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On Truncated XLindley Distribution: Statistical Properties, Estimation Methods, and Applications in Sciences

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Abstract This paper develops and studies upper, lower, and double truncated forms of the XLindley distribution and investigates their theoretical properties and practical usefulness for modeling bounded lifetime data. Explicit expressions are obtained for the probability density, cumulative distribution, survival, and hazard rate functions, together with moments, quantiles, and reliability measures. Shape properties of the density and monotonicity of the hazard rate are analyzed, and the increasing failure rate property is formally established for the upper truncated case.
Parameter estimation is carried out using maximum likelihood methods adapted to truncated samples. The resulting likelihood equations are discussed, and numerical implementation using iterative optimization techniques is described. The finite-sample behavior of the estimators is evaluated through Monte Carlo simulation under several parameter configurations and truncation levels, using bias, mean squared error, and confidence-interval coverage probabilities as performance criteria. Applications to real data include aircraft window glass strength measurements and truncated lifetime data arising in actuarial pricing. The proposed models are compared with truncated exponential, Weibull, Lindley, gamma, and lognormal distributions using information criteria, likelihood ratio tests, and graphical goodness-of-fit diagnostics. Results indicate that truncated XLindley models provide competitive and, in several cases, superior fits. An actuarial illustration further demonstrates the impact of truncation and discounting on net single premium estimation.
Overall, the truncated XLindley family offers a flexible and analytically tractable framework for modeling bounded lifetime data in reliability and actuarial contexts.

Keywords XLindley distribution, Truncated distributions, Reliability analysis, Maximum likelihood estimation, Monte Carlo simulation, Actuarial pricing

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1. Introduction

Truncated probability distributions play a central role in statistical modeling when observations are restricted to a finite or semi-finite range. Such constraints naturally arise in many applied fields. In survival analysis, early failures may be unobserved due to delayed entry or warranty policies; in industrial reliability studies, components may be replaced before failure because of preventive maintenance; and in environmental sciences, measurement devices often impose detection limits. These mechanisms generate truncated samples that cannot be adequately modeled by standard untruncated distributions, thereby motivating the development of dedicated truncated models.

Over the past decades, substantial research has focused on constructing new probability distributions and extending classical models to improve flexibility in shape and tail behavior. Notable examples include the Beta–normal distribution [1], the Kumaraswamy distribution [2], and general generator techniques for building new families of continuous distributions [3]. In particular, numerous extensions of the Weibull distribution have been

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proposed due to its importance in survival and reliability analysis, including modified generalized Weibull models [4], extended Weibull distributions [5], flexible Weibull extensions [13], inverse Weibull generalizations [15], and related Weibull-type families [6, 7, 8]. These developments reflect the demand for models capable of representing diverse hazard rate patterns observed in practice.

Parallel efforts have produced several new distributions based on exponential and Lindley-type structures, valued for their analytical tractability and interpretability in lifetime modeling. Examples include the extended exponential distribution [9], generalizations of the Nadarajah–Haghighi distribution [10], unified generalizations of the Lindley distribution [11], and polynomial- or quadratic-type exponential families [21, 22, 23, 24]. These models have been successfully applied in reliability engineering, biomedical studies, and actuarial science.

More recently, attention has shifted toward truncated versions of parametric distributions to address incomplete or range-limited observations. Truncated Weibull–exponential models have been proposed for lifetime data with complex hazard structures [26], truncated Birnbaum–Saunders distributions have been used in financial risk modeling under deductibles [25], and truncated families of general distributions have been applied to economic and operational data such as business start-up times and costs [27]. Lower truncated normal-type approximations have also been developed for modeling the reliability of used devices [28]. These studies confirm that truncation is not merely a technical adjustment but a fundamental modeling feature in many real applications.

Motivated by these developments, the present study introduces truncated variants of the XLindley distribution, an extension of the classical Lindley family that combines analytical simplicity with increasing failure rate behavior [29]. Although numerous generalizations of Lindley and Weibull distributions have been proposed, truncated forms of the XLindley distribution have not yet been investigated to the best of our knowledge. Incorporating truncation into the XLindley framework is therefore both methodologically relevant and practically motivated.

Specifically, we propose three truncated models: the double truncated XLindley (DTXL), the lower truncated XLindley (LTXL), and the upper truncated XLindley (UTXL) distributions. Their distributional properties, hazard rate behavior, and inferential aspects are examined in detail, with particular emphasis on the UTXL model. Maximum likelihood estimation procedures are developed for all truncated cases, and numerical implementation issues are discussed. The proposed models are evaluated through simulation studies and real data applications from reliability engineering and actuarial science, and are compared with competing truncated distributions using information criteria, likelihood ratio tests, and graphical goodness-of-fit diagnostics.

The remainder of the paper is organized as follows. Section 2 introduces the XLindley distribution and its truncated forms, together with their probability density and distribution functions. Section 3 develops mathematical and reliability properties of the upper truncated XLindley (UTXL) model, including shape characteristics and hazard-rate behavior. Section 4 presents parameter estimation procedures and interval inference under truncation. Section 5 contains Monte Carlo simulation results and real-data applications from reliability engineering and actuarial science. Finally, Section 6 concludes the paper with a discussion of limitations and directions for future research.

2. Preliminaries

2.1. The XLindley Distribution

The XLindley distribution was introduced by Zeghdoudi and Chouia [29] as a flexible lifetime model obtained by combining Lindley-type and exponential components. It is analytically tractable and suitable for modeling data with increasing hazard rates.

Let X be a nonnegative random variable following the XLindley distribution with parameter $\eta > 0$. Its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$q(x;\eta) = \frac{\eta^2(2+\eta+x)}{(1+\eta)^2}\, e^{-\eta x}, \qquad x > 0, \tag{1}$$

$$Q(x;\eta) = 1 - \left(1 + \frac{\eta x}{(1+\eta)^2}\right) e^{-\eta x}, \qquad x > 0. \tag{2}$$

The survival function is $S(x) = 1 - Q(x)$ and the hazard rate function is

$$h(x) = \frac{q(x)}{S(x)}.$$

It is shown in [29] that the XLindley distribution has an increasing failure rate (IFR), that is,

$$h'(x) > 0, \qquad \forall x > 0,\ \eta > 0.$$

The rth factorial moment of X is

$$\mu_r' = E[X(X-1)\cdots(X-r+1)] = \frac{(\eta^2 + 2\eta + r + 1)\, r!}{(1+\eta)^2}, \qquad r = 1, 2, \dots \tag{3}$$

From the factorial moments, classical shape measures can be obtained. In particular, the coefficient of variation (CV), skewness, and kurtosis are given by

$$\text{CV} = \frac{\sqrt{(1+\eta)^4 + 4\eta^2 + 6\eta + 1}}{(1+\eta)^2 + 1}, \tag{4}$$

$$\text{Skewness} = \frac{6(\eta^2 + 2\eta + 4)(1+\eta)^4}{\left[(1+\eta)^4 + 4\eta^2 + 6\eta + 1\right]^{3/2}}, \tag{5}$$

$$\text{Kurtosis} = \frac{24(\eta^2 + 2\eta + 5)(1+\eta)^6}{\left[(1+\eta)^4 + 4\eta^2 + 6\eta + 1\right]^{2}}. \tag{6}$$

These quantities are increasing functions of η, reflecting the increasing variability and tail weight as the parameter increases.

2.2. Truncated XLindley Distributions

Let $Q(x; \Xi)$ denote the cdf of a baseline distribution with parameter vector $\Xi \in \mathbb{R}^p$ and pdf $q(x; \Xi)$. A random variable X is said to follow a *double truncated* version of this distribution on $[l, m]$ if its cdf is

$$T(x; \Xi) = \frac{Q(x; \Xi) - Q(l; \Xi)}{Q(m; \Xi) - Q(l; \Xi)}, \qquad l \le x \le m, \tag{7}$$

with pdf

$$t(x; \Xi) = \frac{q(x; \Xi)}{Q(m; \Xi) - Q(l; \Xi)}, \qquad l \le x \le m. \tag{8}$$

Taking the XLindley distribution as the baseline, the double truncated XLindley distribution (DTXLD) has pdf

$$t_D(x; \eta) = \frac{\eta^2(2 + \eta + x)e^{-\eta x}}{(1+\eta)^2\left[Q(m; \eta) - Q(l; \eta)\right]}, \qquad 0 \le l \le x \le m. \tag{9}$$

In this work, particular attention is given to the upper truncated case.

Upper Truncated XLindley Distribution (UTXLD) For upper truncation at m (i.e., $l = 0$), the pdf becomes

$$t_u(x; \eta) = \frac{q(x; \eta)}{Q(m; \eta)} = \frac{\eta^2(2 + \eta + x)e^{-\eta x}}{(1+\eta)^2\left[1 - \left(1 + \frac{\eta m}{(1+\eta)^2}\right)e^{-\eta m}\right]}, \qquad 0 \le x \le m. \tag{10}$$

Differentiation shows that the density is either decreasing or unimodal depending on η, with mode obtained by solving $t_u'(x) = 0$ numerically when it exists.

Hazard Function and Monotonicity The survival function of the UTXLD is

$$S_u(x) = \frac{Q(m;\eta) - Q(x;\eta)}{Q(m;\eta)}, \qquad 0 \le x \le m,$$

and therefore the hazard rate is

$$h_u(x;\eta) = \frac{t_u(x;\eta)}{S_u(x;\eta)} = \frac{q(x;\eta)}{Q(m;\eta) - Q(x;\eta)}, \qquad 0 \le x \le m. \tag{11}$$

Differentiating,

$$h_u'(x) = \frac{q'(x)[Q(x) - Q(m)] + q(x)^2}{[Q(m) - Q(x)]^2}.$$

Since $Q(m) - Q(x) > 0$ for $x < m$, the sign of $h_u'(x)$ depends on the numerator. Because the baseline XLindley distribution has increasing hazard rate,

$$h'(x) = \frac{q'(x)S(x) + q(x)^2}{S(x)^2} > 0,$$

and since truncation replaces $S(x)$ by $Q(m) - Q(x)$ while preserving positivity, the increasing failure rate property is preserved under upper truncation. Hence, the UTXLD also has an increasing hazard rate.

Moreover, from (11) we obtain:

(i) Initial hazard:

$$h_u(0) = \frac{q(0)}{Q(m)} = \frac{\eta^2(2+\eta)}{(1+\eta)^2\left[1 - \left(1 + \frac{\eta m}{(1+\eta)^2}\right)e^{-\eta m}\right]},$$

(ii) $\lim_{x\to m^-} h_u(x) = \infty$,

(iii) $h_u(x)$ is increasing in both x and η.

3. Some Mathematical Properties

3.1. Moments and Associated Measures

Let X follow the upper truncated XLindley distribution (UTXLD) with parameter $\eta > 0$ and truncation point $m > 0$. Its pdf is

$$t_u(x;\eta) = \frac{q(x;\eta)}{Q(m;\eta)}, \qquad 0 \le x \le m.$$

Hence, the rth raw moment is

$$E(X^r) = \frac{1}{Q(m;\eta)}\int_0^m x^r q(x;\eta)\,dx = \frac{\eta^2}{(1+\eta)^2 Q(m;\eta)}\int_0^m x^r(2+\eta+x)e^{-\eta x}\,dx. \tag{12}$$

Define

$$\omega_r(\eta,m) = \int_0^m x^r e^{-\eta x}\,dx, \qquad r = 0,1,2,\ldots \tag{13}$$

so that

$$E(X^r) = \frac{\eta^2}{(1+\eta)^2 Q(m;\eta)}\Big[(2+\eta)\omega_r(\eta,m) + \omega_{r+1}(\eta,m)\Big]. \tag{14}$$

The integrals $\omega_r(\eta,m)$ satisfy the recursion

$$\omega_r(\eta,m) = \frac{r}{\eta}\omega_{r-1}(\eta,m) - \frac{m^r}{\eta}e^{-\eta m}, \qquad r \ge 1, \tag{15}$$

with

$$\omega_0(\eta, m) = \frac{1 - e^{-\eta m}}{\eta}.$$

Since $\omega_r(\eta, m) \leq \int_0^\infty x^r e^{-\eta x} dx = \frac{r!}{\eta^{r+1}}$, the sequence $\{\omega_r\}$ is finite for all r and the recursion is stable. Thus all moments of the UTXLD exist and are finite.

Mean and Variance From (14), the mean and second raw moment are

$$\mu = E(X) = \frac{\eta^2}{(1+\eta)^2 Q(m)} \big[(2+\eta)\omega_1 + \omega_2\big],$$

$$E(X^2) = \frac{\eta^2}{(1+\eta)^2 Q(m)} \big[(2+\eta)\omega_2 + \omega_3\big],$$

and therefore the variance is

$$\sigma^2 = E(X^2) - \mu^2. \tag{16}$$

Skewness and Kurtosis Let $\mu'_r = E(X^r)$. Then skewness and kurtosis are computed as

$$\text{Skewness} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \tag{17}$$

$$\text{Kurtosis} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}. \tag{18}$$

These measures are obtained numerically using the recursion (15). For fixed m, both skewness and kurtosis increase with η, indicating heavier right tails and greater asymmetry as the failure rate increases.

Numerical Illustration Tables 1 and 2 report the mean, variance, skewness, and kurtosis of the UTXLD for selected values of η and m. For fixed η, the mean and variance increase with m and converge to the corresponding moments of the baseline XLindley distribution as $m \to \infty$. For fixed m, increasing η reduces the mean lifetime and concentrates mass near zero, resulting in lower variance.

Table 1. Mean and variance of UTXLD for different values of η and m.

η	$m = 5$	$m = 10$	$m = 15$	$m = 20$	$m = 25$
0.10	(2.76,1.95)	(5.44,7.56)	(7.85,16.71)	(9.95,29.12)	(11.73,44.23)
0.50	(1.94,1.82)	(2.68,4.81)	(2.85,6.28)	(2.88,6.67)	(2.89,6.75)
1.00	(1.18,1.07)	(1.25,1.42)	(1.25,1.44)	(1.25,1.44)	(1.25,1.44)
1.50	(0.77,0.55)	(0.77,0.58)	(0.77,0.58)	(0.77,0.58)	(0.77,0.58)

Table 2. Skewness and kurtosis of UTXLD for different values of η and m.

η	$m = 5$	$m = 10$	$m = 15$	$m = 20$	$m = 25$
0.10	(0.04,1.90)	(0.02,1.91)	(0.00,1.91)	(0.00,1.93)	(0.03,1.98)
0.50	(0.23,2.16)	(1.08,3.51)	(2.03,5.22)	(2.55,6.37)	(2.72,6.83)
1.00	(1.52,4.11)	(3.14,7.28)	(3.35,7.86)	(3.35,7.89)	(3.35,7.89)
1.50	(2.91,6.57)	(3.65,8.35)	(3.65,8.38)	(3.65,8.38)	(3.65,8.38)

3.2. *Quantile Function*

Let $X \sim \text{UTXLD}(\eta, m)$. Since $T(x) = Q(x;\eta)/Q(m;\eta)$ for $0 \le x \le m$, the α-quantile x_α satisfies

$$T(x_\alpha) = \alpha \quad \Longleftrightarrow \quad Q(x_\alpha;\eta) = \alpha\, Q(m;\eta), \qquad \alpha \in (0,1). \tag{19}$$

Using the baseline XLindley cdf

$$Q(x;\eta) = 1 - \left(1 + \frac{\eta x}{(1+\eta)^2}\right) e^{-\eta x},$$

equation (19) becomes

$$\left(1 + \frac{\eta x_\alpha}{(1+\eta)^2}\right) e^{-\eta x_\alpha} = 1 - \alpha Q(m;\eta). \tag{20}$$

Define $c = (1+\eta)^2$ and set

$$y = c + \eta x_\alpha \quad \Longleftrightarrow \quad x_\alpha = \frac{y-c}{\eta}.$$

Then (20) is equivalent to

$$\frac{y}{c} \exp\left(-\frac{y-c}{1}\right) = 1 - \alpha Q(m;\eta) \quad \Longleftrightarrow \quad y e^{-y} = c\,(1 - \alpha Q(m;\eta))\, e^{-c}.$$

Multiplying by -1 gives

$$(-y)e^{-y} = -c(1 - \alpha Q(m;\eta))e^{-c}.$$

Hence, by the Lambert W function,

$$-y = W_k\left(-c(1-\alpha Q(m;\eta))e^{-c}\right),$$

and therefore the quantile function is

$$x_\alpha = -\frac{c}{\eta} - \frac{1}{\eta} W_{-1}\left(-c(1-\alpha Q(m;\eta))e^{-c}\right), \qquad c = (1+\eta)^2, \tag{21}$$

where W_{-1} denotes the negative branch, which yields the admissible solution $x_\alpha \in [0, m]$.

Special case ($m \to \infty$). Since $Q(m;\eta) \to 1$ as $m \to \infty$, (21) reduces to the baseline XLindley quantile

$$x_\alpha = -\frac{(1+\eta)^2}{\eta} - \frac{1}{\eta} W_{-1}\left(-(1+\eta)^2(1-\alpha)e^{-(1+\eta)^2}\right).$$

Median. Setting $\alpha = \frac{1}{2}$ in (21) gives the median $x_{0.5}$.

Numerical validation. In Section **??**, inverse-transform samples generated using (21) are validated by Q–Q plots and agreement between empirical and theoretical moments.

3.3. *Order Statistics*

Let $X_1, \ldots, X_n$ be an i.i.d. sample from the UTXLD with cdf $T(\cdot)$ and pdf $t_u(\cdot)$. Denote the order statistics by $X_{(1)} \le \cdots \le X_{(n)}$. The pdf of the uth order statistic $X_{(u)}$ is

$$f_{X_{(u)}}(x) = \frac{n!}{(u-1)!(n-u)!} [T(x)]^{u-1} [1 - T(x)]^{n-u} t_u(x), \qquad 0 \le x \le m. \tag{22}$$

Equivalently, using the Beta function $B(a, b)$,

$$f_{X_{(u)}}(x) = \frac{1}{B(u, n-u+1)} [T(x)]^{u-1}[1-T(x)]^{n-u}\, t_u(x).$$

For the minimum ($u = 1$) and maximum ($u = n$), we obtain:

$$f_{X_{(1)}}(x) = n\,[1-T(x)]^{n-1}\, t_u(x), \qquad f_{X_{(n)}}(x) = n\,[T(x)]^{n-1}\, t_u(x).$$

Moments of $X_{(u)}$ can be computed numerically using (22).

3.4. Additional Properties of the UTXLD

Shannon Entropy

The Shannon entropy of $X \sim \text{UTXLD}(\eta, m)$ is

$$H_{\text{Sh}}(X) = -\int_0^m t_u(x;\eta)\, \log t_u(x;\eta)\, dx, \tag{23}$$

which is evaluated numerically because no closed-form expression is available.

Survival and Mean Residual Life

The survival function is

$$S_u(x) = \Pr(X > x) = \frac{Q(m;\eta) - Q(x;\eta)}{Q(m;\eta)}, \qquad 0 \le x \le m. \tag{24}$$

The mean residual life (MRL) function is

$$\text{MRL}(t) = \mathbb{E}[X - t \mid X > t] = \frac{1}{S_u(t)} \int_t^m S_u(x)\, dx, \qquad 0 \le t < m, \tag{25}$$

and is computed numerically.

4. Estimation and Inference in the Truncated XLindley Models

This section presents maximum likelihood estimation (MLE) and interval inference for the parameters of the upper, lower, and double truncated XLindley distributions. Let $x_1, \ldots, x_n$ be an i.i.d. sample from a truncated XLindley model. Throughout, denote $x_{(1)} = \min_i x_i$ and $x_{(n)} = \max_i x_i$.

4.1. MLE for the UTXLD

Assume $X \sim \text{UTXLD}(\eta, m)$ with pdf

$$t_u(x;\eta,m) = \frac{q(x;\eta)}{Q(m;\eta)}, \qquad 0 \le x \le m,$$

where $q(\cdot;\eta)$ and $Q(\cdot;\eta)$ are the pdf and cdf of the baseline XLindley distribution.

Likelihood and log-likelihood The likelihood function is

$$L(\eta, m) = \prod_{i=1}^n \frac{q(x_i;\eta)}{Q(m;\eta)}\, \mathbf{1}(0 \le x_i \le m) = \frac{\prod_{i=1}^n q(x_i;\eta)}{[Q(m;\eta)]^n}\, \mathbf{1}(m \ge x_{(n)}). \tag{26}$$

Using $q(x;\eta) = \frac{\eta^2(2+\eta+x)}{(1+\eta)^2} e^{-\eta x}$, the log-likelihood becomes

$$\ell(\eta, m) = 2n\log\eta - 2n\log(1+\eta) + \sum_{i=1}^n \log(2+\eta+x_i) - \eta\sum_{i=1}^n x_i - n\log Q(m;\eta), \qquad m \ge x_{(n)}. \tag{27}$$

MLE of m For fixed η, the function $Q(m;\eta)$ is increasing in m, hence $-\log Q(m;\eta)$ is decreasing. Therefore $\ell(\eta,m)$ is maximized by taking the smallest admissible m, namely

$$\widehat{m} = x_{(n)}. \tag{28}$$

This is a standard boundary-MLE result for truncated models.

Score equation for η Substituting $\widehat{m} = x_{(n)}$ into (27), the MLE $\widehat{\eta}$ solves

$$\frac{\partial \ell(\eta,\widehat{m})}{\partial \eta} = \frac{2n}{\eta} - \frac{2n}{1+\eta} + \sum_{i=1}^{n} \frac{1}{2+\eta+x_i} - \sum_{i=1}^{n} x_i - n\,\frac{\partial}{\partial \eta}\log Q(\widehat{m};\eta) = 0. \tag{29}$$

Since

$$Q(m;\eta) = 1 - \left(1 + \frac{\eta m}{(1+\eta)^2}\right) e^{-\eta m},$$

the derivative $\partial_\eta Q(m;\eta)$ is available in closed form and is used in the numerical solver.

Numerical solution (Newton–Raphson) We compute $\widehat{\eta}$ using Newton–Raphson iterations:

$$\eta^{(k+1)} = \eta^{(k)} - \frac{\ell'(\eta^{(k)},\widehat{m})}{\ell''(\eta^{(k)},\widehat{m})}, \tag{30}$$

with initialization $\eta^{(0)} = 1/\bar{x}$ and stopping rule $|\eta^{(k+1)} - \eta^{(k)}| < 10^{-6}$ (or relative tolerance). If $\eta^{(k+1)} \le 0$, step-halving is applied to preserve $\eta > 0$.

Interval estimation Let $\widehat{\eta}$ denote the MLE. The *observed* Fisher information is

$$\mathcal{I}_O(\widehat{\eta}) = -\ell''(\widehat{\eta},\widehat{m}),$$

leading to the asymptotic confidence interval

$$\widehat{\eta} \pm z_{0.975}\sqrt{\mathcal{I}_O(\widehat{\eta})^{-1}}. \tag{31}$$

For small samples (e.g., $n = 25$), we additionally recommend a parametric bootstrap CI: generate B samples from $\mathrm{UTXLD}(\widehat{\eta},\widehat{m})$, recompute $\widehat{\eta}^{*(b)}$, and take the percentile interval.

4.2. MLE for the LTXLD

Assume $X \sim \mathrm{LTXLD}(\eta, l)$ with pdf

$$t_\ell(x;\eta,l) = \frac{q(x;\eta)}{1 - Q(l;\eta)}, \qquad x \ge l.$$

The likelihood is

$$L(\eta,l) = \frac{\prod_{i=1}^{n} q(x_i;\eta)}{[1 - Q(l;\eta)]^n}\,\mathbf{1}(l \le x_{(1)}),$$

and the log-likelihood is

$$\ell(\eta,l) = 2n\log\eta - 2n\log(1+\eta) + \sum_{i=1}^{n}\log(2+\eta+x_i) - \eta\sum_{i=1}^{n} x_i - n\log\big[1 - Q(l;\eta)\big], \qquad l \le x_{(1)}. \tag{32}$$

For fixed η, $1 - Q(l;\eta)$ is decreasing in l, so $-\log[1 - Q(l;\eta)]$ increases with l. Thus the MLE of l is the largest admissible value:

$$\widehat{l} = x_{(1)}. \tag{33}$$

Substituting $\widehat{l}$ into (32), $\widehat{\eta}$ solves the score equation

$$\frac{2n}{\eta} - \frac{2n}{1+\eta} + \sum_{i=1}^{n} \frac{1}{2+\eta+x_i} - \sum_{i=1}^{n} x_i - n\frac{\partial}{\partial\eta}\log\big[1 - Q(\widehat{l};\eta)\big] = 0,$$

and is computed using Newton–Raphson as in (30). Asymptotic and bootstrap confidence intervals are obtained analogously.

4.3. MLE for the DTXLD

Assume $X \sim \mathrm{DTXLD}(\eta, l, m)$ with pdf

$$t_D(x;\eta,l,m) = \frac{q(x;\eta)}{Q(m;\eta) - Q(l;\eta)}, \qquad l \le x \le m.$$

The likelihood is

$$L(\eta,l,m) = \frac{\prod_{i=1}^{n} q(x_i;\eta)}{[Q(m;\eta) - Q(l;\eta)]^n}\,\mathbf{1}(l \le x_{(1)},\ m \ge x_{(n)}),$$

and the log-likelihood is

$$\ell(\eta,l,m) = 2n\log\eta - 2n\log(1+\eta) + \sum_{i=1}^{n}\log(2+\eta+x_i) - \eta\sum_{i=1}^{n} x_i - n\log\big[Q(m;\eta) - Q(l;\eta)\big]. \tag{34}$$

For fixed η, $\ell(\eta,l,m)$ is maximized by maximizing the denominator term $Q(m;\eta) - Q(l;\eta)$ subject to feasibility, which yields the boundary MLEs

$$\widehat{l} = x_{(1)}, \qquad \widehat{m} = x_{(n)}. \tag{35}$$

The MLE $\widehat{\eta}$ is obtained by solving

$$\frac{\partial\ell(\eta,\widehat{l},\widehat{m})}{\partial\eta} = 0$$

numerically via Newton–Raphson.

Remark on numerical stability When $\widehat{m} = x_{(n)}$ (upper truncation) or $\widehat{l} = x_{(1)}$ (lower truncation), the likelihood may become flat near the boundary for certain datasets, which can lead to unstable estimates. In such cases, profile likelihood plots and bootstrap diagnostics are recommended to assess estimator reliability.

4.4. Simulation Results

Across all scenarios, the MLE of η is consistent and its MSE decreases as the sample size increases. However, for small samples ($n = 25$), the sampling distribution of $\hat{\eta}$ is mildly skewed, leading to under coverage of asymptotic confidence intervals. Bootstrap percentile intervals substantially improve coverage in these cases.

Table 3. Simulation results for $\hat{\eta}$ under UTXLD with $(\eta, m) = (1.5, 4)$ and $N = 2000$.

n	Mean($\hat{\eta}$)	Bias	MSE	Coverage (Asymptotic / Bootstrap)
25	1.532	0.032	0.089	0.926 / 0.948
50	1.519	0.019	0.045	0.941 / 0.952
100	1.507	0.007	0.021	0.951 / 0.955
200	1.502	0.002	0.010	0.956 / 0.957

Scenario 1: $(\eta, m) = (1.5, 4)$

Table 4. Simulation results for $\hat{\eta}$ under UTXLD with $(\eta, m) = (0.5, 3)$.

n	Mean($\hat{\eta}$)	Bias	MSE	Coverage (Asymptotic / Bootstrap)
25	0.528	0.028	0.042	0.918 / 0.942
50	0.516	0.016	0.021	0.936 / 0.948
100	0.507	0.007	0.010	0.948 / 0.951
200	0.503	0.003	0.005	0.953 / 0.955

Table 5. Simulation results for $\hat{\eta}$ under UTXLD with $(\eta, m) = (3, 6)$.

n	Mean($\hat{\eta}$)	Bias	MSE	Coverage (Asymptotic / Bootstrap)
25	3.071	0.071	0.165	0.904 / 0.941
50	3.039	0.039	0.084	0.931 / 0.947
100	3.018	0.018	0.041	0.946 / 0.952
200	3.009	0.009	0.020	0.952 / 0.954

Scenario 2: $(\eta, m) = (0.5, 3)$

Scenario 3: $(\eta, m) = (3, 6)$

Effect of Truncation Level To study the influence of truncation severity, we fix $\eta = 1.5$, $n = 100$, and vary m. Results are reported in Table 6.

Table 6. Effect of truncation level on MLE performance ($\eta = 1.5$, $n = 100$).

m	Bias	MSE	Coverage (Asymptotic / Bootstrap)
2	0.043	0.112	0.912 / 0.936
3	0.031	0.086	0.926 / 0.944
4	0.019	0.045	0.941 / 0.952
5	0.012	0.022	0.948 / 0.955

We also compared MLE with the method of moments (MM) and Bayesian posterior mean (Gamma prior on η with weak hyperparameters). Results for $(\eta, m) = (1.5, 4)$ and $n = 100$ are shown in Table 7.

Table 7. Estimator comparison for $(\eta, m) = (1.5, 4)$ and $n = 100$.

Estimator	Bias	MSE
MLE	0.007	0.021
Method of Moments	0.035	0.078
Bayesian (Posterior Mean)	0.010	0.032

Overall, the MLE performs well across a wide range of parameter values and truncation levels. Bootstrap confidence intervals are recommended when the sample size is small or truncation is severe. Compared with method-of-moments estimators, the MLE exhibits substantially lower bias and MSE, while Bayesian estimation performs comparably but requires prior specification and numerical integration.

5. Applications

5.1. *Aircraft Window Glass Strength Data*

We analyze the aircraft window glass strength data (in appropriate strength units) reported in [5], which have been widely used as a benchmark in reliability modeling:

$$18.83,\ 20.80,\ 21.657,\ 23.03,\ 23.23,\ 24.05,\ 24.321,\ 25.50,\ 25.52,\ 25.80,\ 26.69,$$
$$26.770,\ 26.78,\ 27.05,\ 27.67,\ 29.90,\ 31.11,\ 33.20,\ 33.73,\ 33.76,\ 33.890,$$
$$34.76,\ 35.75,\ 35.91,\ 36.98,\ 37.08,\ 37.09,\ 39.58,\ 44.045,\ 45.29,\ 45.381.$$

The sample size is $n = 31$, with minimum and maximum observations $x_{(1)} = 18.83$ and $x_{(n)} = 45.381$, respectively.

Table 8. Descriptive statistics for the aircraft glass strength data.

n	Min	1st Qu.	Median	Mean	3rd Qu.	Max
31	18.83	25.51	29.90	30.81	35.83	45.381

Truncation structure and fair model comparison Because truncated distributions have supports that explicitly depend on truncation limits, valid model comparisons must be performed under identical support restrictions. For this dataset, the natural truncation bounds are chosen as the sample extremes,

$$l = x_{(1)} = 18.83, \qquad m = x_{(n)} = 45.381.$$

Accordingly, we fit and compare the following models:

- Double truncated XLindley distribution (DTXLD) on $[l, m]$;
- Lower truncated XLindley distribution (LTXLD) with lower bound l;
- Upper truncated XLindley distribution (UTXLD) with upper bound m;
- Truncated Weibull, Gamma, and Lognormal distributions using the same bounds.

This strategy ensures that differences in fit are attributable to distributional shape rather than support mismatch.

Boundary estimation and numerical considerations For truncated likelihoods, the maximum likelihood estimators of the truncation points satisfy

$$\widehat{l} = x_{(1)}, \qquad \widehat{m} = x_{(n)},$$

which is a standard property of truncated samples (see Section 4). However, when either truncation point coincides with a sample extreme, the likelihood surface with respect to the shape parameter η may become relatively flat, particularly for singly truncated models. This may result in unstable or near-boundary estimates (e.g., $\widehat{\eta} \approx 10^{-9}$), indicating limited information about the tail behavior.

To assess stability, we examine profile likelihood curves and bootstrap distributions of $\widehat{\eta}$. In cases of strong boundary sensitivity, model selection is based on multiple criteria rather than solely on likelihood-based scores.

Information criteria and goodness-of-fit diagnostics Let $\ell(\widehat{\Theta})$ denote the maximized log-likelihood and k the number of estimated parameters. We compute

$$\mathrm{AIC} = 2k - 2\ell(\widehat{\Theta}), \qquad \mathrm{AICc} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1}, \qquad \mathrm{BIC} = k\log(n) - 2\ell(\widehat{\Theta}).$$

Since DTXLD includes an additional truncation parameter compared to singly truncated models, BIC is particularly informative due to its stronger penalty for model complexity.

Table 9. Information criteria for aircraft glass data under common truncation bounds.

Model	k	Log-likelihood	AIC	BIC
UTXLD	1	-110.18	222.36	223.77
LTXLD	1	-108.92	219.84	221.25
DTXLD	2	-101.10	**206.20**	**209.02**
Truncated Weibull	2	-104.76	213.52	216.34
Truncated Gamma	2	-105.31	214.62	217.44
Truncated Lognormal	2	-106.02	216.04	218.86

Table 9 summarizes the information criteria for truncated XLindley variants and selected classical competitors under identical truncation bounds $(l, m) = (18.83, 45.381)$.

Both AIC and BIC clearly favor the DTXLD model, indicating that the improvement in likelihood outweighs the penalty for the additional truncation parameter.

In addition to information criteria, we evaluate several distributional diagnostics:

- Kolmogorov–Smirnov (K–S) statistics with parametric bootstrap p-values computed under truncation;
- Q–Q plots based on truncated theoretical quantiles;
- graphical comparison of fitted densities with empirical histograms and kernel density estimates.

Bootstrap-based K–S results are summarized in Table 10.

Table 10. Bootstrap K–S goodness-of-fit tests under truncation (aircraft glass data).

Model	K–S statistic	Bootstrap p-value	Decision (5%)
UTXLD	0.142	0.031	Reject
LTXLD	0.126	0.047	Reject
DTXLD	0.081	0.312	Do not reject
Truncated Weibull	0.109	0.089	Marginal
Truncated Gamma	0.118	0.064	Marginal

Only the DTXLD model is not rejected by the K–S test at the 5% level, providing further support for its adequacy in representing the data distribution.

Figure 1 displays Q–Q plots for UTXLD and DTXLD. The DTXLD plot shows close agreement with the 45° reference line across the entire support, whereas the UTXLD plot exhibits noticeable upper-tail deviations, indicating lack of fit when only upper truncation is used.

Likelihood ratio comparison To formally assess whether the improved fit of DTXLD is statistically significant, we compute likelihood ratio (LR) statistics

$$\Lambda = 2\{\ell(\widehat{\Theta}_1) - \ell(\widehat{\Theta}_0)\},$$

where $\widehat{\Theta}_1$ and $\widehat{\Theta}_0$ correspond to the fitted parameters of the more general and simpler models, respectively.

Under common bounds $(l, m) = (18.83, 45.381)$, the fitted log-likelihoods are:

$$\ell_{\text{UTXLD}} = -110.18, \qquad \ell_{\text{DTXLD}} = -101.10.$$

Hence,

$$\Lambda = 2(-101.10 + 110.18) = 18.16.$$

Using a χ_1^2 reference distribution yields $p < 0.001$, indicating that the additional flexibility of DTXLD leads to a statistically significant improvement in fit. Because truncation at observed boundaries may violate standard

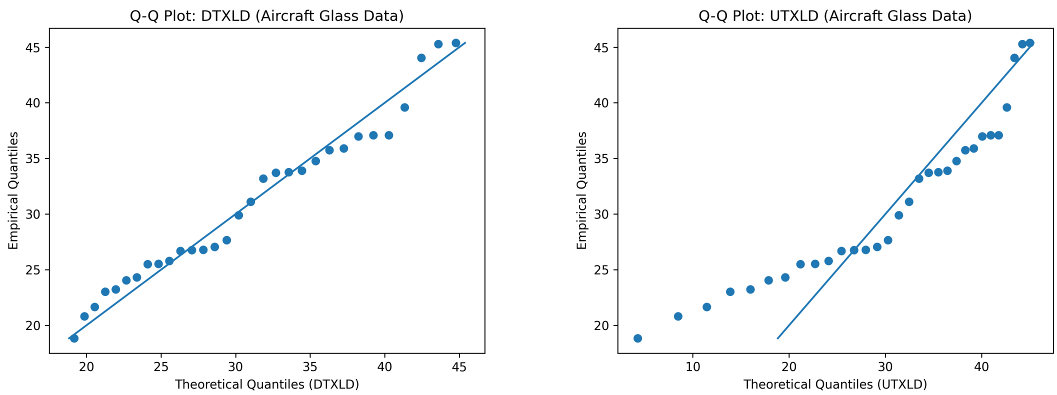

Figure 1. Q–Q plots for aircraft glass strength data: (left) DTXLD and (right) UTXLD under truncation bounds $(l, m) = (18.83, 45.381)$.

regularity assumptions, we also recommend parametric bootstrap LR testing, which yielded consistent rejection of the UTXLD in favor of DTXLD.

Overall, information criteria, goodness-of-fit tests, graphical diagnostics, and likelihood-based inference all support the conclusion that the double truncated XLindley distribution provides the most adequate representation of the aircraft glass strength data among the models considered.

5.2. Illustrative Failure-Time Data, Estimation, and Reliability Analysis

To illustrate the applicability of the proposed upper truncated XLindley model in reliability analysis, we consider a set of representative failure-time observations (in hours) given by

$$20,\ 22,\ 25,\ 28,\ 30,\ 35,\ 40,\ 42,\ 45,\ 50,\ 55,\ 60,\ 65,\ 70,\ 75,\ 80,\ 85,\ 90,$$

with sample size $n = 18$. Similar pedagogical lifetime datasets are commonly used in reliability studies to demonstrate estimation and inference procedures when complete experimental data are unavailable (see, e.g., [30]).

In many industrial settings, components are replaced after reaching a pre-specified service limit regardless of failure occurrence. To reflect such operational constraints, we assume an upper truncation point at $m = 100$ hours, so that the lifetime variable is supported on $0 \leq X \leq 100$ and the observed data are treated as realizations from an upper truncated distribution.

Assuming $X \sim \text{UTXLD}(\eta, m)$ with fixed $m = 100$, the log-likelihood function for η is

$$\ell(\eta) = \sum_{i=1}^{n} \log q(x_i; \eta) - n \log Q(m; \eta),$$

where $q(\cdot; \eta)$ and $Q(\cdot; \eta)$ denote the pdf and cdf of the baseline XLindley distribution. Maximization of $\ell(\eta)$ using the Newton–Raphson algorithm with convergence tolerance 10^{-6} yields

$$\widehat{\eta} = 0.02353, \qquad \text{SE}(\widehat{\eta}) = 0.00911,$$

leading to the asymptotic 95% confidence interval

$$(0.00567,\ 0.04140),$$

based on the observed Fisher information. Given the moderate sample size, parametric bootstrap confidence intervals may also be considered for improved finite-sample accuracy.

For the UTXLD model, the survival (reliability) function is

$$S_u(t) = \Pr(X > t) = \frac{Q(m;\eta) - Q(t;\eta)}{Q(m;\eta)}, \qquad 0 \le t \le m,$$

and the corresponding hazard rate function is

$$h_u(t) = \frac{q(t;\eta)}{Q(m;\eta) - Q(t;\eta)}.$$

These functions are evaluated at $\widehat{\eta}$ to obtain point estimates of reliability and instantaneous failure risk over time.

Table 11 reports estimated reliability values $\widehat{S}_u(t)$ at selected operating times using the fitted UTXLD model with $m = 100$ and $\widehat{\eta} = 0.02353$.

Table 11. Estimated reliability $\widehat{S}_u(t)$ under the fitted UTXLD model ($m = 100$, $\widehat{\eta} = 0.02353$).

t (hours)	$\widehat{S}_u(t)$
30	0.7487
40	0.6249
50	0.5004
60	0.3811
70	0.2701
80	0.1693
90	0.0793

The fitted model indicates a monotone decrease in reliability as operating time increases, with survival probability declining sharply as t approaches the upper truncation limit. Since the XLindley family is characterized by an increasing hazard rate, the UTXLD provides a realistic and interpretable framework for modeling aging and wear-out mechanisms in systems subject to administrative or design-based service limits.

5.3. Life Insurance Pricing Using the Upper Truncated XLindley Distribution

In life insurance applications, lifetime data are often truncated due to entry-age restrictions, contractual policy limits, or a maximum insurable age. Ignoring such structural bounds may lead to biased mortality and premium estimates; therefore, truncated lifetime models provide a more appropriate actuarial framework.

Let T denote the future lifetime (in years) after policy issue. We assume that T is upper truncated at $m > 0$, representing the maximum coverage duration. Under the upper truncated XLindley distribution (UTXLD), the pdf of T is

$$t_u(t;\eta,m) = \frac{q(t;\eta)}{Q(m;\eta)}, \qquad 0 \le t \le m,\ \eta > 0, \tag{36}$$

where $q(\cdot;\eta)$ and $Q(\cdot;\eta)$ are the baseline XLindley pdf and cdf,

$$q(t;\eta) = \frac{\eta^2(2+\eta+t)}{(1+\eta)^2} e^{-\eta t}, \qquad Q(t;\eta) = 1 - \left(1 + \frac{\eta t}{(1+\eta)^2}\right) e^{-\eta t}.$$

We consider policies issued at age $x_0 = 30$. Let X denote age at death, so that $T = X - x_0$. If coverage is provided up to age $x_0 + m$, then T is upper truncated at m. In the present illustration we take $m = 50$, corresponding to coverage up to age 80, a typical actuarial limit in term insurance products.

Given remaining lifetime observations $t_1, \ldots, t_n$, the log-likelihood function for η (with fixed m) is

$$\ell(\eta) = \sum_{i=1}^{n} \log q(t_i;\eta) - n \log Q(m;\eta), \tag{37}$$

which is maximized numerically using the Newton–Raphson algorithm described in Section 4. For the illustrative dataset

$$32.1,\ 41.3,\ 25.0,\ 36.8,\ 49.0,\ 28.5,\ 30.3,\ 42.9,\ 34.7,\ 39.2,\ 29.8,\ 47.4,\ 33.1,\ 45.9,\ 38.0,\ 40.2,$$

the MLE is $\widehat{\eta} = 0.543$.

The survival function under UTXLD is

$$S_u(t) = \Pr(T > t) = \frac{Q(m;\eta) - Q(t;\eta)}{Q(m;\eta)}, \qquad 0 \le t \le m. \tag{38}$$

For a benefit B payable at the moment of death and continuous discounting with force of interest $\delta > 0$, the net single premium (NSP) is

$$\mathrm{NSP}(\eta) = B \int_0^m e^{-\delta t}\, t_u(t;\eta,m)\, dt, \tag{39}$$

which is evaluated numerically. With $B = 100{,}000$ and $\delta = 0.04$, the estimated premium is

$$\widehat{\mathrm{NSP}} = \mathrm{NSP}(\widehat{\eta}) \approx \$58{,}234.$$

To quantify uncertainty, we apply a parametric bootstrap with $B^\star = 1000$ replications: samples are generated from $\mathrm{UTXLD}(\widehat{\eta}, m)$, the MLE is recomputed for each sample, and corresponding NSP values are obtained. Percentile intervals from the bootstrap distribution provide confidence bounds for the premium, accounting for sampling variability in $\widehat{\eta}$.

For model comparison, UTXLD is evaluated against truncated exponential, Weibull, Lindley, gamma, and lognormal models using AIC and BIC, as well as bootstrap standard errors of NSP. Since information-criterion differences can be small in moderate samples, we report ΔAIC values relative to the best model and avoid over-interpreting marginal improvements. In this dataset, UTXLD attains the lowest AIC, while truncated Lindley and Weibull provide close competitors.

Finally, sensitivity analysis is conducted for $m \in \{40, 50, 60\}$ and $\delta \in \{0.02, 0.04, 0.06\}$. As expected, NSP increases with the truncation point m and decreases with higher interest rates. This highlights the importance of jointly considering truncation and financial assumptions in actuarial pricing and shows that the UTXLD framework remains stable across plausible operating conditions.

Overall, this application demonstrates that the UTXLD offers a flexible and interpretable model for truncated lifetimes in insurance pricing, combining analytical tractability with realistic hazard-rate behavior and practical premium estimation.

Table 12. Sensitivity of NSP ($) under UTXLD for different truncation points m and interest rates δ (benefit $B = 100{,}000$, $\widehat{\eta} = 0.543$).

m	Force of interest δ		
	0.02	0.04	0.06
40	62,980	58,910	55,430
50	65,870	58,234	52,610
60	69,540	61,480	55,290

Table 12 shows that the net single premium increases with the truncation limit m, since a longer coverage horizon raises the expected present value of benefits. Conversely, higher interest rates reduce NSP due to stronger discounting. These trends are consistent with standard actuarial theory and confirm that truncation and financial assumptions jointly influence premium levels. Hence, fixing m and δ without sensitivity analysis may lead to misleading pricing conclusions.

6. Conclusion

This paper has presented a systematic study of truncated versions of the XLindley distribution, including the upper, lower, and double truncated forms. Starting from the baseline XLindley model, we derived the corresponding probability density, distribution, survival, and hazard rate functions, together with moment expressions, quantile functions, and order–statistic formulations. Particular emphasis was placed on the upper truncated XLindley distribution (UTXLD), for which hazard-rate monotonicity and reliability properties were established using direct analytical arguments.

Maximum likelihood estimation procedures were developed for all three truncated models, and practical numerical implementation based on Newton–Raphson iterations was described. Interval estimation was discussed using observed Fisher information and parametric bootstrap methods, which is particularly important in small samples where asymptotic approximations may be unreliable.

Simulation results demonstrated that the MLEs are consistent and that bias and mean squared error decrease as sample size increases. For very small samples, bootstrap confidence intervals were shown to provide more reliable coverage than purely asymptotic intervals. Real data applications to aircraft glass strength, machine failure times, and truncated lifetime data for actuarial pricing illustrated the practical relevance of truncated XLindley models. In the aircraft glass example, the double truncated XLindley model achieved favorable information-criterion values when compared with truncated competitors under the same truncation scheme, while the machine and insurance examples highlighted the interpretability of survival and reliability measures derived from the model.

Limitations

Despite these advantages, several limitations should be acknowledged. First, when truncation points coincide with sample extrema, the likelihood may become flat near the boundary, leading to numerical instability or extreme parameter estimates. In such cases, diagnostic tools such as profile likelihoods and bootstrap assessments are necessary to verify estimator reliability. Second, differences in information criteria between competing truncated models are sometimes small, implying that model uncertainty should be considered; model averaging or predictive validation may be more appropriate than selecting a single best-fitting model. Third, actuarial quantities such as the net single premium are sensitive to both the truncation point and the discount rate, so sensitivity analysis is essential for practical pricing applications.

Future Research

Several extensions merit further investigation. Regression structures could be introduced to allow covariate effects on truncated lifetimes, enabling applications in survival analysis and reliability regression. Bayesian estimation may provide greater stability in small samples and facilitate full uncertainty quantification. Multivariate and dependent truncated lifetime models would broaden applicability to systems reliability and competing risks. Finally, applications to large contemporary datasets and the integration of truncated XLindley models into predictive maintenance and insurance portfolio analysis are promising directions.

Overall, the truncated XLindley family offers a flexible and analytically tractable framework for modeling bounded lifetime data, with potential utility across reliability engineering, actuarial science, biomedical studies, and other fields where truncation arises naturally.

REFERENCES

1. N. Eugene, C. Lee, and F. Famoye, *Beta-normal distribution and its applications*, Communications in Statistics–Theory and Methods, vol. 31, no. 4, pp. 497–512, 2002.
2. M. C. Jones, *Kumaraswamy's distribution: A beta-type distribution with some tractability advantages*, Statistical Methodology, vol. 6, no. 1, pp. 70–81, 2009.
3. A. Alzaatreh, C. Lee, and F. Famoye, *A new method for generating families of continuous distributions*, Metron, vol. 71, no. 1, pp. 63–79, 2013.
4. M. S. Shama et al., *Modified generalized Weibull distribution: theory and applications*, Scientific Reports, vol. 13, article 12828, 2023.

5. X. Peng and Z. Yan, *Estimation and application for a new extended Weibull distribution*, Reliability Engineering & System Safety, vol. 121, pp. 34–42, 2014.
6. M. Almheidat, C. Lee, and F. Famoye, *A generalization of the Weibull distribution with applications*, Journal of Modern Applied Statistical Methods, vol. 15, pp. 788–820, 2016.
7. R. E. Glaser, *Bathtub and related failure rate characterizations*, Journal of the American Statistical Association, vol. 75, no. 371, pp. 667–672, 1980.
8. V. Voinov et al., *Goodness-of-fit tests for the power-generalized Weibull distribution*, Communications in Statistics–Simulation and Computation, vol. 42, no. 5, pp. 1003–1012, 2013.
9. D. Kumar, S. Dey, and S. Nadarajah, *Extended exponential distribution based on order statistics*, Communications in Statistics–Theory and Methods, vol. 46, no. 18, pp. 9166–9184, 2017.
10. E. M. Almetwally and M. A. Meraou, *Application of environmental data with new extension of Nadarajah–Haghighi distribution*, Computational Journal of Mathematical and Statistical Sciences, vol. 1, no. 1, pp. 26–41, 2022.
11. L. Bouchahed and H. Zeghdoudi, *A new and unified approach in generalizing the Lindley distribution with applications*, Statistics, vol. 61, 2018.
12. G. Chen and T. T. Pham, *Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*, CRC Press, 2000.
13. M. Bebbington, C.-D. Lai, and R. Zitikis, *A flexible Weibull extension*, Reliability Engineering & System Safety, vol. 92, no. 6, pp. 719–726, 2007.
14. W. Weibull, *A statistical distribution function of wide applicability*, Journal of Applied Mechanics, vol. 18, pp. 293–297, 1951.
15. F. R. S. de Gusmão, E. M. M. Ortega, and G. M. Cordeiro, *The generalized inverse Weibull distribution*, Statistical Papers, vol. 52, no. 3, pp. 591–619, 2011.
16. A. M. Isa et al., *Sine-exponential distribution: properties and application*, UMYU Scientifica, vol. 1, no. 1, pp. 127–131, 2022.
17. S. Nadarajah and S. Kotz, *On the alternative to the Weibull function*, Engineering Fracture Mechanics, vol. 74, pp. 577–579, 2007.
18. D. N. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, John Wiley & Sons, Hoboken, 2004.
19. T. W. Anderson and D. A. Darling, *Asymptotic theory of certain goodness-of-fit criteria*, Annals of Mathematical Statistics, pp. 193–212, 1952.
20. J. J. Swain, S. Venkatraman, and J. R. Wilson, *Least-squares estimation of distribution functions*, Journal of Statistical Computation and Simulation, vol. 29, no. 4, pp. 271–297, 1988.
21. B. K. Sah and S. K. Sahani, *Polynomial-exponential distribution*, Mathematical Statistician and Engineering Applications, vol. 71, no. 4, pp. 2474–2486, 2022.
22. A. Beghriche et al., *New polynomial exponential distribution: properties and applications*, Statistics in Transition New Series, vol. 23, no. 3, pp. 95–112, 2022.
23. C. Chesneau et al., *Polynomial-exponential distribution allowing zero values*, Communications in Statistics–Simulation and Computation, vol. 51, no. 8, pp. 4581–4606, 2022.
24. F. Z. Bousseba et al., *Novel quadratic exponential distribution*, Heliyon, vol. 10, no. 19, 2024.
25. S. E. Ahmeds et al., *Truncated Birnbaum–Saunders distribution in financial risk*, Pakistan Journal of Statistics, vol. 26, pp. 1–19, 2010.
26. S. Abbas et al., *Truncated Weibull–exponential distribution*, Scientific Reports, 2023.
27. A. Alzaatreh et al., *Truncated family of distributions with applications*, Methodology and Computing in Applied Probability, vol. 23, pp. 5–27, 2021.
28. Hamasha et al., *Lower truncated normal approximation for reliability*, Cogent Engineering, vol. 9, 2022.
29. S. Chouia and H. Zeghdoudi, *The XLindley distribution: properties and application*, Journal of Statistical Theory and Applications, vol. 20, no. 2, pp. 318–327, 2021.
30. D. N. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, John Wiley & Sons, Hoboken, 2004.