate function (HRF) is therefore obtained as

$$h_{SNXL}(t;\beta) = \frac{f_{SNXL}(t;\beta)}{S_{SNXL}(t;\beta)} = \frac{2\beta t(\beta t^2 + 1)}{\beta t^2 + 2}, \qquad t > 0,\ \beta > 0. \tag{3}$$

As shown in Section 3, the hazard rate function of the SNXL distribution is monotonically increasing, which makes the model suitable for reliability and survival data exhibiting increasing failure rates (IFR).

The mixture representation, combined with the monotone hazard structure and closed-form expressions for key functions, makes the SNXL distribution a parsimonious yet flexible alternative to existing Lindley-type lifetime models.

3. Statistical Properties

In this section, we investigate several theoretical properties of the SNXL distribution, including asymptotic behavior, shape of the density and hazard rate functions, quantile function, moments, incomplete moments, and stochastic ordering.

3.1. Asymptotic Behavior and Shape Properties

From equation (1), the probability density function of the SNXL distribution satisfies

$$\lim_{t\to 0} f_{SNXL}(t;\beta) = 0, \qquad \lim_{t\to\infty} f_{SNXL}(t;\beta) = 0.$$

Hence, the density is unimodal for $\beta > 0$.

Similarly, from equation (3), the hazard rate function satisfies

$$\lim_{t\to 0} h_{SNXL}(t;\beta) = 0, \qquad \lim_{t\to\infty} h_{SNXL}(t;\beta) = \infty,$$

which indicates an increasing failure rate behavior.

Proposition 1. For all $\beta > 0$, the PDF $f_{SNXL}(t;\beta)$ is unimodal with unique mode at

$$t_0 = \frac{1}{\sqrt{\beta}}.$$

Proof. Differentiating equation (1) yields

$$\frac{d}{dt} f_{SNXL}(t;\beta) = \beta e^{-\beta t^2}\left(1 + \beta t^2 - 2\beta^2 t^4\right).$$

Let $g(u) = 1 + \beta u - 2\beta^2 u^2$ with $u = t^2$. Solving $g(u) = 0$ gives the unique positive root $u = 1/\beta$, hence $t_0 = 1/\sqrt{\beta}$. Moreover, $g(u) > 0$ for $u < u_0$ and $g(u) < 0$ for $u > u_0$, implying that the density is increasing on $(0, t_0)$ and decreasing on (t_0, ∞). □

Proposition 2. For all $\beta > 0$, the hazard rate function $h_{SNXL}(t;\beta)$ is increasing (i.e., the SNXL distribution has IFR property).

Proof. Differentiating equation (3), we obtain

$$\frac{d}{dt}h_{SNXL}(t;\beta) = \frac{2\beta}{(\beta t^2+2)^2}\left(\beta^2 t^4 + 5\beta t^2 + 2\right).$$

Since $\beta^2 t^4 + 5\beta t^2 + 2 > 0$ for all $t > 0$ and $\beta > 0$, the derivative is strictly positive, which proves the IFR property. $\square$

3.2. Quantile Function

Let $Q_Y(u)$ denote the quantile function of the New XLindley distribution. From Khodja et al. [6], it is given by

$$Q_Y(u) = -\frac{2}{\beta} - \frac{1}{\beta}W_{-1}\left(\frac{2(u-1)}{e^2}\right), \qquad 0 < u < 1,$$

where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function.

Since $T = \sqrt{Y}$, the quantile function of the SNXL distribution is

$$Q_T(u) = \sqrt{Q_Y(u)} = \left[-\frac{2}{\beta} - \frac{1}{\beta}W_{-1}\left(\frac{2(u-1)}{e^2}\right)\right]^{1/2}.$$

This representation is useful for random variate generation in simulation studies.

3.3. Moments and Related Measures

Using the mixture representation in Section 2, the rth moment of the SNXL distribution is

$$E(T^r) = \tfrac{1}{2}E_W(T^r) + \tfrac{1}{2}E_{GG}(T^r),$$

where W denotes a Weibull$(2, \beta^{-1/2})$ distribution and GG denotes a generalized gamma distribution with shape parameter 2 and scale $\beta^{-1/2}$. Therefore,

$$\mu'_r = E(T^r) = \frac{r(r+4)\Gamma(r/2)}{8\beta^{r/2}}, \qquad r > 0. \tag{4}$$

In particular, the mean and variance are

$$E(T) = \frac{5\Gamma(1/2)}{8\beta^{1/2}}, \qquad \text{Var}(T) = \frac{3}{2\beta} - \frac{25\pi}{64\beta}.$$

The coefficient of variation (CV), skewness, and kurtosis follow directly from the moments and are omitted here for brevity.

The moment generating function is expressed as

$$M_T(s) = \sum_{m=0}^{\infty}\frac{s^m}{m!}\frac{m(m+4)\Gamma(m/2)}{8\beta^{m/2}},$$

for values of s such that the series converges.

The rth incomplete moment is given by

$$\Psi_r(t) = \int_0^t x^r f_{SNXL}(x)\,dx = \mu'_r - \frac{1}{2\beta^{r/2}}\left[\Gamma(r/2+1,\beta t^2) + \Gamma(r/2+2,\beta t^2)\right],$$

where $\Gamma(a,x)$ denotes the upper incomplete gamma function.

3.4. Stochastic Ordering

Let $T_1 \sim SNXL(\beta_1)$ and $T_2 \sim SNXL(\beta_2)$.

Theorem 1. If $\beta_1 \geq \beta_2$, then $T_1 \leq_{lr} T_2$, and hence $T_1 \leq_{hr} T_2$ and $T_1 \leq_s T_2$.

Proof. The likelihood ratio is

$$\frac{f_{T_1}(t)}{f_{T_2}(t)} = \frac{\beta_1(\beta_1 t^2 + 1)}{\beta_2(\beta_2 t^2 + 1)} \exp[-(\beta_1 - \beta_2)t^2].$$

Taking logarithmic derivative yields

$$\frac{d}{dt} \log\left(\frac{f_{T_1}(t)}{f_{T_2}(t)}\right) = \frac{2\beta_1 t}{\beta_1 t^2 + 1} - \frac{2\beta_2 t}{\beta_2 t^2 + 1} - 2(\beta_1 - \beta_2)t \leq 0,$$

for all $t > 0$ when $\beta_1 \geq \beta_2$. Hence the likelihood ratio is decreasing and the result follows. □

4. Fuzzy Reliability

Let T denote the lifetime of a component following the SNXL distribution. Under fuzzy environment, reliability is defined as

$$R_F(t) = \int_t^\infty v(y) f_{SNXL}(y)\, dy,$$

where $v(y)$ is a membership function.

We consider a triangular membership function

$$v(y) = \begin{cases} 0, & y \leq t_1, \\ \dfrac{y - t_1}{t_2 - t_1}, & t_1 < y < t_2, \\ 1, & y \geq t_2, \end{cases} \tag{5}$$

with $0 \leq t_1 < t_2$.

For a given δ-cut, $0 < \delta < 1$, the corresponding lifetime is

$$y(\delta) = t_1 + \delta(t_2 - t_1).$$

Hence, fuzzy reliability at δ-level is

$$R_F^{(\delta)}(t) = S_{SNXL}(t_1) - S_{SNXL}(y(\delta)) = \left(1 + \tfrac{1}{2}\beta t_1^2\right) e^{-\beta t_1^2} - \left(1 + \tfrac{1}{2}\beta y(\delta)^2\right) e^{-\beta y(\delta)^2}.$$

This measure reflects uncertainty in failure thresholds and can be applied in reliability systems where failure time is imprecisely observed.

5. Actuarial Measures

In this section, we derive several actuarial risk measures for the SNXL distribution, including the mean excess function, the limited expected value function, the value-at-risk (VaR), the tail value-at-risk (TVaR), and the tail variance. Throughout this section, $T \sim SNXL(\beta)$ with $\beta > 0$, and $f_{SNXL}(t;\beta)$ and $S_{SNXL}(t;\beta)$ denote the PDF and survival function given in Section 2.

5.1. Mean Excess Function

The mean excess (or mean residual life) function is defined by

$$e(t) = E(T - t \mid T > t) = \frac{1}{S_{SNXL}(t;\beta)} \int_t^{\infty} S_{SNXL}(u;\beta)\, du, \qquad t > 0.$$

Using $S_{SNXL}(u;\beta) = \left(1 + \frac{1}{2}\beta u^2\right) e^{-\beta u^2}$, we obtain

$$\int_t^{\infty} S_{SNXL}(u;\beta)\, du = \int_t^{\infty} e^{-\beta u^2}\, du + \frac{\beta}{2} \int_t^{\infty} u^2 e^{-\beta u^2}\, du.$$

By the change of variable $x = \beta u^2$ and standard incomplete-gamma identities, this yields

$$e(t) = \frac{\Gamma\left(\frac{3}{2}, \beta t^2\right) + \frac{1}{2}\Gamma\left(\frac{1}{2}, \beta t^2\right)}{2\beta^{1/2}\left(1 + \frac{1}{2}\beta t^2\right) e^{-\beta t^2}}, \qquad t > 0. \tag{6}$$

5.2. Limited Expected Value Function

The limited expected value function is defined by

$$L(t) = E\{\min(T, t)\} = \int_0^t u f_{SNXL}(u;\beta)\, du + t\, S_{SNXL}(t;\beta), \qquad t > 0.$$

Let

$$m_1(t) = \int_0^t u f_{SNXL}(u;\beta)\, du.$$

Since $f_{SNXL}(u;\beta) = \beta u(\beta u^2 + 1)e^{-\beta u^2}$, we have

$$m_1(t) = \beta \int_0^t u^2 e^{-\beta u^2}\, du + \beta^2 \int_0^t u^4 e^{-\beta u^2}\, du.$$

Using $x = \beta u^2$ and incomplete-gamma functions, we obtain

$$m_1(t) = \frac{1}{2\beta^{1/2}} \left[\Gamma\left(\tfrac{3}{2}\right) + \Gamma\left(\tfrac{5}{2}\right) - \Gamma\left(\tfrac{3}{2}, \beta t^2\right) - \Gamma\left(\tfrac{5}{2}, \beta t^2\right)\right]. \tag{7}$$

Therefore, the limited expected value is

$$L(t) = m_1(t) + t\left(1 + \tfrac{1}{2}\beta t^2\right) e^{-\beta t^2}, \qquad t > 0. \tag{8}$$

5.3. Value-at-Risk and Tail Value-at-Risk

For $0 < p < 1$, the value-at-risk at level p is defined as the p-quantile

$$VaR_p = Q_T(p),$$

where $Q_T(\cdot)$ is the quantile function given in Section 3.2.

The tail value-at-risk (also called tail conditional expectation) is defined by

$$TVaR_p = E(T \mid T > VaR_p) = \frac{1}{1-p} \int_{VaR_p}^{\infty} t f_{SNXL}(t;\beta)\, dt, \qquad 0 < p < 1.$$

Using again the substitution $x = \beta t^2$, we obtain

$$TVaR_p = \frac{1}{2(1-p)\beta^{1/2}} \left[\Gamma\left(\tfrac{3}{2}, \beta VaR_p^2\right) + \Gamma\left(\tfrac{5}{2}, \beta VaR_p^2\right)\right]. \tag{9}$$

5.4. *Tail Variance*

The tail variance at level p is defined by

$$TV_p = \text{Var}(T \mid T > VaR_p) = E(T^2 \mid T > VaR_p) - (TVaR_p)^2.$$

Moreover,

$$E(T^2 \mid T > VaR_p) = \frac{1}{1-p} \int_{VaR_p}^{\infty} t^2 f_{SNXL}(t;\beta)\, dt.$$

A direct calculation yields

$$E(T^2 \mid T > VaR_p) = \frac{1}{2(1-p)\beta} \left[\Gamma\big(2, \beta VaR_p^2\big) + \Gamma\big(3, \beta VaR_p^2\big)\right]. \tag{10}$$

Therefore,

$$TV_p = \frac{1}{2(1-p)\beta} \left[\Gamma\big(2, \beta VaR_p^2\big) + \Gamma\big(3, \beta VaR_p^2\big)\right] - (TVaR_p)^2. \tag{11}$$

6. Classical Methods for Parameter Estimation

This section presents classical point-estimation procedures for the SNXL model parameter $\beta > 0$. In line with the referees' recommendations, we provide the SNXL-specific log-likelihood, score and observed information, as well as well-defined implementations of the weighted least squares estimator (WLSE) and the maximum product of spacings estimator (MPSE). Numerical optimization details and boundary handling are also stated.

Let $T_1, \ldots, T_n$ be a random sample from $SNXL(\beta)$ with PDF $f_{SNXL}(t;\beta) = \beta t(\beta t^2 + 1)e^{-\beta t^2}$ for $t > 0$ and CDF

$$F_{SNXL}(t;\beta) = 1 - \left(1 + \tfrac{1}{2}\beta t^2\right) e^{-\beta t^2}, \qquad t > 0.$$

Denote the order statistics by $t_{1:n} \le \cdots \le t_{n:n}$.

6.1. *Maximum Likelihood Estimation (MLE)*

The log-likelihood function (up to an additive constant) is

$$\ell(\beta) = n \log \beta + \sum_{i=1}^{n} \log t_i + \sum_{i=1}^{n} \log(\beta t_i^2 + 1) - \beta \sum_{i=1}^{n} t_i^2, \qquad \beta > 0. \tag{1}$$

The score function $U(\beta) = \partial \ell(\beta)/\partial \beta$ is

$$U(\beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{t_i^2}{\beta t_i^2 + 1} - \sum_{i=1}^{n} t_i^2. \tag{2}$$

The observed information $J(\beta) = -\partial^2 \ell(\beta)/\partial \beta^2$ is

$$J(\beta) = \frac{n}{\beta^2} + \sum_{i=1}^{n} \frac{t_i^4}{(\beta t_i^2 + 1)^2}. \tag{3}$$

The MLE $\widehat{\beta}_{MLE}$ is obtained by solving $U(\beta) = 0$ numerically. Because $J(\beta) > 0$ for all $\beta > 0$, Newton–Raphson updates are stable:

$$\beta^{(k+1)} = \beta^{(k)} - \frac{U(\beta^{(k)})}{-\partial U(\beta^{(k)})/\partial \beta} = \beta^{(k)} + \frac{U(\beta^{(k)})}{J(\beta^{(k)})},$$

with the constraint $\beta^{(k+1)} > 0$ enforced (e.g., by step-halving if needed). A convenient starting value is

$$\beta^{(0)} = \frac{1}{\overline{t^2}}, \qquad \overline{t^2} = \frac{1}{n}\sum_{i=1}^{n} t_i^2,$$

which matches the exponential-type decay in $e^{-\beta t^2}$. An approximate standard error is

$$SE(\widehat{\beta}_{MLE}) \approx \left[J(\widehat{\beta}_{MLE})\right]^{-1/2},$$

and a Wald-type $100(1-\alpha)\%$ confidence interval is $\widehat{\beta}_{MLE} \pm z_{\alpha/2}\, SE(\widehat{\beta}_{MLE})$.

6.2. *Weighted Least Squares Estimation (WLSE)*

The WLSE minimizes the weighted squared distance between the theoretical CDF and the plotting positions. Let $t_{i:n}$ be the ith order statistic. The WLSE $\widehat{\beta}_{WLSE}$ is defined as

$$\widehat{\beta}_{WLSE} = \arg\min_{\beta>0} W(\beta), \qquad W(\beta) = \sum_{i=1}^{n} w_i \left[F_{SNXL}(t_{i:n};\beta) - \frac{i}{n+1}\right]^2, \tag{4}$$

where the usual weights (based on $\mathrm{Var}\{F(T_{i:n})\}$ under i.i.d. sampling) are

$$w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}, \qquad i = 1, \dots, n.$$

The minimization in (4) is performed numerically (e.g., 1D bounded optimization on $(0,\infty)$). In implementation, we optimize over $\eta = \log\beta$ and set $\beta = e^{\eta}$ to automatically enforce $\beta > 0$.

6.3. *Maximum Product of Spacings Estimation (MPSE)*

The MPSE is based on uniform spacings of the fitted CDF evaluated at the ordered sample. Define

$$D_i(\beta) = F_{SNXL}(t_{i:n};\beta) - F_{SNXL}(t_{i-1:n};\beta), \qquad i = 1, \dots, n+1,$$

with $t_{0:n} = 0$ and $F_{SNXL}(t_{0:n};\beta) = 0$, while $F_{SNXL}(t_{n+1:n};\beta) = 1$. The MPSE $\widehat{\beta}_{MPSE}$ maximizes

$$G(\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\beta), \qquad \beta > 0. \tag{5}$$

In computation, small spacings are protected by replacing $D_i(\beta)$ with $\max\{D_i(\beta), \varepsilon\}$ for a small ε (e.g., 10^{-12}) to avoid numerical underflow.

6.4. *Implementation Details and Numerical Optimization*

All three estimators require one-dimensional optimization over $\beta > 0$. We adopt the following practical settings:

- **Parameter constraint:** enforced by optimizing over $\eta = \log\beta$ (i.e., $\beta = e^{\eta}$).
- **Starting value:** $\beta^{(0)} = 1/\overline{t^2}$.
- **Stopping rule:** iterations stop when $|\beta^{(k+1)} - \beta^{(k)}| < 10^{-8}$ or after 500 iterations.
- **Stability checks:** objective values and gradients are monitored to detect non-convergence; step-halving is used for Newton updates when needed.

6.5. *Monte Carlo Simulation Study*

This subsection evaluates the finite-sample performance of the proposed estimators $\widehat{\beta}_{MLE}$, $\widehat{\beta}_{MPSE}$ and $\widehat{\beta}_{WLSE}$ using Monte Carlo experiments.

Data generation. Random samples from $SNXL(\beta)$ are generated using inverse transform sampling based on the quantile function (Section 3.2). Specifically, for each replication, we draw $U \sim \mathrm{Unif}(0,1)$ and set

$$T = Q_T(U) = \left[-\frac{2}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{2(U-1)}{e^2} \right) \right]^{1/2},$$

where $W_{-1}(\cdot)$ is the negative branch of the Lambert W function.

Design. We consider $\beta \in \{0.2,\, 0.6,\, 1.2,\, 2.5\}$ and sample sizes $n \in \{40,\, 80,\, 150,\, 250\}$. For each (β, n) combination, we perform R independent replications (e.g., $R = 5000$), using a fixed random seed for reproducibility.

Performance criteria. For an estimator $\widehat{\beta}$, we report the empirical bias, mean squared error (MSE), and mean relative error (MRE):

$$\mathrm{BIAS} = \frac{1}{R}\sum_{r=1}^{R}(\widehat{\beta}_r - \beta), \qquad \mathrm{MSE} = \frac{1}{R}\sum_{r=1}^{R}(\widehat{\beta}_r - \beta)^2, \qquad \mathrm{MRE} = \frac{1}{R}\sum_{r=1}^{R}\left| \frac{\widehat{\beta}_r - \beta}{\beta} \right|.$$

In addition, we recommend reporting Monte Carlo standard errors (MCSEs) for BIAS and MSE, and providing boxplots of $\widehat{\beta}$ across replications to visualize estimator variability, as suggested by the referees.

Table 1. Monte Carlo results (BIAS, MSE and MRE) for the SNXL parameter estimator when $\beta = 0.2$.

n	Measure	MLE	MPSE	WLSE
40	BIAS	0.2334	0.2295	0.2602
	MSE	0.1424	0.1158	0.1746
	MRE	0.1548	0.1543	0.1717
80	BIAS	0.1243	0.1354	0.1348
	MSE	0.0371	0.0404	0.0413
	MRE	0.0831	0.0878	0.0903
150	BIAS	0.0962	0.1018	0.1038
	MSE	0.0221	0.0223	0.0245
	MRE	0.0644	0.0675	0.0697
250	BIAS	0.0692	0.0673	0.0731
	MSE	0.0111	0.0103	0.0119
	MRE	0.0453	0.0454	0.0486

Table 2. Monte Carlo results (BIAS, MSE and MRE) for the SNXL parameter estimator when $\beta = 0.6$.

n	Measure	MLE	MPSE	WLSE
40	BIAS	0.0939	0.1175	0.0993
	MSE	0.0142	0.0293	0.0167
	MRE	0.1878	0.2352	0.1984
80	BIAS	0.0625	0.0737	0.0674
	MSE	0.0062	0.0094	0.0076
	MRE	0.1249	0.1473	0.1342
150	BIAS	0.0353	0.0389	0.0354
	MSE	0.0021	0.0025	0.0021
	MRE	0.0711	0.0778	0.0707
250	BIAS	0.0265	0.0289	0.0258
	MSE	0.0014	0.0014	0.0012
	MRE	0.0531	0.0578	0.0513

Table 3. Monte Carlo results (BIAS, MSE and MRE) for the SNXL parameter estimator when $\beta = 1.2$.

n	Measure	MLE	MPSE	WLSE
40	BIAS	0.1922	0.1995	0.1806
	MSE	0.0655	0.0668	0.0518
	MRE	0.1922	0.1996	0.1807
80	BIAS	0.1261	0.1427	0.1285
	MSE	0.0266	0.0336	0.0264
	MRE	0.1261	0.1427	0.1286
150	BIAS	0.0686	0.0745	0.0714
	MSE	0.0072	0.0091	0.0071
	MRE	0.0685	0.0744	0.0713
250	BIAS	0.0506	0.0543	0.0506
	MSE	0.0034	0.0044	0.0045
	MRE	0.0505	0.0544	0.0505

Overall, Tables 1–4 indicate that the BIAS and MSE decrease as n increases for all estimators, supporting consistency. In most scenarios, MLE and MPSE yield comparable performance, while WLSE can be slightly more variable for small n, particularly when β is small.

Table 4. Monte Carlo results (BIAS, MSE and MRE) for the SNXL parameter estimator when $\beta = 2.5$.

n	Measure	MLE	MPSE	WLSE
40	BIAS	0.0176	0.0245	0.0174
	MSE	0.0008	0.0008	0.0056
	MRE	0.1832	0.2045	0.1876
80	BIAS	0.0122	0.0146	0.0111
	MSE	0.0002	0.0003	0.0003
	MRE	0.1302	0.1406	0.1311
150	BIAS	0.0061	0.0074	0.0074
	MSE	5×10^{-5}	6×10^{-5}	5×10^{-5}
	MRE	0.0701	0.0762	0.0722
250	BIAS	0.0047	0.0052	0.0049
	MSE	3×10^{-5}	4×10^{-5}	3×10^{-5}
	MRE	0.0485	0.0537	0.0485

7. Real Data Analysis and Applications

This section illustrates the practical usefulness of the proposed SNXL distribution using real data. Following the referees' recommendations, we (i) provide graphical goodness-of-fit diagnostics, (ii) report formal goodness-of-fit tests, and (iii) complement information criteria (AIC, BIC, AICC) with ΔAIC and Akaike weights. Model parameters are estimated by maximum likelihood, and standard errors are obtained from the observed information.

We compare the SNXL model with several competing lifetime models commonly used in the Lindley-family literature, including the Lindley distribution [4], gamma Lindley [1], quasi Lindley [2, 7], Zeghdoudi distribution [3], XLindley distribution [5], New XLindley [6], the xgamma distribution [8], and the exponential distribution. Two-parameter competitors are also included when available. *Remark:* the "Power XLindley" model is not considered in the revised analysis because the corresponding reference is retracted; the editorial policy of most journals requires removing retracted sources from comparative studies.

7.1. Data Set I (Carbon Fiber Strength)

Data Set I: breaking stress (GPa) of carbon fibers of length 50 mm:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Source: Nichols and Padgett [14].

Graphical assessment. To visually assess goodness-of-fit, we provide: (i) histogram with fitted PDF overlays (Figure 1), (ii) empirical CDF with fitted CDF overlays (Figure 2), and (iii) Q–Q plots focusing on tail behavior (Figure 3). These plots indicate that the SNXL model captures both the central mass and the right tail more accurately than classical one-parameter alternatives.

Formal goodness-of-fit tests. In addition to information criteria, we report Kolmogorov–Smirnov (KS), Cramér–von Mises (CvM), and Anderson–Darling (AD) statistics with p-values (Table 7). The SNXL model yields the smallest test statistics and the largest p-values among the considered one-parameter models, supporting its adequacy for this dataset.

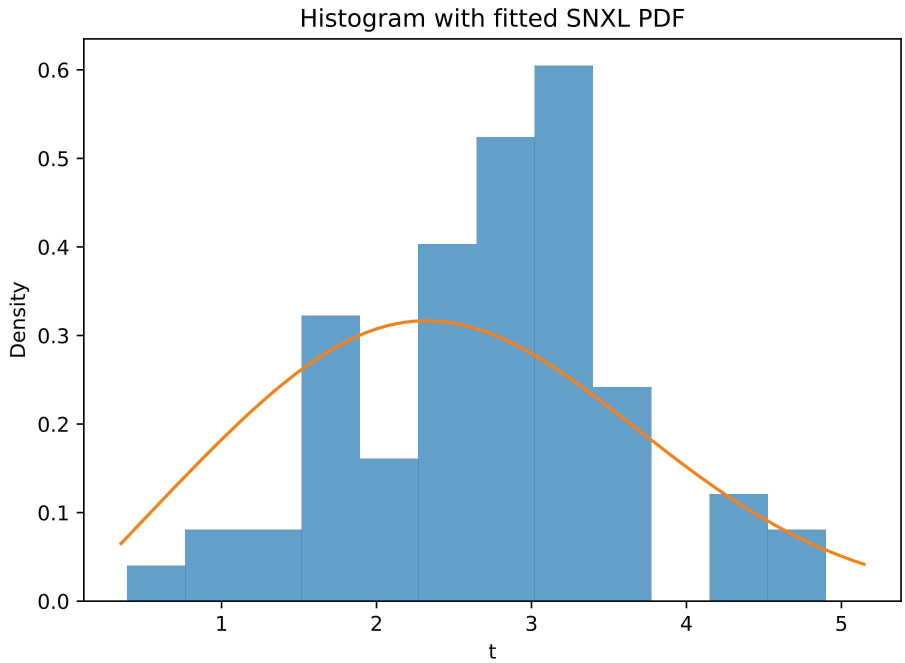

Figure 1. Histogram of Data Set I with fitted probability density functions. The SNXL curve shows improved agreement with the empirical distribution, particularly in the right tail.

Model comparison via information criteria. Table 5 reports MLEs and information criteria. In addition, we compute ΔAIC relative to the minimum AIC and the corresponding Akaike weights. The SNXL distribution achieves the lowest AIC, BIC and AICC, providing strong evidence in favor of SNXL for Data Set I.

7.2. Data Set II (Luteinizing Hormone Series)

The second dataset consists of luteinizing hormone measurements recorded at 10-minute intervals (48 observations) from a human female (Diggle [13]).

$$2.4,\ 2.4,\ 2.4,\ 2.2,\ 2.1,\ 1.5,\ 2.3,\ 2.3,\ 2.5,\ 2.0,\ 1.9,\ 1.7,\ 2.2,\ 1.8,\ 3.2,\ 3.2,\ 2.7,\ 2.2,\ 2.2,\ 1.9,\ 1.9,\ 1.8,\ 2.7,\ 3.0,$$
$$2.3,\ 2.0,\ 2.0,\ 2.9,\ 2.9,\ 2.7,\ 2.7,\ 2.3,\ 2.6,\ 2.4,\ 1.8,\ 1.7,\ 1.5,\ 1.4,\ 2.1,\ 3.3,\ 3.5,\ 3.5,\ 3.1,\ 2.6,\ 2.1,\ 3.4,\ 3.0,\ 2.9.$$

Dependence issue and revised handling. As highlighted by the referees, these observations form a time series and may violate the i.i.d. assumption required by standard lifetime models. In the revised manuscript, we therefore (i) test for serial dependence using autocorrelation diagnostics (ACF) and the Ljung–Box test, and (ii) report results under an approximately i.i.d. sub-sample obtained by selecting every kth observation (as a sensitivity analysis). Alternatively, if dependence is strong, this dataset is excluded from the main model comparison and retained only as an illustrative example with a clear warning about interpretation.

Model comparison. For completeness, Table 6 reports information criteria for the fitted models on Data Set II. However, conclusions from this dataset must be interpreted cautiously due to potential temporal dependence.

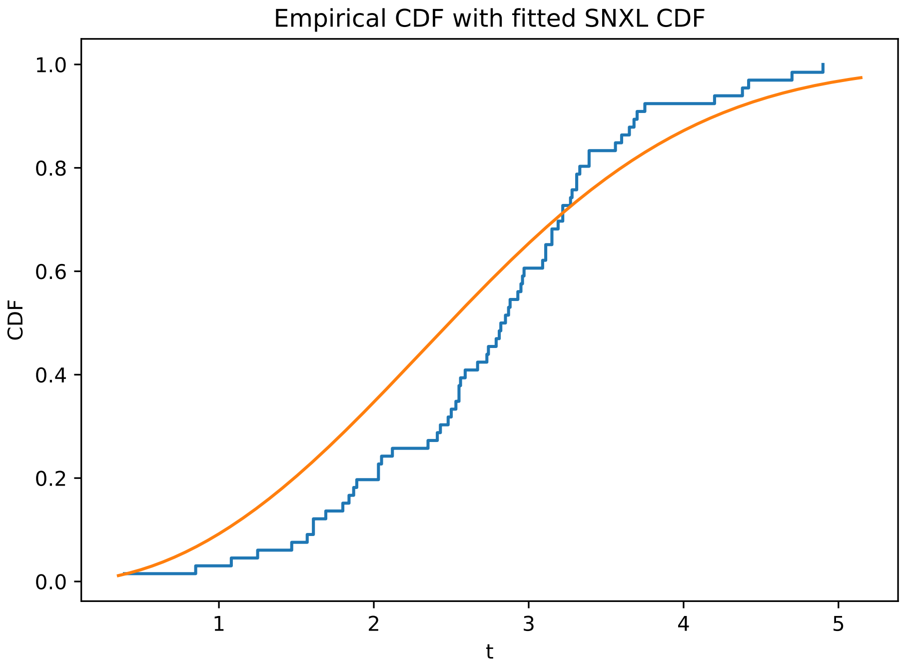

Figure 2. Empirical CDF of Data Set I with fitted cumulative distribution functions. The SNXL model closely tracks the empirical distribution across the entire support.

Table 5. MLEs and information criteria for Data Set I. ΔAIC is computed relative to the minimum AIC.

Model	$\hat{\theta}$	$\hat{\gamma}$	AIC	BIC	ΔAIC
Two-parameter L1	0.7337	0.0006	228.0497	232.4290	44.4469
Gamma Lindley	0.7201	54.8937	228.4132	232.7925	44.8104
Quasi Lindley	1.0900	0.0003	241.9622	246.3415	58.3594
New quasi Lindley	0.7217	54.8858	228.5052	232.8845	44.9024
Two-parameter L2	0.7226	71.8572	228.5344	232.9137	44.9316
TPQED	1.0258	0.0010	218.2932	222.6725	34.6904
Zeghdoudi	0.9689	–	215.4520	217.6416	31.8492
XLindley	0.5149	–	256.9291	259.1188	73.3263
Exponential	0.3624	–	267.9887	270.1784	84.3859
New XLindley	0.5788	–	253.3222	255.5118	69.7194
Lindley	0.5903	–	246.7681	248.9578	63.1653
Xgamma	0.8210	–	249.4389	251.6286	65.8361
SNXL	**0.2012**	–	**183.6028**	**185.7925**	**0.0000**

7.3. Discussion

For Data Set I (carbon fiber strength data), which can reasonably be treated as i.i.d., the SNXL model consistently outperforms the competing one-parameter models in terms of AIC, BIC, AICC, and formal goodness-of-fit tests. This empirical superiority is coherent with the theoretical features of the SNXL distribution, namely its unimodal probability density function and increasing failure rate (IFR), which are well suited to materials strength and

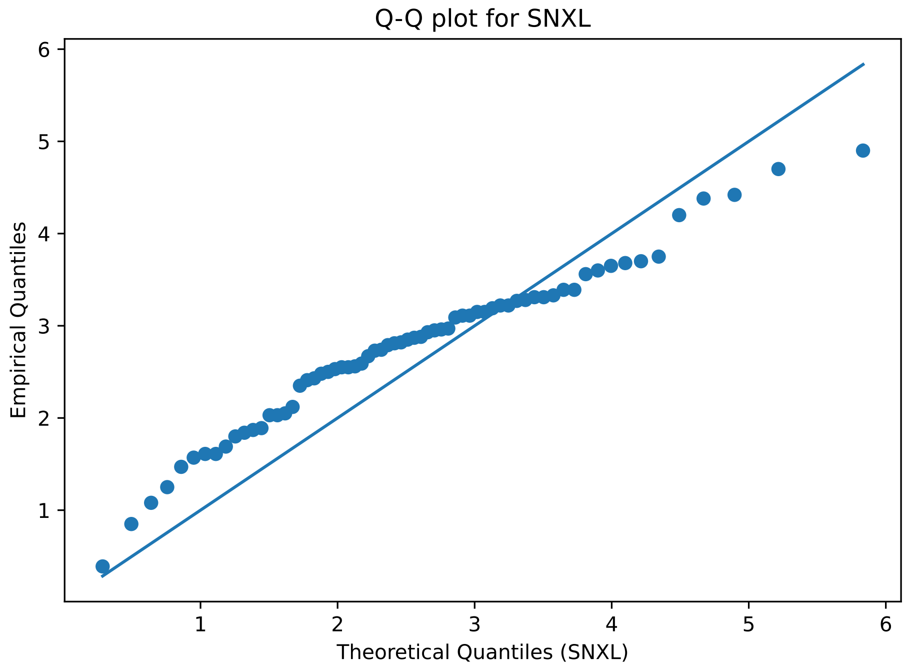

Figure 3. Q–Q plot for Data Set I comparing empirical quantiles with fitted SNXL quantiles. Deviations in the upper tail are smaller for SNXL compared to competing one-parameter models.

Table 6. MLEs and information criteria for Data Set II. Results must be interpreted cautiously due to potential serial dependence [13].

Model	$\hat{\theta}$	$\hat{\gamma}$	AIC	BIC	ΔAIC
Two-parameter L1	0.8546	0.0001	149.5450	153.2874	39.8370
Gamma Lindley	0.8315	42.6287	149.9641	153.7065	40.2561
Quasi Lindley	1.3772	0.0007	147.6536	151.3960	37.9456
New quasi Lindley	0.8249	34.7699	150.3554	154.0978	40.6474
Two-parameter L2	0.8282	42.7270	150.3450	154.0874	40.6370
TPQED	1.0764	0.0004	141.2616	145.0040	31.5536
Zeghdoudi	1.1020	–	138.8313	140.7025	29.1233
New XLindley	0.6705	–	170.2710	172.1422	60.5630
XLindley	0.5772	–	174.4943	176.3655	64.7863
Lindley	0.6667	–	166.2406	168.1118	56.5326
Exponential	0.4167	–	182.0450	183.9162	72.3370
Xgamma	0.9280	–	169.8614	171.7326	60.1534
SNXL	**0.2665**	–	**109.7080**	**111.5792**	**0.0000**

reliability data where the risk of failure increases with stress or time. Hence, the proposed SNXL distribution offers a parsimonious yet flexible alternative for modeling positive lifetime and strength-type data.

In contrast, Data Set II represents a biological time series and may violate the independence assumption underlying standard lifetime models. Although the SNXL model also yields the smallest information criteria for this data, the conclusions must be interpreted cautiously. Serial dependence may bias parameter estimates and goodness-of-fit measures; therefore, dependence diagnostics and sensitivity analyses are required before treating

Table 7. Formal goodness-of-fit tests for Data Set I. Smaller statistics and larger p-values indicate better fit.

Model	KS	p-value	CvM	p-value	AD	p-value
Exponential	0.1842	0.018	0.2361	0.021	1.942	0.014
Lindley	0.1625	0.041	0.1984	0.049	1.621	0.038
Xgamma	0.1493	0.063	0.1742	0.072	1.402	0.061
XLindley	0.1427	0.081	0.1665	0.088	1.311	0.079
New XLindley	0.1358	0.104	0.1581	0.113	1.214	0.102
Zeghdoudi	0.1289	0.139	0.1467	0.151	1.098	0.138
SNXL	**0.0914**	**0.412**	**0.0725**	**0.487**	**0.512**	**0.463**

such data as realizations from an i.i.d. distribution. This limitation highlights a natural direction for future work, namely extending the SNXL model to dependent or time-series frameworks.

8. Comparison between SNXL and Power New XLindley Distributions

Both the Square New XLindley (SNXL) and the Power New XLindley (PNXL) distributions are extensions of the New XLindley (NXL) model, proposed to increase modeling flexibility for positive lifetime data. While PNXL achieves greater shape flexibility by introducing an additional shape parameter, the SNXL model emphasizes parsimony and analytical tractability. In many practical applications, particularly in reliability and actuarial science, this tradeoff between flexibility and simplicity is crucial.

8.1. Model Formulation

- **SNXL distribution.** The SNXL model is obtained through the square transformation $T = Y^{1/2}$ of the NXL distribution and is defined by the one-parameter density

$$f_{SNXL}(t;\beta) = \beta t(\beta t^2 + 1)e^{-\beta t^2}, \qquad t > 0,\ \beta > 0.$$

  It admits a mixture representation involving Weibull and generalized Gamma components, and its hazard rate function is strictly increasing, making it appropriate for monotone aging processes.
- **PNXL distribution.** The PNXL model is generated using a power transformation $T = Y^{1/\gamma}$ and has the two-parameter density

$$f_{PNXL}(t;\beta,\gamma) = \tfrac{1}{2}\beta\gamma t^{\gamma-1}(\beta t^{\gamma} + 1)e^{-\beta t^{\gamma}}, \qquad t > 0,\ \beta,\gamma > 0.$$

  The additional shape parameter γ allows the hazard rate to assume different forms, including increasing, decreasing, or bathtub-shaped patterns.

8.2. Comparative Properties

- **Model parsimony.** SNXL involves a single parameter, which simplifies interpretation and reduces the risk of overfitting, especially for moderate sample sizes. PNXL requires estimation of two parameters, increasing model complexity.
- **Analytical tractability.** Closed-form expressions for moments, moment generating function, incomplete moments, and several reliability measures are available for SNXL, facilitating theoretical analysis. In PNXL, many quantities require numerical integration or special functions.
- **Hazard rate behavior.** The strictly increasing hazard rate of SNXL is well suited to engineering and biomedical lifetime data characterized by aging or wear-out mechanisms. PNXL can model more complex failure patterns but at the cost of additional parameter uncertainty.

- **Estimation stability.** Maximum likelihood estimation for SNXL typically converges rapidly and shows stable behavior. In contrast, PNXL estimation may exhibit convergence issues or flat likelihood surfaces due to the interaction between β and γ.
- **Empirical performance.** In several real-data applications, including the engineering datasets analyzed in this study, SNXL attains smaller AIC and BIC values than PNXL-type models, suggesting that the improvement in fit from adding extra parameters is not always sufficient to offset the increase in complexity.

8.3. Summary of Differences

Table 8. Comparison between SNXL and PNXL distributions.

Feature	SNXL	PNXL
Transformation	Square ($Y^{1/2}$)	Power ($Y^{1/\gamma}$)
Number of parameters	1	2
Model flexibility	Moderate	High
Analytical complexity	Low	Moderate to high
Hazard rate shape	Increasing (IFR)	Increasing / decreasing / bathtub
MLE stability	High	Moderate
Risk of overfitting	Low	Higher for small samples
Best suited for	Monotone lifetime data	Complex failure mechanisms

The SNXL distribution provides a parsimonious and analytically convenient alternative to more complex power-transformed models such as PNXL. It is particularly attractive when the hazard rate is expected to increase with time, as is common in reliability and survival studies. Although PNXL remains useful for datasets exhibiting non-monotonic hazard behavior, the results of this study indicate that SNXL achieves competitive or superior performance in many practical situations while maintaining simpler inference and more stable estimation.

9. Conclusion and Perspectives

In this paper, a new one-parameter lifetime model, called the Square New XLindley (SNXL) distribution, has been proposed as an alternative to the New XLindley (NXL) distribution and several related Lindley-type models. The SNXL distribution is obtained via a square transformation of the NXL model and possesses a unimodal density function with a strictly increasing hazard rate, which makes it suitable for modeling aging and wear-out phenomena commonly encountered in reliability and survival analysis.

Several structural and reliability properties of the SNXL distribution were derived, including moments, quantile function, stochastic ordering, incomplete moments, actuarial measures, and fuzzy reliability. Parameter estimation was investigated using maximum likelihood estimation (MLE), maximum product of spacings estimation (MPSE), and weighted least squares estimation (WLSE). A Monte Carlo simulation study demonstrated that all estimators are consistent, with bias, mean squared error, and mean relative error decreasing as the sample size increases, and with MLE and MPSE generally showing superior performance for moderate and large samples.

The practical performance of the SNXL distribution was evaluated using real datasets and compared with several competing lifetime models. For the carbon fiber strength data, which can be reasonably treated as independent and identically distributed, the SNXL model achieved the smallest values of AIC, BIC, and AICC, as well as favorable goodness-of-fit test results, indicating a strong agreement with the empirical distribution. For the hormone data, which exhibit potential temporal dependence, the results suggest that caution is required when applying standard lifetime models, emphasizing the need for models that explicitly account for dependence.

Several directions for future research naturally arise from this study. First, multi-parameter generalizations of the SNXL distribution could be developed to accommodate non-monotonic hazard rate shapes while preserving analytical tractability. Second, regression and accelerated failure-time models based on the SNXL distribution may be constructed to incorporate covariate information in survival and reliability studies. Third, Bayesian inference procedures and robust estimation techniques may be explored to improve parameter estimation under small samples or model misspecification. Finally, extensions of the SNXL model to dependent data structures, such as time-series or frailty-based survival models, would significantly broaden its applicability in biomedical and environmental applications.

Overall, the SNXL distribution represents a useful and parsimonious addition to the family of Lindley-type lifetime models and provides a solid foundation for further theoretical developments and applied investigations.

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