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Linear Diophantine HyperFuzzy Set and SuperHyperFuzzy Set

Takaaki Fujita [1], Ahmed Salem Heilat [2,*], Raed Hatamleh [3] and Arkan A. Ghaib [4]

[1] *Independent Researcher, Tokyo, Japan. Email: Takaaki.fujita060@gmail.com*
[2] *Department of Mathematics, Faculty of Science, Jadara University, Irbid-Jordan. Email: a.heilat@jadara.edu.jo*
[3] *Department of Mathematics, Faculty of Science, Jadara University, Irbid-Jordan. Email: raed@jadara.edu.jo*
[4] *Department of Information Technology, Management Technical College, Southern Technical University, Basrah, 61004, Iraq. Email: arkan.ghaib@stu.edu.iq*

Abstract Uncertainty modeling underpins decision-making across diverse domains, and numerous frameworks—such as Fuzzy Sets, Rough Sets, Hesitant Fuzzy Sets, and Plithogenic Sets—have been developed to capture different facets of imprecision. Hyperfuzzy Sets and their recursive generalization, SuperHyperfuzzy Sets, assign set-valued membership degrees at multiple hierarchical levels to represent uncertainty more richly. The Linear Diophantine Fuzzy Set further refines this approach by imposing weighted linear Diophantine constraints on membership and non-membership grades. In this paper, we define two new constructs—the Linear Diophantine Hyperfuzzy Set and the Linear Diophantine SuperHyperfuzzy Set—by integrating Diophantine constraints with hyperfuzzy and superhyperfuzzy frameworks, and we present a concise application example.
A Linear Diophantine HyperFuzzy Set assigns each element set-valued membership and nonmembership grades, constrained by a linear Diophantine relation. A $(m, n)-$Linear Diophantine SuperHyperFuzzy Set assigns each element set-valued membership and nonmembership grades, constrained by a linear Diophantine relation. We also examine the algorithms associated with these notions.
These extensions offer a more structured, hierarchical means of applying Linear Diophantine Fuzzy Set methodology in practical uncertain environments.

Keywords Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, Linear Diophantine Fuzzy Set

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1. Introduction

1.1. Fuzzy, HyperFuzzy, and SuperHyperFuzzy Sets

In classical set theory, objects that share a given property are collected into a set, and membership is strictly binary: an element either belongs to the set or it does not[1]. When one tries to model vague boundaries, partial truth, or

*Correspondence to: Ahmed Heilat (Email: a.heilat@jadara.edu.jo). Department of Mathematics, Faculty of Science, Jadara University, Irbid-Jordan.

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pieces of evidence that are incomplete or even conflicting, such crisp sets are too rigid to capture the full spectrum of real-world uncertainty. To overcome this limitation in a systematic manner, many kinds of *uncertain set* formalisms have been introduced[2, 3]. In the fuzzy-set framework, each element of a universe U is assigned a membership grade in $[0, 1]$, so that belonging to a set becomes a gradual notion rather than an all-or-nothing decision[4]. Typical generalizations include pythagorean fuzzy sets [5, 6], neutrosophic sets[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], bipolar fuzzy sets[19], bipolar neutrosophic sets[20], neutrosophic soft sets[21], neutrosophic resolving sets[22], pentapartitioned neutrosophic sets[23], heptapartitioned neutrosophic sets[24], hesitant neutrosophic sets[25], complex neutrosophic sets[26, 27, 28, 29, 30], shadowed sets[31], plithogenic sets[32, 33, 34, 35], double-valued neutrosophic sets[36, 37], and intuitionistic fuzzy sets[38]. These fuzzy-type models and their variants have been studied extensively and successfully applied in decision making, machine learning, and many areas of computer science.

Hyperfuzzy sets refine this picture by assigning to each element a *nonempty subset* of $[0, 1]$, representing several plausible membership degrees and hence uncertainty about the membership value itself[39, 40]. In an even more structured, hierarchical setting, an (m, n)-superhyperfuzzy set associates each nonempty m-level subset of U with a family of nonempty n-level sets of membership grades, thereby encoding multiple layers of uncertainty within a single unified framework[41, 42]. For convenience, Table 1 summarizes the main viewpoints behind Uncertain, HyperUncertain, and SuperHyperUncertain Sets.

Table 1. Summary of Uncertain, HyperUncertain, and SuperHyperUncertain Sets.

Notion	**Underlying domain**	**Core idea**	**Typical instances / references**
Uncertain Set	S (or a model-dependent subfamily of $\mathcal{P}(S)$)	Attach to elements or subsets some representation of uncertainty, such as graded membership, parameters, lower/upper approximations, neutrosophic triples, or contradiction-aware degrees.	Fuzzy Sets[4]; Soft Sets[21]; Neutrosophic Sets[43]; Plithogenic Sets[44].
HyperUncertain Set	$\mathcal{P}(S)$	Treat uncertainty directly on the powerset; evaluations on $\mathcal{P}(S)$ become set-valued, capturing hesitation, multiple compatible grades, and richer patterns of appurtenance.	HyperFuzzy[45, 46]; HyperNeutrosophic[47], HyperSoft[48].
SuperHyperUncertain Set	$\mathcal{P}^n(S), n \geq 1$	Lift the uncertainty semantics to iterated powersets, so that hierarchical and multi-level uncertainty over multi-ary inputs/outputs can be modeled across the different layers of $\mathcal{P}^n(S)$.	SuperHyperFuzzy, SuperHyper-Vague, SuperHyperNeutrosophic, and related models[49, 50].

Note. $\mathcal{P}(S)$ denotes the powerset of S, and $\mathcal{P}^n(S)$ the n-fold iterated powerset ($n \geq 1$). The n-fold iterated powerset of S is obtained by applying the powerset operator n times: $\mathcal{P}^1(S) = \mathcal{P}(S)$.

Several notions are closely connected to SuperHyperFuzzy sets, including Type-n fuzzy sets (such as Type-2 and Type-3 fuzzy sets), n-dimensional or multidimensional fuzzy sets, and interval-valued fuzzy sets. A concise comparison of these models appears in Table 2. Type-n fuzzy sets[51], n-dimensional/multidimensional fuzzy sets[52, 53], and interval-valued fuzzy sets [54] are all foundational components of fuzzy theory, have broad applicability, and have been extensively studied in the literature.

1.2. Linear Diophantine fuzzy set

Within this fuzzy landscape, a Linear Diophantine fuzzy set provides an additional algebraic constraint: each element is equipped with a membership and a nonmembership degree whose weighted sum satisfies a prescribed linear Diophantine relation[56, 57, 58]. Related concepts include Spherical Linear Diophantine Fuzzy Sets[59, 57], Linear Diophantine Neutrosophic Sets[60], and q-rung orthopair fuzzy sets [61, 62, 63]. Linear Diophantine fuzzy sets integrate algebraic constraints with uncertainty, enabling structured decision models, consistency checks, and richer parameterized reasoning capabilities. For reference, Table 3 provides a brief comparison of the Fuzzy Set and the Linear Diophantine Fuzzy Set.

Table 2. Comparison of SuperHyperFuzzy, Type-n, n-dimensional/multidimensional, and interval-valued fuzzy sets.

Notion	**Membership mapping**	**Value structure**	**Relation / remark**
SuperHyperFuzzy Set [41]	$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(U) \to \mathcal{P}(\mathcal{P}_n([0,1])) \setminus \{\emptyset\}$	Each m-level subset of U is mapped to a nonempty family of n-level sets of grades in $[0,1]$.	Provides hierarchical, set-valued membership on iterated powersets; extends hyperfuzzy sets and models multi-layer uncertainty on groups of elements.
Type-n Fuzzy Set [51, 55]	$\mu^{(n)} : X \to \mathcal{F}^{(n-1)}([0,1])$, where $\mathcal{F}^{(k)}([0,1])$ denotes type-k fuzzy sets on $[0,1]$	The membership of $x \in X$ is a type-$(n-1)$ fuzzy set on $[0,1]$, defined recursively from type-1 fuzzy sets.	For $n = 1$ one recovers an ordinary fuzzy set; for $n = 2$ a general type-2 fuzzy set. Uncertainty is encoded via $(n-1)$ nested fuzzy layers over $[0,1]$.
n-Dimensional / Multidimensional Fuzzy Set [52, 53]	$\nu : X \to J_\infty([0,1])$, where $J_\infty([0,1]) = \bigcup_{k\geq 1} J_k([0,1])$ and $\nu(x) = (z_1, \dots, z_{k(x)})$ with $0 \leq z_1 \leq \cdots \leq z_{k(x)} \leq 1$	Each element carries a finite, ordered tuple of membership grades (with element-dependent length), representing several evaluation levels of a single attribute.	Extends n-dimensional fuzzy sets by allowing variable tuple length per element; yields vector-valued memberships.
Interval-valued Fuzzy Set [54]	$\mu_{\mathrm{IV}} : X \to \{[a,b] \subseteq [0,1] \mid 0 \leq a \leq b \leq 1\}$	Each element is assigned a closed interval of admissible membership grades in $[0,1]$.	Can be seen as a special type-2 fuzzy set where the secondary membership is 1 on a single interval and 0 outside; provides lower and upper bounds without additional hierarchical layers.

Table 3. Brief comparison of Fuzzy Set and Linear Diophantine Fuzzy Set

Aspect	Fuzzy Set	Linear Diophantine Fuzzy Set
Membership representation	Each x has $\mu(x) \in [0,1]$.	Each x has $A_D(x), S_D(x) \in [0,1]$.
Additional parameters	No global parameters.	Uses fixed weights (α, β) with $0 \leq \alpha + \beta \leq 1$.
Core constraint	Only $0 \leq \mu(x) \leq 1$.	Requires $0 \leq \alpha A_D(x) + \beta S_D(x) \leq 1$.
Hesitation / indeterminacy	Often omitted or $1 - \mu(x)$.	$\pi_D(x) = 1 - (\alpha A_D(x) + \beta S_D(x))$.
Modeling focus	Graded membership for vague concepts.	Structured balance of support and opposition.

1.3. *Our Contributions*

Although fuzzy, hyperfuzzy, and superhyperfuzzy sets have been widely analyzed, the systematic combination of these structures with the Linear Diophantine fuzzy–set paradigm has, to the best of our knowledge, not yet been investigated. Because hyperfuzzy and superhyperfuzzy sets are important in that they allow clear representation of concepts with multi-level uncertainty, it is meaningful to extend the Linear Diophantine fuzzy set to the hyperfuzzy and superhyperfuzzy settings as well, and we believe that these extensions likewise have the potential to represent concepts with multi-level uncertainty.

In this paper, we close this gap by introducing two new notions: the *Linear Diophantine Hyperfuzzy Set* and the *Linear Diophantine SuperHyperfuzzy Set*. Both are obtained by embedding Diophantine–type linear constraints into the hyperfuzzy and superhyperfuzzy settings, respectively. We also present a compact application example to demonstrate how these constructions can be used to model complex, hierarchically organized uncertainty in practical decision scenarios. For reference, Table 4 provides a concise comparison of Linear Diophantine Fuzzy, HyperFuzzy, and SuperHyperFuzzy Sets.

Table 4. Linear Diophantine Fuzzy, HyperFuzzy, and SuperHyperFuzzy Sets (brief overview).

Notion	**Domain / mapping**	**Brief description**
Linear Diophantine Fuzzy Set (LDFS)	$x \in Q \mapsto (A_D(x), S_D(x)) \in [0,1]^2$	Single membership and nonmembership degrees with $0 \leq \alpha A_D(x) + \beta S_D(x) \leq 1$. Residual hesitation $1 - (\alpha A_D + \beta S_D)$.
Linear Diophantine HyperFuzzy Set (LDHFS)	$x \in U \mapsto (\widetilde{\mu}(x), \widetilde{\nu}(x))$, $\widetilde{\mu}(x), \widetilde{\nu}(x) \subseteq [0,1]$ nonempty	Set–valued membership and nonmembership. For all $u \in \widetilde{\mu}(x), v \in \widetilde{\nu}(x), 0 \leq \alpha u + \beta v \leq 1$.
(m,n)-Linear Diophantine SuperHyperFuzzy Set	$A \in \mathcal{P}_m^*(U) \mapsto (\widetilde{\mu}_{m,n}(A), \widetilde{\nu}_{m,n}(A))$, values in $\mathcal{P}(\mathcal{P}_n([0,1])) \setminus \{\emptyset\}$	Hierarchical (multi-level) set–valued membership/nonmembership. For all $s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)$, $0 \leq \alpha s + \beta t \leq 1$.

The proposed construction enforces Diophantine consistency across all hierarchical hyperfuzzy levels. This allows global feasibility checks and a tunable trade-off between membership and nonmembership, yielding sharper and more structurally constrained decision models than those obtained from Diophantine fuzzy or (separately defined) (super)hyperfuzzy frameworks. Moreover, it can encode a global policy requiring that, across different levels (panels, scenarios, expert groups), the balance between support and opposition must remain within a prescribed range. From a decision-making viewpoint, at a suitable stage of the hierarchical aggregation one can switch between risk-averse and risk-seeking strategies by adjusting α and β.

1.4. Structure of This Paper

This subsection outlines the structure of the paper. Section 2 presents the preliminaries, introducing the required background on Fuzzy, Hyperfuzzy, and SuperHyperfuzzy Sets, as well as the definitions of Linear Diophantine Fuzzy Sets. Section 3 develops the concepts of the Linear Diophantine HyperFuzzy Set and the (m,n)-Linear Diophantine SuperHyperFuzzy Set. Section 4 provides the concluding remarks of the paper.

2. Preliminaries: SuperHyperFuzzy Sets and Linear Diophantine Fuzzy Sets

In this section, we summarize the key definitions and notation used throughout this paper. Unless otherwise specified, all sets are assumed finite.

2.1. Fuzzy, Hyperfuzzy, and SuperHyperfuzzy Sets

The definitions of Fuzzy, Hyperfuzzy, and SuperHyperfuzzy Sets are presented below.

Definition 2.1 (Fuzzy Set)**.** [4] A *fuzzy set* F on a universe U is specified by a membership function

$$\mu_F : U \longrightarrow [0,1],$$

so that each element $x \in U$ is assigned a degree of membership $\mu_F(x)$.

Definition 2.2 (Fuzzy Relation)**.** [64] Let F be a fuzzy set on U. A *fuzzy relation* R on U is a map

$$R : U \times U \longrightarrow [0,1],$$

satisfying

$$R(x,y) \ \leq \ \min\{\mu_F(x),\, \mu_F(y)\} \quad \text{for all } x, y \in U.$$

Definition 2.3 (Hyperfuzzy Set)**.** [39] A *hyperfuzzy set* $\widetilde{F}$ on U is given by a function

$$\widetilde{\mu}\colon U \longrightarrow \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

where for each $x \in U$, the nonempty subset $\widetilde{\mu}(x) \subseteq [0,1]$ represents all possible membership grades of x.

We provide below a brief overview of the iterated powerset and the superhyperfuzzy set.

Definition 2.4 (Iterated Powerset)**.** [65, 66] For each integer $n \geq 1$, the n-fold iterated powerset of U is defined by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \quad \mathcal{P}^{n+1}(U) = \mathcal{P}\big(\mathcal{P}^n(U)\big).$$

If one wishes to exclude the empty set at each iteration, replace $\mathcal{P}$ with

$$\mathcal{P}^*(X) \ = \ \mathcal{P}(X) \setminus \{\emptyset\}.$$

Definition 2.5 ((m,n)-SuperHyperfuzzy Set)**.** [41] Fix integers $m, n \geq 0$. Define

$$\mathcal{P}^*_m(U) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{m \text{ times}})(U), \quad \mathcal{P}^*_n([0,1]) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{n \text{ times}})([0,1]),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. An (m,n)-*superhyperfuzzy set* on U is a mapping

$$\widetilde{\mu}_{m,n}\colon \mathcal{P}^*_m(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\},$$

which assigns each nonempty m-level subset of U a nonempty family of n-level membership-value sets, thereby capturing hierarchical uncertainty.

Example 2.6 (Multi-Expert Product Reliability Assessment)**.** We demonstrate an (m,n)-SuperHyperfuzzy set with $m = 1$, $n = 2$ in the context of assessing product reliability by two independent expert panels. Let

$$U = \{\text{Smartphone, Laptop, Headphones}\}, \qquad m = 1,\ n = 2.$$

Recall that

$$\mathcal{P}^*_1(U) \ = \ \mathcal{P}(U) \setminus \{\emptyset\},$$

so in particular the singletons $\{\text{Smartphone}\}$, $\{\text{Laptop}\}$, $\{\text{Headphones}\}$ all lie in $\mathcal{P}^*_1(U)$. Moreover,

$$\mathcal{P}^*_2([0,1]) \ = \ \big\{\, S \subseteq \mathcal{P}([0,1]) \setminus \{\emptyset\} \mid S \neq \emptyset \big\},$$

whose elements are nonempty families of nonempty sets of membership grades.

For each product $x \in U$ we now specify two elements

$$H_1(x),\ H_2(x) \in \mathcal{P}^*_2([0,1]),$$

representing the hyper-membership information supplied by Expert Panel 1 and Expert Panel 2, respectively. Concretely, we set

$$\begin{aligned}
H_1(\text{Smartphone}) &= \big\{\{0.80,\ 0.85,\ 0.90\}\big\}, & H_2(\text{Smartphone}) &= \big\{\{0.75,\ 0.82\}\big\},\\
H_1(\text{Laptop}) &= \big\{\{0.65,\ 0.70,\ 0.75\}\big\}, & H_2(\text{Laptop}) &= \big\{\{0.60,\ 0.68\}\big\},\\
H_1(\text{Headphones}) &= \big\{\{0.50,\ 0.55,\ 0.60\}\big\}, & H_2(\text{Headphones}) &= \big\{\{0.45,\ 0.52\}\big\}.
\end{aligned}$$

Each inner set (for instance $\{0.80, 0.85, 0.90\}$) collects the possible normalized reliability grades reported by different sub-surveys within a single panel, while the outer braces form a (here singleton) family of such grade-sets.

We define the mapping

$$\widetilde{\mu}_{1,2}\colon \mathcal{P}_1^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_2([0,1])\big) \setminus \{\emptyset\}$$

by

$$\widetilde{\mu}_{1,2}\big(\{\text{Smartphone}\}\big) = \big\{\, H_1(\text{Smartphone}),\ H_2(\text{Smartphone}) \,\big\},$$
$$\widetilde{\mu}_{1,2}\big(\{\text{Laptop}\}\big) = \big\{\, H_1(\text{Laptop}),\ H_2(\text{Laptop}) \,\big\},$$
$$\widetilde{\mu}_{1,2}\big(\{\text{Headphones}\}\big) = \big\{\, H_1(\text{Headphones}),\ H_2(\text{Headphones}) \,\big\}.$$

Thus for each singleton $\{x\} \in \mathcal{P}_1^*(U)$, the value $\widetilde{\mu}_{1,2}(\{x\})$ is a nonempty subset of $\mathcal{P}_2([0,1])$, whose elements $H_1(x), H_2(x)$ encode, at the second level, the hyper-membership information provided by the two panels.

Consequently,

$$\widetilde{D} \;=\; \big\{ \big(\{x\},\, \widetilde{\mu}_{1,2}(\{x\})\big) \;\big|\; x \in U \big\}$$

forms a $(1,2)$-SuperHyperfuzzy set capturing two levels of hierarchical uncertainty: one level for the choice of product ($m = 1$), and one level for the expert-panel structure and its internal survey variability ($n = 2$).

2.2. *Linear Diophantine Fuzzy Set*

A Linear Diophantine Fuzzy Set assigns to each element a membership and a nonmembership grade whose weighted sum satisfies a specified linear Diophantine equation [56, 57, 58, 67]. The definitions and a brief illustrative example are given below.

Definition 2.7 (Linear Diophantine Fuzzy Set)**.** [56, 57] Let Q be a nonempty universe. Fix reference parameters $\alpha, \beta \in [0,1]$ with

$$0 \;\le\; \alpha + \beta \;\le\; 1.$$

A *linear Diophantine fuzzy set* (LDFS) $\widetilde{D}$ on Q is a collection of triples

$$\widetilde{D} = \big\{\, \big(x, \langle A_D(x),\, S_D(x)\rangle, \langle \alpha, \beta\rangle\big) \;\big|\; x \in Q \,\big\},$$

where

- $A_D, S_D\colon Q \to [0,1]$ assign to each $x \in Q$ its membership grade $A_D(x)$ and non-membership grade $S_D(x)$, and
- these must satisfy
$$0 \;\le\; \alpha\, A_D(x) \;+\; \beta\, S_D(x) \;\le\; 1, \quad \forall\, x \in Q.$$

The residual (hesitation) degree is then

$$\pi_D(x) = 1 - \big(\alpha\, A_D(x) + \beta\, S_D(x)\big).$$

We often abbreviate each triple $\big(x, (A_D(x), S_D(x)), (\alpha, \beta)\big)$ simply as the *linear Diophantine fuzzy number*

$$\big\langle A_D(x),\, S_D(x)\big\rangle_{(\alpha,\beta)}.$$

3. Main Results: Linear Diophantine HyperFuzzy Set and Linear Diophantine SuperHyperFuzzy Set

This paper presents the main findings concerning the Linear Diophantine HyperFuzzy Set and the Linear Diophantine SuperHyperFuzzy Set.

3.1. Linear Diophantine HyperFuzzy Set

A Linear Diophantine HyperFuzzy Set assigns each element set-valued membership and nonmembership grades, constrained by a linear Diophantine relation.

Definition 3.1 (Linear Diophantine HyperFuzzy Set)**.** Let U be a nonempty universe. Fix real weights $\alpha, \beta \in [0,1]$ with

$$\alpha + \beta \ \leq \ 1.$$

A *Linear Diophantine HyperFuzzy Set* (LDHFS) $\widetilde{D}$ on U with parameters (α, β) is given by two set-valued mappings

$$\widetilde{\mu}, \ \widetilde{\nu} \colon U \ \longrightarrow \ \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

called the *hyper-membership* and *hyper-nonmembership* functions, respectively, such that for every $x \in U$ and for all $u \in \widetilde{\mu}(x), \ v \in \widetilde{\nu}(x)$ we have the *Linear Diophantine condition*

$$0 \ \leq \ \alpha\, u \ + \ \beta\, v \ \leq \ 1.$$

The associated *hyper-hesitation set* $\widetilde{\pi}$ is defined by

$$\widetilde{\pi}(x) = \left\{ 1 - (\alpha\, u + \beta\, v) \;\middle|\; u \in \widetilde{\mu}(x), \ v \in \widetilde{\nu}(x) \right\}, \quad x \in U.$$

Example 3.2 (Credit Risk Assessment)**.** Consider three loan applicants evaluated by three models and two expert committees. Let

$$U = \{\text{Ayuka, Masahiro, Carol}\}, \quad (\alpha, \beta) = (0.6, \ 0.3).$$

For each $x \in U$:

- $\widetilde{\mu}(x) \subseteq [0,1]$ is the set of possible *creditworthiness* grades from three sources.
- $\widetilde{\nu}(x) \subseteq [0,1]$ is the set of possible *default-risk* grades from two experts.
- All $u \in \widetilde{\mu}(x)$, $v \in \widetilde{\nu}(x)$ must satisfy $0 \leq 0.6\, u + 0.3\, v \leq 1$.

Suppose the assessments are:

$$\begin{aligned} \widetilde{\mu}(\text{Ayuka}) &= \{0.80, \ 0.85, \ 0.90\}, & \widetilde{\nu}(\text{Ayuka}) &= \{0.10, \ 0.15\}, \\ \widetilde{\mu}(\text{Masahiro}) &= \{0.60, \ 0.65, \ 0.70\}, & \widetilde{\nu}(\text{Masahiro}) &= \{0.25, \ 0.30\}, \\ \widetilde{\mu}(\text{Carol}) &= \{0.40, \ 0.50\}, & \widetilde{\nu}(\text{Carol}) &= \{0.40, \ 0.45, \ 0.50\}. \end{aligned}$$

For example, for Ayuka with $u = 0.85$ and $v = 0.15$:

$$0.6 \times 0.85 + 0.3 \times 0.15 = 0.51 \quad (\text{lies in } [0,1]).$$

The *hyper-hesitation set* is

$$\widetilde{\pi}(x) = \{ 1 - (0.6\, u + 0.3\, v) \mid u \in \widetilde{\mu}(x), \, v \in \widetilde{\nu}(x) \}.$$

Hence the LDHFS $\widetilde{D}$ is

$$\left\{ (\, x, \ \widetilde{\mu}(x), \ \widetilde{\nu}(x), \ \widetilde{\pi}(x)) \;\middle|\; x \in \{\text{Ayuka,Masahiro,Carol}\} \right\}.$$

This example captures multiple credit-scoring opinions (hyper-membership), expert default-risk judgments (hyper-nonmembership), and the residual uncertainty (hyper-hesitation) in a coherent LDHFS framework.

Theorem 3.3 (Generalization of LDFS and HyperFuzzy Set)
Every Linear Diophantine Fuzzy Set and every HyperFuzzy Set can be obtained as a special case of an LDHFS:

(i) *(LDFS case)* If for each $x \in U$ the sets $\widetilde{\mu}(x)$ and $\widetilde{\nu}(x)$ are singletons, $\widetilde{\mu}(x) = \{A_D(x)\}$, $\widetilde{\nu}(x) = \{S_D(x)\}$, then $\widetilde{D}$ reduces to the usual Linear Diophantine Fuzzy Set $\{(x, A_D(x), S_D(x))\}$.

(ii) *(HyperFuzzy case)* If we choose $\alpha = 1$, $\beta = 0$, and set $\widetilde{\nu}(x) = \{0\}$ for all x, then the only nontrivial data is the mapping $\widetilde{\mu}\colon U \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$, recovering exactly a HyperFuzzy Set.

Proof

(i) Under the singleton assumption $\widetilde{\mu}(x) = \{A_D(x)\}$, $\widetilde{\nu}(x) = \{S_D(x)\}$, the Linear Diophantine condition

$$0 \le \alpha u + \beta v = \alpha A_D(x) + \beta S_D(x) \le 1 \quad \big(u = A_D(x),\ v = S_D(x)\big)$$

is exactly the feasibility requirement for a Linear Diophantine Fuzzy Set. The hyper-hesitation set $\widetilde{\pi}(x) = \{1 - (\alpha A_D(x) + \beta S_D(x))\}$ collapses to the single hesitation degree of the LDFS. Hence $\widetilde{D}$ coincides with the standard LDFS.

(ii) If $\alpha = 1$ and $\beta = 0$, then for any $\widetilde{\mu}(x) \subseteq [0,1]$ nonempty and $\widetilde{\nu}(x) = \{0\}$, the condition

$$0 \le 1 \cdot u + 0 \cdot v = u \le 1 \quad \forall u \in \widetilde{\mu}(x)$$

holds automatically. The mapping $\widetilde{\mu}$ alone carries all uncertainty information, exactly as in a HyperFuzzy Set. The auxiliary sets $\widetilde{\nu}(x)$ and $\widetilde{\pi}(x)$ play no substantive role. Thus $\widetilde{D}$ restricts to a HyperFuzzy Set.

□

3.2. $(m,n)-$*Linear Diophantine SuperHyperFuzzy Set*

A $(m,n)-$Linear Diophantine SuperHyperFuzzy Set assigns each element set-valued membership and nonmembership grades, constrained by a linear Diophantine relation.

Notation 3.4
Define the *flattening operator* $\mathrm{flat}_n\colon \mathcal{P}^n([0,1]) \to \mathcal{P}([0,1])$ recursively by

$$\mathrm{flat}_0(x) = \{x\}, \qquad \mathrm{flat}_{k+1}(S) = \bigcup_{T \in S} \mathrm{flat}_k(T).$$

Definition 3.5 ((m,n)-Linear Diophantine SuperHyperfuzzy Set)**.** Let U be a nonempty universe. Fix integers $m, n \ge 0$ and weights $\alpha, \beta \in [0,1]$ with $\alpha + \beta \le 1$. Recall $\mathcal{P}^*_m(U)$ and $\mathcal{P}^*_n([0,1])$ from Definition 2.5. An (m,n)-*linear Diophantine superhyperfuzzy set* on U is a pair of mappings

$$\widetilde{\mu}_{m,n},\ \widetilde{\nu}_{m,n} \ :\ \mathcal{P}^*_m(U) \ \longrightarrow\ \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\},$$

called the *hyper-membership* and *hyper-nonmembership* functions, such that for every $A \in \mathcal{P}^*_m(U)$, every $u \in \widetilde{\mu}_{m,n}(A)$, every $v \in \widetilde{\nu}_{m,n}(A)$, and every $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$, the following *linear Diophantine condition* holds:

$$0 \le \alpha s + \beta t \le 1.$$

The associated *hyper-hesitation set* $\widetilde{\pi}_{m,n}\colon \mathcal{P}^*_m(U) \to \mathcal{P}([0,1])$ is defined by

$$\widetilde{\pi}_{m,n}(A) = \big\{1 - \big(\alpha s + \beta t\big) \bigm| u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\}.$$

Remark 3.6. Classical HyperFuzzy and SuperHyperFuzzy sets allow the membership and nonmembership hypergrades to vary largely independently, so that any pair $(s,t) \in [0,1]^2$ is, in principle, admissible. The Linear Diophantine condition

$$0 \le \alpha s + \beta t \le 1$$

couples these components through the weights (α, β) and is intended to encode a global trade–off between support and opposition that can be tuned from domain knowledge or calibration. In the normalized setting adopted here (with $\alpha + \beta \le 1$ and $s, t \in [0,1]$), this inequality is automatically satisfied; we nevertheless make it explicit so that the same Diophantine template can be used in more general, non–normalized variants where it genuinely rules out infeasible (s,t)–combinations and thereby adds structure beyond standard HyperFuzzy/SuperHyperFuzzy models.

For illustration, consider a non-normalized variant in which membership and nonmembership grades lie in an interval $[0, L]$ with $L > 1$, while we still impose $0 \le \alpha u + \beta v \le 1$. Suppose $L = 2$, $(\alpha, \beta) = (1, 1)$, and a candidate pair $(u, v) = (1.8, 0.6)$ arises in some assessment. Then $\alpha u + \beta v = 2.4 > 1$, so this pair is rejected by the Diophantine constraint. In this way, the constraint rules out jointly extreme support–opposition combinations and enforces a global budget on admissible uncertainty patterns.

Several concrete examples are presented below.

Example 3.7 (Hierarchical Credit-Risk Evaluation)**.** Consider two loan applicants:

$$U = \{\text{Ayuka, Masahiro}\}, \quad m = 1,\ n = 2, \quad (\alpha, \beta) = (0.6, 0.3).$$

Then

$$\mathcal{P}_1^*(U) = \big\{\{\text{Ayuka}\}, \{\text{Masahiro}\}\big\}, \quad \mathcal{P}_2^*([0,1]) = \big\{S \subseteq \mathcal{P}([0,1]) \setminus \{\emptyset\} \mid S \neq \emptyset\big\}.$$

We define two set–valued mappings $\widetilde{\mu}_{1,2}, \widetilde{\nu}_{1,2} \colon \mathcal{P}_1^*(U) \to \mathcal{P}\big(\mathcal{P}_2([0,1])\big) \setminus \{\emptyset\}$ by, for each $x \in \{\text{Ayuka, Masahiro}\}$,

$$\widetilde{\mu}_{1,2}(\{x\}) = \big\{\{\,0.80,\ 0.85\},\ \{\,0.78,\ 0.82\}\big\},$$
$$\widetilde{\nu}_{1,2}(\{x\}) = \big\{\{\,0.10,\ 0.15\},\ \{\,0.12,\ 0.18\}\big\}.$$

Here each inner set is a sub-expert's fuzzy grade, and the outer family groups two expert committees.

By Definition 3.5, for any $u \in \widetilde{\mu}_{1,2}(\{x\})$, $v \in \widetilde{\nu}_{1,2}(\{x\})$ and any $s \in \mathrm{flat}_2(u)$, $t \in \mathrm{flat}_2(v)$, the Linear Diophantine constraint

$$0 \ \le\ \alpha\, s + \beta\, t = 0.6\, s + 0.3\, t \ \le\ 1$$

must hold. For instance, taking $u = \{0.85, 0.80\}$, $v = \{0.18, 0.12\}$, and $s = 0.85$, $t = 0.18$, we get

$$0.6 \cdot 0.85 + 0.3 \cdot 0.18 = 0.51 + 0.054 = 0.564 \in [0,1].$$

The associated hyper-hesitation set is

$$\widetilde{\pi}_{1,2}(\{x\}) = \big\{\, 1 - (0.6\, s + 0.3\, t) \,\big|\, s \in \mathrm{flat}_2(u),\ t \in \mathrm{flat}_2(v)\big\}.$$

Thus the pair $\big(\widetilde{\mu}_{1,2},\, \widetilde{\nu}_{1,2}\big)$ defines a $(1,2)$-Linear Diophantine SuperHyperfuzzy Set capturing two hierarchical levels of expert uncertainty under weighted membership vs. nonmembership constraints.

Example 3.8 (Project Portfolio Risk–Reward Evaluation)**.** Consider a company evaluating portfolios of two projects chosen from

$$U = \{\text{Project A, Project B, Project C}\}, \quad m = 2,\ n = 2,\ (\alpha, \beta) = (0.7, 0.2).$$

Then

$$\mathcal{P}_2^*(U) = \big\{\{\,\text{A}, \text{B}\},\ \{\text{A}, \text{C}\},\ \{\text{B}, \text{C}\}\big\}.$$

For each pair $A \in \mathcal{P}_2^*(U)$, two independent expert panels estimate the *success probability* (hyper-membership) and two risk committees estimate the *failure risk* (hyper-nonmembership). We set

$$\widetilde{\mu}_{2,2}(A) = \big\{U_1(A),\, U_2(A)\big\}, \quad \widetilde{\nu}_{2,2}(A) = \big\{V_1(A),\, V_2(A)\big\},$$

where each $U_i(A) \subseteq [0,1]$ and $V_j(A) \subseteq [0,1]$ are nonempty.

Case $\{A, B\}$:

$$U_1(\{A,B\}) = \{0.80,\ 0.85\}, \quad V_1(\{A,B\}) = \{0.10,\ 0.15\},$$
$$U_2(\{A,B\}) = \{0.75,\ 0.78\}, \quad V_2(\{A,B\}) = \{0.12,\ 0.18\}.$$

Here:

- Panel 1 forecasts success rates of 0.80 or 0.85.
- Panel 2 forecasts success rates of 0.75 or 0.78.
- Committee 1 assesses risk at 0.10 or 0.15.
- Committee 2 assesses risk at 0.12 or 0.18.

For each $u \in U_i(\{A,B\}),\ v \in V_j(\{A,B\}),$ and each $s \in \mathrm{flat}_2(u) = u,\ t \in \mathrm{flat}_2(v) = v,$ we verify the Linear Diophantine condition:

$$0 \ \le\ 0.7\,s + 0.2\,t \ \le\ 1.$$

For example, taking $s = 0.85$ and $t = 0.18$ gives

$$0.7 \cdot 0.85 + 0.2 \cdot 0.18 = 0.595 + 0.036 = 0.631\ \in [0,1].$$

The associated hyper-hesitation set is

$$\widetilde{\pi}_{2,2}(\{A,B\}) = \big\{\, 1 - (0.7\,s + 0.2\,t) \ \big|\ s \in U_i(\{A,B\}),\ t \in V_j(\{A,B\})\big\},$$

which concretely contains values such as $1 - (0.7 \cdot 0.85 + 0.2 \cdot 0.18) = 0.369$, etc.

Other pairs: One similarly defines

$$U_1(\{A,C\}) = \{0.82,\, 0.88\}, \quad V_1(\{A,C\}) = \{0.08,\, 0.12\},$$
$$U_2(\{A,C\}) = \{0.78,\, 0.81\}, \quad V_2(\{A,C\}) = \{0.10,\, 0.14\},$$

$$U_1(\{B,C\}) = \{0.70,\, 0.75\}, \quad V_1(\{B,C\}) = \{0.15,\, 0.20\},$$
$$U_2(\{B,C\}) = \{0.68,\, 0.72\}, \quad V_2(\{B,C\}) = \{0.18,\, 0.22\}.$$

Each case satisfies $0 \le 0.7\,s + 0.2\,t \le 1$, and one computes $\widetilde{\pi}_{2,2}(A)$ accordingly.

Therefore, the pair $\big(\widetilde{\mu}_{2,2},\, \widetilde{\nu}_{2,2}\big)$ defines a $(2,2)$-Linear Diophantine SuperHyperfuzzy Set on U, modeling *hierarchical success and risk estimates* for every two-project portfolio under weighted Diophantine constraints.

Example 3.9 (Team Formation Performance–Risk Assessment)**.** A company has four specialists $U = \{E_1, E_2, E_3, E_4\}$. It must form teams of three for a high-impact project, balancing predicted performance against interpersonal risk. We set

$$m = 3, \quad n = 2, \quad (\alpha, \beta) = (0.5, 0.4),$$

so

$$\mathcal{P}_3^*(U) = \big\{\{E_1, E_2, E_3\},\ \{E_1, E_2, E_4\},\ \{E_1, E_3, E_4\},\ \{E_2, E_3, E_4\}\big\}.$$

Two management panels estimate each team's *performance* (hyper-membership) and two HR committees estimate *conflict risk* (hyper-nonmembership). Define

$$\widetilde{\mu}_{3,2}(T) = \{\, M_1(T),\ M_2(T)\}, \quad \widetilde{\nu}_{3,2}(T) = \{\, V_1(T),\ V_2(T)\}, \quad T \in \mathcal{P}_3^*(U),$$

where each $M_i(T), V_j(T) \subseteq [0, 1]$.

Team $\{E_1, E_2, E_3\}$**:**

$$M_1(\{E_1, E_2, E_3\}) = \{0.80,\ 0.82\}, \quad V_1(\{E_1, E_2, E_3\}) = \{0.10,\ 0.12\},$$
$$M_2(\{E_1, E_2, E_3\}) = \{0.78,\ 0.80\}, \quad V_2(\{E_1, E_2, E_3\}) = \{0.15,\ 0.18\}.$$

For any $u \in M_i(T)$, $v \in V_j(T)$, and $s \in \operatorname{flat}_2(u) = u$, $t \in \operatorname{flat}_2(v) = v$, the Linear Diophantine condition

$$0 \ \leq\ 0.5\, s + 0.4\, t \ \leq\ 1$$

holds. For example, $s = 0.82$, $t = 0.18$ gives

$$0.5 \cdot 0.82 + 0.4 \cdot 0.18 = 0.41 + 0.072 = 0.482 \in [0, 1].$$

The hyper-hesitation set is

$$\widetilde{\pi}_{3,2}(\{E_1, E_2, E_3\}) = \{\, 1 - (0.5\, s + 0.4\, t) \mid s \in M_i,\ t \in V_j\},$$

which concretely includes values like $1 - (0.5 \cdot 0.82 + 0.4 \cdot 0.18) = 0.518$.

Team $\{E_1, E_2, E_4\}$**:**

$$M_1(\{E_1, E_2, E_4\}) = \{0.75,\ 0.78\}, \quad V_1(\{E_1, E_2, E_4\}) = \{0.12,\ 0.14\},$$
$$M_2(\{E_1, E_2, E_4\}) = \{0.73,\ 0.76\}, \quad V_2(\{E_1, E_2, E_4\}) = \{0.16,\ 0.20\}.$$

Team $\{E_1, E_3, E_4\}$**:**

$$M_1(\{E_1, E_3, E_4\}) = \{0.82,\ 0.85\}, \quad V_1(\{E_1, E_3, E_4\}) = \{0.08,\ 0.10\},$$
$$M_2(\{E_1, E_3, E_4\}) = \{0.80,\ 0.83\}, \quad V_2(\{E_1, E_3, E_4\}) = \{0.13,\ 0.17\}.$$

Team $\{E_2, E_3, E_4\}$**:**

$$M_1(\{E_2, E_3, E_4\}) = \{0.70,\ 0.72\}, \quad V_1(\{E_2, E_3, E_4\}) = \{0.18,\ 0.22\},$$
$$M_2(\{E_2, E_3, E_4\}) = \{0.68,\ 0.71\}, \quad V_2(\{E_2, E_3, E_4\}) = \{0.20,\ 0.24\}.$$

Thus

$$(\widetilde{\mu}_{3,2}, \widetilde{\nu}_{3,2})$$

defines a $(3, 2)$-Linear Diophantine SuperHyperfuzzy Set modeling hierarchical team-performance and conflict-risk estimates under weighted Diophantine constraints.

Theorem 3.10 (Generalization of LDFS, LDHFS, and (m,n)-SHF Set)
Let $\widetilde{D} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set on U. Then:

(i) *(LDFS)* If $m = n = 0$ and for each $x \in U$, both $\widetilde{\mu}_{0,0}(x)$ and $\widetilde{\nu}_{0,0}(x)$ are singletons $\{A_D(x)\}$, $\{S_D(x)\}$, then $\widetilde{D}$ reduces to the Linear Diophantine Fuzzy Set of Definition 3.5.

(ii) *(LDHFS)* If $m = n = 0$, $\alpha = 1$, $\beta = 0$, and $\widetilde{\nu}_{0,0}(x) = \{0\}$ for all $x \in U$, then the essential data of $\widetilde{D}$ is exactly a HyperFuzzy Set as in Definition 2.3.

(iii) *((m,n)-SHF Set)* If $\beta = 0$ and for each $A \in \mathcal{P}^*_m(U)$, the set $\widetilde{\nu}_{m,n}(A)$ is chosen to be the trivial n-fold nested zero $\{\{\ldots\{0\}\ldots\}\}$, then the only nontrivial mapping is $\widetilde{\mu}_{m,n}$, so $\widetilde{D}$ reduces to the (m,n)-SuperHyperfuzzy Set of Definition 2.5.

Proof
Recall that, by the definition of an (m,n)-Linear Diophantine SuperHyperfuzzy Set, we have

$$\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n} : \mathcal{P}^*_m(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\},$$

and for every $A \in \mathcal{P}^*_m(U)$, for every

$$u \in \widetilde{\mu}_{m,n}(A), \quad v \in \widetilde{\nu}_{m,n}(A),$$

and every

$$s \in \flat_n(u), \quad t \in \flat_n(v),$$

the Linear Diophantine inequality

$$0 \le \alpha s + \beta t \le 1 \tag{1}$$

holds. Here $\flat_n : \mathcal{P}_n([0,1]) \to \mathcal{P}([0,1])$ is the flattening operator defined recursively by

$$\flat_0(x) = \{x\}, \qquad \flat_{k+1}(S) = \bigcup_{T \in S} \flat_k(T).$$

We treat each item separately.

(i) LDFS case.

Assume $m = n = 0$ and that for each $x \in U$ the images are singletons

$$\widetilde{\mu}_{0,0}(x) = \{A_D(x)\}, \qquad \widetilde{\nu}_{0,0}(x) = \{S_D(x)\}.$$

By the convention for iterated nonempty powersets, when $m = 0$ we have

$$\mathcal{P}^*_0(U) = U, \qquad \mathcal{P}^*_0([0,1]) = [0,1],$$

and for the flattening operator

$$\flat_0(z) = \{z\} \quad \text{for every } z \in [0,1].$$

Fix any $x \in U$. By assumption there is a unique

$$u \in \widetilde{\mu}_{0,0}(x) \quad \text{and} \quad v \in \widetilde{\nu}_{0,0}(x),$$

namely $u = A_D(x)$, $v = S_D(x)$. The only possible choices of $(s,t) \in \flat_0(u) \times \flat_0(v)$ are

$$s = u = A_D(x), \qquad t = v = S_D(x),$$

because

$$\flat_0(u) = \{u\}, \quad \flat_0(v) = \{v\}.$$

Substituting these into the Diophantine constraint (1), we obtain

$$0 \;\le\; \alpha s + \beta t \;\le\; 1 \quad\Longleftrightarrow\quad 0 \;\le\; \alpha A_D(x) + \beta S_D(x) \;\le\; 1.$$

Thus the pair of functions

$$A_D, S_D : U \longrightarrow [0,1]$$

satisfies the basic LDFS feasibility condition

$$0 \le \alpha A_D(x) + \beta S_D(x) \le 1 \quad \text{for all } x \in U.$$

Next, the associated hyper-hesitation mapping in the $(0,0)$-case is

$$\widetilde{\pi}_{0,0}(x) = \big\{1 - (\alpha s + \beta t) \,\big|\, u \in \widetilde{\mu}_{0,0}(x),\, v \in \widetilde{\nu}_{0,0}(x),\, s \in \flat_0(u),\, t \in \flat_0(v)\big\}.$$

Again there is only one admissible quadruple

$$(u,v,s,t) = (A_D(x), S_D(x), A_D(x), S_D(x)),$$

hence

$$\widetilde{\pi}_{0,0}(x) = \big\{1 - (\alpha A_D(x) + \beta S_D(x))\big\}.$$

If we denote the (scalar) hesitation degree by

$$\pi_D(x) := 1 - (\alpha A_D(x) + \beta S_D(x)),$$

then the data

$$\big\{(x, \langle A_D(x), S_D(x)\rangle, (\alpha,\beta), \pi_D(x)) \mid x \in U\big\}$$

coincide with the Linear Diophantine Fuzzy Set of Definition 2.1 (up to the harmless identification of the universe with U). Therefore $\widetilde{D}$ reduces to an LDFS, proving (i).

(ii) LDHFS / HyperFuzzy case.

Assume now $m = n = 0$, $\alpha = 1$, $\beta = 0$, and

$$\widetilde{\nu}_{0,0}(x) = \{0\} \quad \text{for all } x \in U.$$

Again $\mathcal{P}_0^*(U) = U$, $\mathcal{P}_0^*([0,1]) = [0,1]$ and $\flat_0(z) = \{z\}$.

Fix $x \in U$ and take any $u \in \widetilde{\mu}_{0,0}(x)$, so $u \in [0,1]$. Because $\widetilde{\nu}_{0,0}(x) = \{0\}$, its unique element is

$$v = 0 \in [0,1].$$

The only choices for s and t in $\flat_0(u) \times \flat_0(v)$ are

$$s = u, \qquad t = v = 0.$$

Plugging these into (1) with $\alpha = 1$, $\beta = 0$, we get

$$0 \le \alpha s + \beta t \le 1 \iff 0 \le 1 \cdot u + 0 \cdot 0 \le 1,$$

that is,

$$0 \le u \le 1.$$

Since this holds for every $u \in \widetilde{\mu}_{0,0}(x)$, we conclude that

$$\widetilde{\mu}_{0,0}(x) \subseteq [0,1] \quad \text{and} \quad \widetilde{\mu}_{0,0}(x) \neq \emptyset$$

for all $x \in U$. Therefore the mapping

$$x \longmapsto \widetilde{\mu}_{0,0}(x) \in \mathcal{P}([0,1]) \setminus \{\emptyset\}$$

is exactly a hyper-membership function in the sense of Definition 2.3, i.e. it defines a HyperFuzzy Set on U.

The nonmembership sets are fixed as

$$\widetilde{\nu}_{0,0}(x) = \{0\},$$

and the induced hyper-hesitation sets are

$$\widetilde{\pi}_{0,0}(x) = \big\{1 - (\alpha s + \beta t) \mid s \in \widetilde{\mu}_{0,0}(x),\, t \in \flat_0(0)\big\} = \big\{1 - s \mid s \in \widetilde{\mu}_{0,0}(x)\big\},$$

which are completely determined by $\widetilde{\mu}_{0,0}$ and contain no additional independent information.

Hence, up to forgetting these trivial nonmembership and hesitation components, the data of $\widetilde{D}$ coincide with the HyperFuzzy Set given by the mapping

$$x \longmapsto \widetilde{\mu}_{0,0}(x).$$

This establishes that, in this parameter regime, $\widetilde{D}$ corresponds exactly to a HyperFuzzy Set as in Definition 2.3, proving (ii).

(iii) (m,n)-SHF case.

Now let $m, n \ge 0$ be arbitrary and assume $\beta = 0$. Suppose moreover that for each $A \in \mathcal{P}^*_m(U)$ the hyper-nonmembership value $\widetilde{\nu}_{m,n}(A)$ is the "trivial n-fold nested zero"

$$\widetilde{\nu}_{m,n}(A) = \{z_n\},$$

where we define

$$z_0 := 0 \in [0,1], \qquad z_{k+1} := \{z_k\} \quad (k = 0, 1, \dots, n-1).$$

By construction, $z_n \in \mathcal{P}_n([0,1])$.

We first show that every element of $\flat_n(z_n)$ is equal to 0. We proceed by induction on n.

Base case $n = 0$. Here $z_0 = 0 \in [0,1]$ and by definition

$$\flat_0(z_0) = \{z_0\} = \{0\}.$$

Induction step. Assume for some $k \ge 0$ that

$$\flat_k(z_k) = \{0\}.$$

For $n = k+1$ we have $z_{k+1} = \{z_k\}$, and therefore

$$\flat_{k+1}(z_{k+1}) = \flat_{k+1}(\{z_k\}) = \bigcup_{T \in \{z_k\}} \flat_k(T) = \flat_k(z_k) = \{0\}.$$

Thus $\flat_{k+1}(z_{k+1}) = \{0\}$, completing the induction.

Consequently, for our fixed n we have

$$\flat_n(z_n) = \{0\},$$

so any $t \in \flat_n(v)$ with $v = z_n$ must satisfy $t = 0$.

Now fix an arbitrary $A \in \mathcal{P}_m^*(U)$ and choose any

$$u \in \widetilde{\mu}_{m,n}(A), \quad v \in \widetilde{\nu}_{m,n}(A) = \{z_n\},$$

so $v = z_n$. Let $s \in \flat_n(u)$ (so $s \in [0,1]$ by definition of $\flat_n$) and let $t \in \flat_n(v) = \flat_n(z_n)$. By the previous argument, $t = 0$.

The Linear Diophantine constraint (1) with $\beta = 0$ thus becomes

$$0 \le \alpha s + \beta t \le 1 \iff 0 \le \alpha s + 0 \cdot 0 \le 1,$$

that is,

$$0 \le \alpha s \le 1 \quad \text{for every } s \in \flat_n(u),\ u \in \widetilde{\mu}_{m,n}(A),\ A \in \mathcal{P}_m^*(U).$$

Since $s \in [0,1]$ and $\alpha \in [0,1]$, the inequality $0 \le \alpha s \le 1$ is automatically satisfied for all admissible s. Therefore the nonmembership component $\widetilde{\nu}_{m,n}$ (fixed as the nested zero) does not impose any additional restriction beyond the trivial requirement that its bottom-level value is 0. All nontrivial, freely chosen hierarchical uncertainty information resides in the hyper-membership mapping

$$\widetilde{\mu}_{m,n} : \mathcal{P}_m^*(U) \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}.$$

But this is precisely the data of an (m,n)-SuperHyperfuzzy Set as in Definition 2.5, where each $A \in \mathcal{P}_m^*(U)$ is assigned a nonempty family of n-level membership-value sets in $[0,1]$. Hence, after disregarding the trivial constant nonmembership part $\widetilde{\nu}_{m,n}$ and its induced hesitation, $\widetilde{D}$ coincides with the (m,n)-SuperHyperfuzzy Set defined by $\widetilde{\mu}_{m,n}$.

This proves (iii) and completes the proof of the theorem. □

Theorem 3.11 (Nonemptiness and boundedness of hesitation degrees)
Let $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set on a nonempty universe U with parameters $\alpha, \beta \in [0,1]$ and $\alpha + \beta \le 1$. For every $A \in \mathcal{P}_m^*(U)$, the associated hesitation set

$$\widetilde{\pi}_{m,n}(A) = \big\{\, 1 - (\alpha s + \beta t) \mid u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v) \big\}$$

is nonempty and satisfies $\widetilde{\pi}_{m,n}(A) \subseteq [0,1]$.

Proof
Fix $A \in \mathcal{P}_m^*(U)$. By the definition of (m,n)-Linear Diophantine SuperHyperfuzzy Set, both $\widetilde{\mu}_{m,n}(A)$ and $\widetilde{\nu}_{m,n}(A)$ are nonempty. We first show that for any $u \in \mathcal{P}_n([0,1])$,

$$\mathrm{flat}_n(u) \subseteq [0,1] \quad \text{and} \quad \mathrm{flat}_n(u) \neq \emptyset.$$

We argue by induction on n.

- Base case $n = 0$: By definition, $\mathrm{flat}_0(x) = \{x\}$ for $x \in [0,1]$. Thus $\mathrm{flat}_0(x) \subseteq [0,1]$ and is clearly nonempty.

- Induction step: Assume the claim holds for some $n \geq 0$. Let $u \in \mathcal{P}_{n+1}([0,1])$. By definition,

$$\mathrm{flat}_{n+1}(u) = \bigcup_{T \in u} \mathrm{flat}_n(T).$$

Each $T \in u$ lies in $\mathcal{P}_n([0,1])$, so by the induction hypothesis $\mathrm{flat}_n(T) \subseteq [0,1]$ and $\mathrm{flat}_n(T) \neq \emptyset$. Hence

$$\mathrm{flat}_{n+1}(u) = \bigcup_{T \in u} \mathrm{flat}_n(T) \subseteq [0,1],$$

and since $u \neq \emptyset$, the union over at least one nonempty $\mathrm{flat}_n(T)$ is nonempty. Thus the claim holds for $n+1$.

By induction, for any $u \in \mathcal{P}_n([0,1])$ and $v \in \mathcal{P}_n([0,1])$, we have

$$\emptyset \neq \mathrm{flat}_n(u) \subseteq [0,1], \qquad \emptyset \neq \mathrm{flat}_n(v) \subseteq [0,1].$$

Now fix any $u_0 \in \widetilde{\mu}_{m,n}(A)$ and $v_0 \in \widetilde{\nu}_{m,n}(A)$. Choose $s_0 \in \mathrm{flat}_n(u_0)$ and $t_0 \in \mathrm{flat}_n(v_0)$; such choices are possible by the nonemptiness just shown. By the Linear Diophantine condition in the definition of (m,n)-Linear Diophantine SuperHyperfuzzy Set,

$$0 \ \leq\ \alpha s_0 + \beta t_0 \ \leq\ 1.$$

Define

$$h_0 := 1 - (\alpha s_0 + \beta t_0).$$

From $0 \leq \alpha s_0 + \beta t_0 \leq 1$ we obtain

$$1 - 1 \ \leq\ 1 - (\alpha s_0 + \beta t_0) \ \leq\ 1 - 0,$$

that is,

$$0 \ \leq\ h_0 \ \leq\ 1.$$

Hence $h_0 \in [0,1]$. By definition of $\widetilde{\pi}_{m,n}(A)$, this h_0 is an element of $\widetilde{\pi}_{m,n}(A)$:

$$h_0 = 1 - (\alpha s_0 + \beta t_0) \in \widetilde{\pi}_{m,n}(A),$$

so $\widetilde{\pi}_{m,n}(A)$ is nonempty. In addition, every $h \in \widetilde{\pi}_{m,n}(A)$ has the form $h = 1 - (\alpha s + \beta t)$ with $0 \leq \alpha s + \beta t \leq 1$, and the same inequality as above gives $h \in [0,1]$. Thus $\widetilde{\pi}_{m,n}(A) \subseteq [0,1]$, as required. □

Theorem 3.12 (Monotonicity with respect to hyper-membership and hyper-nonmembership)
Let

$$(\widetilde{\mu}^{(1)}_{m,n}, \widetilde{\nu}^{(1)}_{m,n})$$

and

$$(\widetilde{\mu}^{(2)}_{m,n}, \widetilde{\nu}^{(2)}_{m,n})$$

be two (m,n)-Linear Diophantine SuperHyperfuzzy Sets on the same universe U with the same parameters α, β. Assume that for every $A \in \mathcal{P}^*_m(U)$,

$$\widetilde{\mu}^{(1)}_{m,n}(A) \ \subseteq\ \widetilde{\mu}^{(2)}_{m,n}(A), \qquad \widetilde{\nu}^{(1)}_{m,n}(A) \ \subseteq\ \widetilde{\nu}^{(2)}_{m,n}(A).$$

Let $\widetilde{\pi}^{(1)}_{m,n}$ and $\widetilde{\pi}^{(2)}_{m,n}$ be the corresponding hesitation mappings. Then for all $A \in \mathcal{P}^*_m(U)$,

$$\widetilde{\pi}^{(1)}_{m,n}(A) \ \subseteq\ \widetilde{\pi}^{(2)}_{m,n}(A).$$

Proof
Fix $A \in \mathcal{P}_m^*(U)$ and let $h \in \widetilde{\pi}^{(1)}_{m,n}(A)$. By definition, there exist

$$u \in \widetilde{\mu}^{(1)}_{m,n}(A), \quad v \in \widetilde{\nu}^{(1)}_{m,n}(A), \quad s \in \mathrm{flat}_n(u), \quad t \in \mathrm{flat}_n(v)$$

such that

$$h = 1 - (\alpha s + \beta t).$$

Since $\widetilde{\mu}^{(1)}_{m,n}(A) \subseteq \widetilde{\mu}^{(2)}_{m,n}(A)$ and $\widetilde{\nu}^{(1)}_{m,n}(A) \subseteq \widetilde{\nu}^{(2)}_{m,n}(A)$, the same u and v also satisfy

$$u \in \widetilde{\mu}^{(2)}_{m,n}(A), \qquad v \in \widetilde{\nu}^{(2)}_{m,n}(A).$$

The flattening operator flat_n depends only on u and v, so $s \in \mathrm{flat}_n(u)$ and $t \in \mathrm{flat}_n(v)$ remain valid. Thus, in the definition of $\widetilde{\pi}^{(2)}_{m,n}(A)$, the same quadruple (u, v, s, t) contributes the value $1 - (\alpha s + \beta t) = h$. Hence $h \in \widetilde{\pi}^{(2)}_{m,n}(A)$. Since h was arbitrary in $\widetilde{\pi}^{(1)}_{m,n}(A)$, we conclude $\widetilde{\pi}^{(1)}_{m,n}(A) \subseteq \widetilde{\pi}^{(2)}_{m,n}(A)$. □

Theorem 3.13 (Stability under intersection)
Let $(\widetilde{\mu}^{(1)}_{m,n}, \widetilde{\nu}^{(1)}_{m,n})$ and $(\widetilde{\mu}^{(2)}_{m,n}, \widetilde{\nu}^{(2)}_{m,n})$ be two (m, n)-Linear Diophantine SuperHyperfuzzy Sets on U with the same parameters α, β. Assume that for every $A \in \mathcal{P}_m^*(U)$ the intersections

$$\widetilde{\mu}^{(\cap)}_{m,n}(A) := \widetilde{\mu}^{(1)}_{m,n}(A) \cap \widetilde{\mu}^{(2)}_{m,n}(A), \qquad \widetilde{\nu}^{(\cap)}_{m,n}(A) := \widetilde{\nu}^{(1)}_{m,n}(A) \cap \widetilde{\nu}^{(2)}_{m,n}(A)$$

are nonempty. Then:

(i) The pair $(\widetilde{\mu}^{(\cap)}_{m,n}, \widetilde{\nu}^{(\cap)}_{m,n})$ is again an (m, n)-Linear Diophantine SuperHyperfuzzy Set on U with parameters (α, β).

(ii) If $\widetilde{\pi}^{(1)}_{m,n}$, $\widetilde{\pi}^{(2)}_{m,n}$ and $\widetilde{\pi}^{(\cap)}_{m,n}$ denote the corresponding hesitation mappings, then

$$\widetilde{\pi}^{(\cap)}_{m,n}(A) \ \subseteq\ \widetilde{\pi}^{(1)}_{m,n}(A) \cap \widetilde{\pi}^{(2)}_{m,n}(A) \quad \text{for all } A \in \mathcal{P}_m^*(U).$$

Proof
(i) Fix $A \in \mathcal{P}_m^*(U)$ and take any

$$u \in \widetilde{\mu}^{(\cap)}_{m,n}(A), \qquad v \in \widetilde{\nu}^{(\cap)}_{m,n}(A).$$

By definition of the intersection,

$$u \in \widetilde{\mu}^{(1)}_{m,n}(A) \cap \widetilde{\mu}^{(2)}_{m,n}(A), \quad v \in \widetilde{\nu}^{(1)}_{m,n}(A) \cap \widetilde{\nu}^{(2)}_{m,n}(A),$$

so

$$u \in \widetilde{\mu}^{(1)}_{m,n}(A), \quad u \in \widetilde{\mu}^{(2)}_{m,n}(A), \qquad v \in \widetilde{\nu}^{(1)}_{m,n}(A), \quad v \in \widetilde{\nu}^{(2)}_{m,n}(A).$$

Let $s \in \mathrm{flat}_n(u)$ and $t \in \mathrm{flat}_n(v)$. Because each of $(\widetilde{\mu}^{(1)}_{m,n}, \widetilde{\nu}^{(1)}_{m,n})$ and $(\widetilde{\mu}^{(2)}_{m,n}, \widetilde{\nu}^{(2)}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperfuzzy Set with parameters (α, β), we have

$$0 \ \le\ \alpha s + \beta t \ \le\ 1$$

from both definitions. Thus the Diophantine condition holds for all choices $u \in \widetilde{\mu}^{(\cap)}_{m,n}(A)$, $v \in \widetilde{\nu}^{(\cap)}_{m,n}(A)$, $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$, and hence $(\widetilde{\mu}^{(\cap)}_{m,n}, \widetilde{\nu}^{(\cap)}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperfuzzy Set.

(ii) Let $h \in \widetilde{\pi}^{(\cap)}_{m,n}(A)$. Then there exist

$$u \in \widetilde{\mu}^{(\cap)}_{m,n}(A),\ v \in \widetilde{\nu}^{(\cap)}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)$$

such that

$$h = 1 - (\alpha s + \beta t).$$

As in (i), u and v belong simultaneously to the corresponding sets of both original structures. Therefore the same quadruple (u, v, s, t) is admissible in the definitions of $\widetilde{\pi}^{(1)}_{m,n}(A)$ and $\widetilde{\pi}^{(2)}_{m,n}(A)$, so

$$h \in \widetilde{\pi}^{(1)}_{m,n}(A) \quad \text{and} \quad h \in \widetilde{\pi}^{(2)}_{m,n}(A).$$

Hence $h \in \widetilde{\pi}^{(1)}_{m,n}(A) \cap \widetilde{\pi}^{(2)}_{m,n}(A)$, and the inclusion $\widetilde{\pi}^{(\cap)}_{m,n}(A) \subseteq \widetilde{\pi}^{(1)}_{m,n}(A) \cap \widetilde{\pi}^{(2)}_{m,n}(A)$ follows. □

Theorem 3.14 (Scaling of the Diophantine parameters)
Let $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m, n)-Linear Diophantine SuperHyperfuzzy Set on U with parameters $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Let $\lambda \in [0, 1]$ and define

$$\alpha' := \lambda\alpha, \qquad \beta' := \lambda\beta.$$

Then $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ is also an (m, n)-Linear Diophantine SuperHyperfuzzy Set on U with parameters (α', β').

Proof
Fix $A \in \mathcal{P}^*_m(U)$ and choose

$$u \in \widetilde{\mu}_{m,n}(A), \qquad v \in \widetilde{\nu}_{m,n}(A), \qquad s \in \mathrm{flat}_n(u), \qquad t \in \mathrm{flat}_n(v).$$

Since $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperfuzzy Set with parameters (α, β), we have

$$0 \leq \alpha s + \beta t \leq 1.$$

Multiplying this inequality by $\lambda \in [0, 1]$ gives

$$0 \leq \lambda(\alpha s + \beta t) = (\lambda\alpha)s + (\lambda\beta)t = \alpha' s + \beta' t \leq \lambda \cdot 1 \leq 1.$$

Hence, for every admissible choice of u, v, s, t, the Diophantine condition

$$0 \leq \alpha' s + \beta' t \leq 1$$

holds. Therefore the same pair $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ satisfies the definition of an (m, n)-Linear Diophantine SuperHyperfuzzy Set with parameters (α', β'). □

Theorem 3.15 (Monotonicity of hesitation with respect to the parameters)
Fix $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ on U. Let (α_1, β_1) and (α_2, β_2) be two pairs of parameters in $[0, 1]^2$ such that

$$0 \leq \alpha_1 \leq \alpha_2, \quad 0 \leq \beta_1 \leq \beta_2, \quad \alpha_i + \beta_i \leq 1 \quad (i = 1, 2).$$

Assume $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperfuzzy Set for both (α_1, β_1) and (α_2, β_2). Let $\widetilde{\pi}^{(i)}_{m,n}$ be the corresponding hesitation mappings. Then for every $A \in \mathcal{P}^*_m(U)$ and every admissible choice $u \in \widetilde{\mu}_{m,n}(A)$, $v \in \widetilde{\nu}_{m,n}(A)$, $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$, we have

$$1 - (\alpha_2 s + \beta_2 t) \leq 1 - (\alpha_1 s + \beta_1 t),$$

that is, each hesitation degree produced with (α_2, β_2) is less than or equal to the corresponding degree produced with (α_1, β_1).

Proof
Fix $A \in \mathcal{P}^*_m(U)$ and choose any admissible

$$u \in \widetilde{\mu}_{m,n}(A), \quad v \in \widetilde{\nu}_{m,n}(A), \quad s \in \mathrm{flat}_n(u), \quad t \in \mathrm{flat}_n(v).$$

Since $s, t \in [0,1]$ (by the argument in the proof of Theorem 3.11), we have

$$\alpha_1 s \ \le\ \alpha_2 s, \qquad \beta_1 t \ \le\ \beta_2 t,$$

because $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$ and $s, t \ge 0$. Adding these inequalities gives

$$\alpha_1 s + \beta_1 t \ \le\ \alpha_2 s + \beta_2 t.$$

Define

$$h_1 := 1 - (\alpha_1 s + \beta_1 t), \qquad h_2 := 1 - (\alpha_2 s + \beta_2 t).$$

From $\alpha_1 s + \beta_1 t \le \alpha_2 s + \beta_2 t$ we obtain

$$-(\alpha_1 s + \beta_1 t) \ \ge\ -(\alpha_2 s + \beta_2 t),$$

and by adding 1 to both sides,

$$1 - (\alpha_1 s + \beta_1 t) \ \ge\ 1 - (\alpha_2 s + \beta_2 t),$$

that is,

$$h_1 \ \ge\ h_2.$$

Thus, for each admissible pair (s, t), the hesitation degree h_2 obtained with (α_2, β_2) is less than or equal to the degree h_1 obtained with (α_1, β_1). Since the sets $\widetilde{\pi}^{(1)}_{m,n}(A)$ and $\widetilde{\pi}^{(2)}_{m,n}(A)$ are generated by all such pairs (s, t), the claimed pointwise monotonicity holds for every element arising from the same choice of u, v, s, t. □

3.3. *Basic operations for (m, n)-Linear Diophantine SuperHyperfuzzy Sets*

In this subsection we introduce three basic tools for (m, n)-Linear Diophantine SuperHyperfuzzy Sets: an aggregation operator, a score function, and a distance measure. Throughout, we assume that U is finite and that $(\alpha, \beta) \in [0,1]^2$ satisfies $\alpha + \beta > 0$ and $\alpha + \beta \le 1$. We use Definition 3.5 and the flattening operator flat_n introduced above.

Definition 3.16 (Union-type aggregation of (m, n)-LD SuperHyperfuzzy Sets)**.** Let $\{\mathcal{F}_j\}_{j \in J}$ be a nonempty family of (m, n)-Linear Diophantine SuperHyperfuzzy Sets on the same universe U, where

$$\mathcal{F}_j = \big(\widetilde{\mu}^{(j)}_{m,n}, \widetilde{\nu}^{(j)}_{m,n}\big) \quad (j \in J).$$

Define the aggregated pair $\mathcal{F}^{\bigvee} = (\widetilde{\mu}^{\bigvee}_{m,n}, \widetilde{\nu}^{\bigvee}_{m,n})$ by

$$\widetilde{\mu}^{\bigvee}_{m,n}(A) = \bigcup_{j \in J} \widetilde{\mu}^{(j)}_{m,n}(A), \qquad \widetilde{\nu}^{\bigvee}_{m,n}(A) = \bigcup_{j \in J} \widetilde{\nu}^{(j)}_{m,n}(A)$$

for all $A \in \mathcal{P}^*_m(U)$.

Theorem 3.17 (Closure under union-type aggregation)
The aggregated pair $\mathcal{F}^{\bigvee} = (\widetilde{\mu}^{\bigvee}_{m,n}, \widetilde{\nu}^{\bigvee}_{m,n})$ is again an (m, n)-Linear Diophantine SuperHyperfuzzy Set on U with the same weights (α, β).

Proof
First, for each $A \in \mathcal{P}_m^*(U)$, every $\widetilde{\mu}^{(j)}_{m,n}(A)$ and $\widetilde{\nu}^{(j)}_{m,n}(A)$ is a nonempty subset of $\mathcal{P}_n([0,1])$ by Definition 3.5. Hence their unions $\widetilde{\mu}^{\bigvee}_{m,n}(A)$ and $\widetilde{\nu}^{\bigvee}_{m,n}(A)$ are also nonempty subsets of $\mathcal{P}_n([0,1])$. Thus

$$\widetilde{\mu}^{\bigvee}_{m,n}, \widetilde{\nu}^{\bigvee}_{m,n} \,:\, \mathcal{P}_m^*(U) \to \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}$$

are well-defined.

Next, fix $A \in \mathcal{P}_m^*(U)$ and take any

$$u \in \widetilde{\mu}^{\bigvee}_{m,n}(A), \quad v \in \widetilde{\nu}^{\bigvee}_{m,n}(A), \quad s \in \mathrm{flat}_n(u), \quad t \in \mathrm{flat}_n(v).$$

By definition of the union, there exist indices $j_1, j_2 \in J$ such that $u \in \widetilde{\mu}^{(j_1)}_{m,n}(A)$ and $v \in \widetilde{\nu}^{(j_2)}_{m,n}(A)$. Since every $\mathcal{F}_j$ is an (m,n)-Linear Diophantine SuperHyperfuzzy Set with the same (α, β), we know that

$$0 \le \alpha s + \beta t \le 1$$

for every $s \in \mathrm{flat}_n(u)$ and $t \in \mathrm{flat}_n(v)$. In particular, the above inequality holds for our chosen s and t. Therefore the linear Diophantine condition in Definition 3.5 is satisfied for all admissible tuples (A, u, v, s, t) in the aggregated pair.

Finally, the associated hyper-hesitation set

$$\widetilde{\pi}^{\bigvee}_{m,n}(A) = \big\{1 - (\alpha s + \beta t) \mid u \in \widetilde{\mu}^{\bigvee}_{m,n}(A),\ v \in \widetilde{\nu}^{\bigvee}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\}$$

is well-defined and takes values in $[0,1]$ because $0 \le \alpha s + \beta t \le 1$. Thus $\mathcal{F}^{\bigvee}$ satisfies all requirements of Definition 3.5. □

Definition 3.18 (Flattened Diophantine profile set)**.** Let $\mathcal{F} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set on U. For each $A \in \mathcal{P}_m^*(U)$, define the *Diophantine profile set*

$$\Gamma_{\mathcal{F}}(A) = \big\{(s,t) \in [0,1]^2 \,\big|\, u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\}.$$

By Definition 3.5 and the fact that U is finite, each $\Gamma_{\mathcal{F}}(A)$ is a nonempty finite subset of $[0,1]^2$, and every $(s,t) \in \Gamma_{\mathcal{F}}(A)$ satisfies $0 \le \alpha s + \beta t \le 1$.

Definition 3.19 (Score function for (m,n)-LD SuperHyperfuzzy Sets)**.** Let $\mathcal{F} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set. For each $A \in \mathcal{P}_m^*(U)$ we define its *score* as

$$S_{\mathcal{F}}(A) = \max_{(s,t) \in \Gamma_{\mathcal{F}}(A)} \frac{\alpha s - \beta t + \beta}{\alpha + \beta}.$$

Theorem 3.20 (Basic properties of the score function)
Let $\mathcal{F}$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set.

1. For every $A \in \mathcal{P}_m^*(U)$, the score $S_{\mathcal{F}}(A)$ is well-defined and lies in $[0,1]$.
2. If the hyper-membership and hyper-nonmembership sets collapse to single points, i.e. for all $A \in \mathcal{P}_m^*(U)$ there exist $s_A, t_A \in [0,1]$ such that

$$\widetilde{\mu}_{m,n}(A) = \{\{s_A\}\}, \qquad \widetilde{\nu}_{m,n}(A) = \{\{t_A\}\},$$

then

$$S_{\mathcal{F}}(A) = \frac{\alpha s_A - \beta t_A + \beta}{\alpha + \beta},$$

which coincides with the natural Linear Diophantine score of the underlying single-valued pair (s_A, t_A).

3. If $(s_1, t_1), (s_2, t_2) \in [0,1]^2$ satisfy $s_1 \le s_2$ and $t_1 \ge t_2$, then
$$\frac{\alpha s_1 - \beta t_1 + \beta}{\alpha + \beta} \;\le\; \frac{\alpha s_2 - \beta t_2 + \beta}{\alpha + \beta}.$$
Hence, increasing membership and decreasing nonmembership never lowers the score.

Proof
(1) Fix $A \in \mathcal{P}_m^*(U)$. Since U is finite and each $\widetilde{\mu}_{m,n}(A)$ and $\widetilde{\nu}_{m,n}(A)$ is nonempty and finite, $\Gamma_{\mathcal{F}}(A)$ is a nonempty finite set. Therefore the maximum in the definition of $S_{\mathcal{F}}(A)$ exists.

Next, for any $(s,t) \in [0,1]^2$ we have
$$-\beta \;\le\; \alpha s - \beta t \;\le\; \alpha.$$
Indeed,
$$\alpha s - \beta t \ge \alpha \cdot 0 - \beta \cdot 1 = -\beta, \qquad \alpha s - \beta t \le \alpha \cdot 1 - \beta \cdot 0 = \alpha.$$
Adding β yields
$$0 \;\le\; \alpha s - \beta t + \beta \;\le\; \alpha + \beta.$$
Since $\alpha + \beta > 0$, dividing by $\alpha + \beta$ gives
$$0 \;\le\; \frac{\alpha s - \beta t + \beta}{\alpha + \beta} \;\le\; 1.$$
Therefore every candidate value inside the maximum lies in $[0,1]$; hence $S_{\mathcal{F}}(A) \in [0,1]$.

(2) If for each A we have $\widetilde{\mu}_{m,n}(A) = \{\{s_A\}\}$ and $\widetilde{\nu}_{m,n}(A) = \{\{t_A\}\}$, then
$$\Gamma_{\mathcal{F}}(A) = \big\{(s_A, t_A)\big\},$$
because $\mathrm{flat}_n(\{s_A\}) = \{s_A\}$ and similarly for t_A. Thus
$$S_{\mathcal{F}}(A) = \frac{\alpha s_A - \beta t_A + \beta}{\alpha + \beta},$$
which is exactly the claimed expression.

(3) Suppose $(s_1, t_1), (s_2, t_2) \in [0,1]^2$ with $s_1 \le s_2$ and $t_1 \ge t_2$. Then
$$\alpha s_1 \;\le\; \alpha s_2 \quad \text{and} \quad -\beta t_1 \;\le\; -\beta t_2,$$
because $\alpha, \beta \ge 0$. Adding the two inequalities gives
$$\alpha s_1 - \beta t_1 \;\le\; \alpha s_2 - \beta t_2.$$
Adding β to both sides preserves the inequality, and dividing by the positive constant $\alpha + \beta$ also does not change the order. Hence
$$\frac{\alpha s_1 - \beta t_1 + \beta}{\alpha + \beta} \;\le\; \frac{\alpha s_2 - \beta t_2 + \beta}{\alpha + \beta},$$
which proves the monotonicity claim. □

The score $S_{\mathcal{F}}(A)$ can thus be used as a crisp evaluation index for each $A \in \mathcal{P}_m^*(U)$, summarizing the hierarchical Diophantine profiles $(s,t) \in \Gamma_{\mathcal{F}}(A)$ into a single number in $[0,1]$.

Definition 3.21 (Distance between (m,n)-LD SuperHyperfuzzy Sets)**.** Let $\mathcal{F}_1 = (\widetilde{\mu}^{(1)}_{m,n}, \widetilde{\nu}^{(1)}_{m,n})$ and $\mathcal{F}_2 = (\widetilde{\mu}^{(2)}_{m,n}, \widetilde{\nu}^{(2)}_{m,n})$ be two (m,n)-Linear Diophantine SuperHyperfuzzy Sets on the same finite universe U, and write $S_{\mathcal{F}_1}$ and $S_{\mathcal{F}_2}$ for their score functions. We define the *score-based distance* between $\mathcal{F}_1$ and $\mathcal{F}_2$ by

$$d(\mathcal{F}_1, \mathcal{F}_2) = \max_{A \in \mathcal{P}^*_m(U)} \big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big|.$$

Theorem 3.22 (Metric-type properties of the score-based distance)
The function d defined above has the following properties.

1. $d(\mathcal{F}_1, \mathcal{F}_2) \geq 0$ for all $\mathcal{F}_1, \mathcal{F}_2$.
2. $d(\mathcal{F}_1, \mathcal{F}_2) = d(\mathcal{F}_2, \mathcal{F}_1)$ for all $\mathcal{F}_1, \mathcal{F}_2$ (symmetry).
3. $d(\mathcal{F}_1, \mathcal{F}_3) \leq d(\mathcal{F}_1, \mathcal{F}_2) + d(\mathcal{F}_2, \mathcal{F}_3)$ for all $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ (triangle inequality).
4. $d(\mathcal{F}_1, \mathcal{F}_2) = 0$ if and only if $S_{\mathcal{F}_1}(A) = S_{\mathcal{F}_2}(A)$ for all $A \in \mathcal{P}^*_m(U)$. In particular, if $\mathcal{F}_1 = \mathcal{F}_2$ then $d(\mathcal{F}_1, \mathcal{F}_2) = 0$.

Thus d is a pseudo-metric on the class of (m,n)-LD SuperHyperfuzzy Sets, and becomes a genuine metric on the quotient space modulo score equivalence.

Proof
(1) For each A, the quantity $\big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big|$ is nonnegative, hence its maximum over a finite set is also nonnegative.

(2) For every A we have

$$\big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big| = \big| S_{\mathcal{F}_2}(A) - S_{\mathcal{F}_1}(A) \big|.$$

Taking the maximum over $A \in \mathcal{P}^*_m(U)$ on both sides yields $d(\mathcal{F}_1, \mathcal{F}_2) = d(\mathcal{F}_2, \mathcal{F}_1)$.

(3) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be given. For any fixed $A \in \mathcal{P}^*_m(U)$, the standard triangle inequality for real numbers gives

$$\big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_3}(A) \big| \leq \big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big| + \big| S_{\mathcal{F}_2}(A) - S_{\mathcal{F}_3}(A) \big|.$$

Taking the maximum over all A on both sides yields

$$d(\mathcal{F}_1, \mathcal{F}_3) = \max_A \big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_3}(A) \big| \leq \max_A \big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big| + \max_A \big| S_{\mathcal{F}_2}(A) - S_{\mathcal{F}_3}(A) \big|.$$

The right-hand side equals $d(\mathcal{F}_1, \mathcal{F}_2) + d(\mathcal{F}_2, \mathcal{F}_3)$, which proves the triangle inequality.

(4) If $d(\mathcal{F}_1, \mathcal{F}_2) = 0$, then

$$0 = d(\mathcal{F}_1, \mathcal{F}_2) = \max_A \big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big|.$$

A maximum of nonnegative numbers can be zero only if every term is zero, so

$$\big| S_{\mathcal{F}_1}(A) - S_{\mathcal{F}_2}(A) \big| = 0 \quad \text{for all } A,$$

which is equivalent to $S_{\mathcal{F}_1}(A) = S_{\mathcal{F}_2}(A)$ for all A. Conversely, if $S_{\mathcal{F}_1}(A) = S_{\mathcal{F}_2}(A)$ for all A, then each absolute difference is zero and hence $d(\mathcal{F}_1, \mathcal{F}_2) = 0$. In particular, taking $\mathcal{F}_1 = \mathcal{F}_2$ gives $d(\mathcal{F}_1, \mathcal{F}_1) = 0$. □

Definition 3.23 (Weighted aggregation of scores)**.** Let $\mathcal{F}_1, \ldots, \mathcal{F}_K$ be (m,n)-LD SuperHyperfuzzy Sets with scores $S_{\mathcal{F}_1}, \ldots, S_{\mathcal{F}_K}$, and let $\lambda_1, \ldots, \lambda_K \in [0,1]$ satisfy $\sum_{k=1}^{K} \lambda_k = 1$. The *weighted aggregated score* of $A \in \mathcal{P}_m^*(U)$ is defined by

$$S_{\text{agg}}(A) = \sum_{k=1}^{K} \lambda_k \, S_{\mathcal{F}_k}(A).$$

Proposition 3.24 (Basic properties of weighted score aggregation)
For every $A \in \mathcal{P}_m^*(U)$, the aggregated score satisfies $0 \le S_{\text{agg}}(A) \le 1$. Moreover, if all $S_{\mathcal{F}_k}(A)$ coincide, then $S_{\text{agg}}(A)$ equals this common value (idempotence).

Proof
For each k and each A we have $S_{\mathcal{F}_k}(A) \in [0,1]$ by Theorem 3.20(1). Since $\lambda_k \ge 0$ and $\sum_{k=1}^{K} \lambda_k = 1$, we obtain

$$0 \le \sum_{k=1}^{K} \lambda_k \, S_{\mathcal{F}_k}(A) \le \sum_{k=1}^{K} \lambda_k \cdot 1 = 1,$$

so $S_{\text{agg}}(A) \in [0,1]$. If $S_{\mathcal{F}_1}(A) = \cdots = S_{\mathcal{F}_K}(A) = \sigma$, then

$$S_{\text{agg}}(A) = \sum_{k=1}^{K} \lambda_k \, \sigma = \sigma \sum_{k=1}^{K} \lambda_k = \sigma,$$

which proves idempotence. □

3.4. (m,n)*-Linear Diophantine SuperHyperFuzzy OverSet*

Here we define the (m,n)-Linear Diophantine SuperHyperFuzzy OverSet, which can be regarded as a non-normalized version of the (m,n)-Linear Diophantine SuperHyperFuzzy Set.

Definition 3.25 (Linear Diophantine (m,n)-SuperHyperfuzzy Overset)**.** Let U be a nonempty crisp universe and let $m,n \in \mathbb{N} \cup \{0\}$. Fix a constant $L > 1$ and set

$$I_L := [0, L].$$

For each $k \ge 0$, denote by $\mathcal{P}_k^*(I_L)$ the k-fold nonempty iterated powerset of I_L, and let $\flat_k : \mathcal{P}_k^*(I_L) \to \mathcal{P}(I_L)$ be the flattening operator defined recursively by

$$\flat_0(s) \;=\; \{s\} \quad (s \in I_L), \qquad \flat_{k+1}(S) \;=\; \bigcup_{T \in S} \flat_k(T) \quad (S \in \mathcal{P}_{k+1}^*(I_L)).$$

Let $\alpha, \beta \in [0, \infty)$ be fixed (we do *not* assume $\alpha + \beta \le 1$). A pair

$$\widetilde{D}^O = \left(\widetilde{\mu}^O_{m,n}, \widetilde{\nu}^O_{m,n}\right)$$

is called a *Linear Diophantine* (m,n)*-SuperHyperfuzzy Overset* (abbrev. LD-(m,n)-SHFO) on U if the following conditions hold:

(LD1) $\widetilde{\mu}^{O}_{m,n}, \widetilde{\nu}^{O}_{m,n}$ are mappings

$$\widetilde{\mu}^{O}_{m,n}, \widetilde{\nu}^{O}_{m,n} : \mathcal{P}^{*}_{m}(U) \longrightarrow \mathcal{P}\big(\mathcal{P}^{*}_{n}(I_L)\big) \setminus \{\emptyset\},$$

that is, for each $A \in \mathcal{P}^{*}_{m}(U)$, both $\widetilde{\mu}^{O}_{m,n}(A)$ and $\widetilde{\nu}^{O}_{m,n}(A)$ are nonempty families of n-fold nested "overset" membership values lying in $I_L = [0, L]$.

(LD2) For every $A \in \mathcal{P}^{*}_{m}(U)$ and every

$$u \in \widetilde{\mu}^{O}_{m,n}(A), \qquad v \in \widetilde{\nu}^{O}_{m,n}(A),$$

the *Linear Diophantine feasible slice*

$$F_{m,n}(A; u, v) := \Big\{(s,t) \in \flat_n(u) \times \flat_n(v) \;\Big|\; 0 \le \alpha s + \beta t \le 1\Big\}$$

is nonempty.

(LD3) The associated *Linear Diophantine hyper-hesitation overset* for $A \in \mathcal{P}^{*}_{m}(U)$ is defined by

$$\widetilde{\pi}^{O}_{m,n}(A) := \Big\{1 - (\alpha s + \beta t) \;\Big|\; u \in \widetilde{\mu}^{O}_{m,n}(A),\, v \in \widetilde{\nu}^{O}_{m,n}(A),\, (s,t) \in F_{m,n}(A; u, v)\Big\}.$$

If, in addition, $L = 1$ and all nested values lie in $[0, 1]$, then $\widetilde{D}^{O}$ reduces to the usual (m, n)-Linear Diophantine SuperHyperfuzzy Set (without overset).

Example 3.26 (A numerical LD-$(0, 0)$-SuperHyperfuzzy Overset)**.** Let $U = \{x\}$ be a singleton universe and choose

$$m = n = 0, \qquad L = 2, \qquad I_L = [0, 2].$$

Fix the Linear Diophantine parameters

$$\alpha = \frac{4}{5} = 0.8, \qquad \beta = \frac{1}{2} = 0.5,$$

so that $\alpha + \beta = \frac{13}{10} = 1.3 > 1$; in particular we do *not* have $\alpha + \beta \le 1$.

For the unique element $x \in U$, define the overset hyper-membership and overset hyper-nonmembership by

$$\widetilde{\mu}^{O}_{0,0}(x) = \left\{\frac{9}{10}, \frac{6}{5}\right\}, \qquad \widetilde{\nu}^{O}_{0,0}(x) = \left\{\frac{9}{10}, 0\right\}.$$

Note that

$$\frac{6}{5} = 1.2 > 1,$$

so $\frac{6}{5}$ is an *overset* membership value lying outside the classical fuzzy interval $[0, 1]$.

Step 1: flattening. Since $n = 0$, the flattening operator is

$$\flat_0(s) = \{s\} \quad (s \in I_L),$$

hence

$$\flat_0\left(\frac{9}{10}\right) = \left\{\frac{9}{10}\right\}, \qquad \flat_0\left(\frac{6}{5}\right) = \left\{\frac{6}{5}\right\}, \qquad \flat_0(0) = \{0\}.$$

Step 2: all candidate pairs (s,t). The possible pairs $(s,t) \in \flat_0(u) \times \flat_0(v)$ with $u \in \widetilde{\mu}^O_{0,0}(x)$, $v \in \widetilde{\nu}^O_{0,0}(x)$ are

$$(s,t) = \left(\frac{9}{10}, \frac{9}{10}\right), \quad \left(\frac{6}{5}, \frac{9}{10}\right),$$
$$\left(\frac{9}{10}, 0\right), \quad \left(\frac{6}{5}, 0\right).$$

Step 3: checking the Linear Diophantine condition. For each of these four pairs, we compute $\alpha s + \beta t$ explicitly.

- Pair $\left(\frac{9}{10}, \frac{9}{10}\right)$:
$$\alpha s + \beta t = \frac{4}{5} \cdot \frac{9}{10} + \frac{1}{2} \cdot \frac{9}{10} = \frac{36}{50} + \frac{9}{20} = \frac{72}{100} + \frac{45}{100} = \frac{117}{100} = 1.17 > 1.$$
Therefore
$$0 \leq \alpha s + \beta t \leq 1$$
is not satisfied, and the pair $(s,t) = \left(\frac{9}{10}, \frac{9}{10}\right)$ is *excluded* by the Linear Diophantine condition.
- Pair $\left(\frac{6}{5}, \frac{9}{10}\right)$:
$$\alpha s + \beta t = \frac{4}{5} \cdot \frac{6}{5} + \frac{1}{2} \cdot \frac{9}{10} = \frac{24}{25} + \frac{9}{20} = \frac{96}{100} + \frac{45}{100} = \frac{141}{100} = 1.41 > 1.$$
Hence this pair also fails the inequality and is excluded.
- Pair $\left(\frac{9}{10}, 0\right)$:
$$\alpha s + \beta t = \frac{4}{5} \cdot \frac{9}{10} + \frac{1}{2} \cdot 0 = \frac{36}{50} + 0 = \frac{72}{100} = 0.72.$$
This satisfies
$$0 \leq 0.72 \leq 1,$$
so the pair $\left(\frac{9}{10}, 0\right)$ is an *admissible* pair with respect to the Linear Diophantine condition.
- Pair $\left(\frac{6}{5}, 0\right)$:
$$\alpha s + \beta t = \frac{4}{5} \cdot \frac{6}{5} + \frac{1}{2} \cdot 0 = \frac{24}{25} = \frac{96}{100} = 0.96.$$
This satisfies
$$0 \leq 0.96 \leq 1,$$
so this pair is also admissible.

Consequently, for this element x the Linear Diophantine feasible slice is

$$F_{0,0}(x) = \left\{ \left(\frac{9}{10}, 0\right), \left(\frac{6}{5}, 0\right) \right\},$$

and the two pairs

$$\left(\frac{9}{10}, \frac{9}{10}\right), \quad \left(\frac{6}{5}, \frac{9}{10}\right)$$

are *explicitly excluded* by the inequality

$$0 \leq \alpha s + \beta t \leq 1.$$

Step 4: hyper-hesitation overset. The corresponding hyper-hesitation overset is

$$\widetilde{\pi}^{O}_{0,0}(x) = \Big\{1 - (\alpha s + \beta t) \,\Big|\, (s,t) \in F_{0,0}(x)\Big\}$$
$$= \Big\{1 - 0.72,\ 1 - 0.96\Big\} = \{0.28,\ 0.04\}.$$

This example illustrates two key points:

- The membership value $\frac{6}{5} = 1.2 > 1$ is allowed as an *overset* membership degree and lies outside the classical fuzzy interval $[0,1]$.
- The Linear Diophantine constraint $0 \le \alpha s + \beta t \le 1$ genuinely *filters out* some pairs (s,t), such as $(s,t) = \left(\frac{9}{10}, \frac{9}{10}\right)$ with $\alpha s + \beta t = \frac{117}{100} > 1$.

Thus, this example provides a concrete LD-$(0,0)$-SuperHyperfuzzy Overset in which u, v, s, t can take values in $[0, L]$ with $L > 1$ and the inequality $0 \le \alpha s + \beta t \le 1$ actively excludes part of the candidate pairs (s,t).

I will state the theorems as follows.

Theorem 3.27 (Well-definedness of the Linear Diophantine hyper-hesitation overset)
Let $\widetilde{D}^O = \left(\widetilde{\mu}^O_{m,n}, \widetilde{\nu}^O_{m,n}\right)$ be a Linear Diophantine (m,n)-SuperHyperfuzzy Overset on U as in Definition (LD1)–(LD3) above, with parameters $\alpha, \beta \in [0,\infty)$ and $L > 1$. Then for every $A \in \mathcal{P}^*_m(U)$ the set

$$\widetilde{\pi}^O_{m,n}(A) = \Big\{1 - (\alpha s + \beta t) \,\Big|\, u \in \widetilde{\mu}^O_{m,n}(A),\ v \in \widetilde{\nu}^O_{m,n}(A),\ (s,t) \in F_{m,n}(A; u, v)\Big\}$$

is a nonempty subset of the closed unit interval $[0,1] \subset \mathbb{R}$. In particular, the assignment $A \mapsto \widetilde{\pi}^O_{m,n}(A)$ is well-defined.

Proof
Fix $A \in \mathcal{P}^*_m(U)$. By (LD1), the sets $\widetilde{\mu}^O_{m,n}(A)$ and $\widetilde{\nu}^O_{m,n}(A)$ are nonempty. Take arbitrary

$$u \in \widetilde{\mu}^O_{m,n}(A), \qquad v \in \widetilde{\nu}^O_{m,n}(A).$$

By (LD2), the feasible slice

$$F_{m,n}(A; u, v) = \Big\{(s,t) \in \flat_n(u) \times \flat_n(v) \,\Big|\, 0 \le \alpha s + \beta t \le 1\Big\}$$

is nonempty, so there exists at least one pair $(s,t) \in \flat_n(u) \times \flat_n(v)$ with

$$0 \ \le\ \alpha s + \beta t \ \le\ 1. \tag{2}$$

By Definition (LD1), we have $u, v \in \mathcal{P}^*_n(I_L)$ with $I_L = [0, L]$, hence $\flat_n(u) \subseteq I_L$ and $\flat_n(v) \subseteq I_L$. Thus

$$s, t \in I_L = [0, L].$$

Since $\alpha, \beta \in [0,\infty)$ and $s, t \in [0, L]$, the quantity $\alpha s + \beta t$ is a well-defined real number and (2) implies

$$0 \ \le\ \alpha s + \beta t \ \le\ 1.$$

Therefore

$$0 \ \le\ 1 - (\alpha s + \beta t) \ \le\ 1,$$

so each value $1 - (\alpha s + \beta t)$ lies in $[0,1]$. Varying (s,t) over $F_{m,n}(A; u, v)$ and (u,v) over $\widetilde{\mu}^O_{m,n}(A) \times \widetilde{\nu}^O_{m,n}(A)$, we obtain exactly the elements of $\widetilde{\pi}^O_{m,n}(A)$. Because $F_{m,n}(A; u, v)$ is nonempty for every such (u,v), we conclude that $\widetilde{\pi}^O_{m,n}(A)$ is nonempty, and by the above inequality every element of $\widetilde{\pi}^O_{m,n}(A)$ belongs to $[0,1]$. Hence the map $A \mapsto \widetilde{\pi}^O_{m,n}(A)$ is well-defined with values in $[0,1]$. □

We now show that every Linear Diophantine (m, n)-SuperHyperfuzzy Overset admits a canonical transformation into a usual Linear Diophantine (m, n)-SuperHyperfuzzy Set (i.e. with all nested values contained in the unit interval).

Definition 3.28 (Leafwise truncation on I_L)**.** Let $L > 1$ and set $I_L = [0, L]$. Define the truncation map $\varphi : I_L \to [0, 1]$ by

$$\varphi(x) \ := \ \min\{x, 1\} \qquad (x \in I_L).$$

For each $k \geq 0$ we define recursively a map

$$T_k : \mathcal{P}_k^*(I_L) \longrightarrow \mathcal{P}_k^*\big([0, 1]\big)$$

by

$$T_0(s) \ := \ \varphi(s) \quad (s \in I_L), \qquad T_{k+1}(S) \ := \ \big\{ T_k(T) \mid T \in S \big\} \quad (S \in \mathcal{P}_{k+1}^*(I_L)).$$

Lemma 3.29 (Compatibility of truncation and flattening)
For every $k \geq 0$ and every $u \in \mathcal{P}_k^*(I_L)$ one has

$$\flat_k\big(T_k(u)\big) \ = \ \big\{\varphi(s) \mid s \in \flat_k(u)\big\}.$$

In particular, $\flat_k(T_k(u)) \subseteq [0, 1]$.

Proof
We proceed by induction on k.

For $k = 0$, we have $u \in I_L$ and

$$T_0(u) = \varphi(u), \qquad \flat_0(u) = \{u\}, \qquad \flat_0\big(T_0(u)\big) = \{\varphi(u)\},$$

so

$$\flat_0\big(T_0(u)\big) = \{\varphi(u)\} = \{\varphi(s) \mid s \in \flat_0(u)\},$$

and the claim holds.

Assume the statement holds for some $k \geq 0$, and let $u \in \mathcal{P}_{k+1}^*(I_L)$. Then

$$T_{k+1}(u) = \big\{T_k(T) \mid T \in u\big\},$$

and by the definition of $\flat_{k+1}$ we have

$$\flat_{k+1}\big(T_{k+1}(u)\big) = \bigcup_{W \in T_{k+1}(u)} \flat_k(W) = \bigcup_{T \in u} \flat_k\big(T_k(T)\big).$$

By the induction hypothesis,

$$\flat_k\big(T_k(T)\big) = \big\{\varphi(s) \mid s \in \flat_k(T)\big\},$$

so

$$\flat_{k+1}\big(T_{k+1}(u)\big) = \bigcup_{T \in u} \big\{\varphi(s) \mid s \in \flat_k(T)\big\} = \big\{\varphi(s) \mid s \in \flat_{k+1}(u)\big\},$$

which proves the desired equality. Since $\varphi : I_L \to [0, 1]$, the last set is contained in $[0, 1]$, and the lemma follows. □

Definition 3.30 (Truncation of a Linear Diophantine (m,n)-SuperHyperfuzzy Overset)**.** Let $\widetilde{D}^O = \left(\widetilde{\mu}^O_{m,n}, \widetilde{\nu}^O_{m,n}\right)$ be a Linear Diophantine (m,n)-SuperHyperfuzzy Overset on U with parameters (α, β, L). Define new mappings

$$\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n} : \mathcal{P}^*_m(U) \longrightarrow \mathcal{P}\big(\mathcal{P}^*_n([0,1])\big) \setminus \{\emptyset\}$$

by

$$\widetilde{\mu}_{m,n}(A) := \left\{\, T_n(u) \mid u \in \widetilde{\mu}^O_{m,n}(A) \,\right\}, \qquad \widetilde{\nu}_{m,n}(A) := \left\{\, T_n(v) \mid v \in \widetilde{\nu}^O_{m,n}(A) \,\right\}$$

for every $A \in \mathcal{P}^*_m(U)$. We call

$$\widetilde{D} := \left(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n}\right)$$

the *truncation* of $\widetilde{D}^O$.

Theorem 3.31 (Linear Diophantine (m,n)-SuperHyperfuzzy Oversets can be transformed into Sets)
Let $\widetilde{D}^O = \left(\widetilde{\mu}^O_{m,n}, \widetilde{\nu}^O_{m,n}\right)$ be a Linear Diophantine (m,n)-SuperHyperfuzzy Overset on U with parameters $\alpha, \beta \in [0,\infty)$ and $L > 1$, and let $\widetilde{D} = \left(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n}\right)$ be its truncation as above. Then:

1. For every $A \in \mathcal{P}^*_m(U)$, the sets $\widetilde{\mu}_{m,n}(A)$ and $\widetilde{\nu}_{m,n}(A)$ are nonempty families of n-fold nested membership values lying in the unit interval $[0,1]$.
2. For every $A \in \mathcal{P}^*_m(U)$ and every

$$u' \in \widetilde{\mu}_{m,n}(A), \qquad v' \in \widetilde{\nu}_{m,n}(A),$$

the Diophantine feasible slice

$$F_{m,n}(A; u', v') := \left\{ (s', t') \in \flat_n(u') \times \flat_n(v') \;\middle|\; 0 \le \alpha s' + \beta t' \le 1 \right\}$$

is nonempty.

3. If we define, for each $A \in \mathcal{P}^*_m(U)$,

$$\widetilde{\pi}_{m,n}(A) := \left\{ 1 - (\alpha s' + \beta t') \;\middle|\; u' \in \widetilde{\mu}_{m,n}(A),\, v' \in \widetilde{\nu}_{m,n}(A),\, (s', t') \in F_{m,n}(A; u', v') \right\},$$

then $\widetilde{\pi}_{m,n}(A) \subseteq [0,1]$ and $\widetilde{\pi}_{m,n}(A) \neq \emptyset$.

Consequently, the truncated pair $\widetilde{D} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$, together with $\widetilde{\pi}_{m,n}$, satisfies the axioms (LD1)–(LD3) with $L = 1$ and defines a Linear Diophantine (m,n)-SuperHyperfuzzy Set. In other words, every Linear Diophantine (m,n)-SuperHyperfuzzy Overset can be canonically transformed into a Linear Diophantine (m,n)-SuperHyperfuzzy Set.

Proof
(1) Fix $A \in \mathcal{P}^*_m(U)$. By (LD1) for $\widetilde{D}^O$, the sets $\widetilde{\mu}^O_{m,n}(A)$ and $\widetilde{\nu}^O_{m,n}(A)$ are nonempty. By definition of $\widetilde{\mu}_{m,n}(A)$, every element of $\widetilde{\mu}_{m,n}(A)$ has the form $T_n(u)$ with $u \in \widetilde{\mu}^O_{m,n}(A)$. Since $u \in \mathcal{P}^*_n(I_L)$ and T_n maps $\mathcal{P}^*_n(I_L)$ into $\mathcal{P}^*_n([0,1])$, we have $T_n(u) \in \mathcal{P}^*_n([0,1])$. Thus $\widetilde{\mu}_{m,n}(A)$ is a family of n-fold nested sets whose leaves lie in $[0,1]$. Nonemptiness follows because $\widetilde{\mu}^O_{m,n}(A)$ is nonempty and T_n is defined on every element of $\mathcal{P}^*_n(I_L)$. The same argument applies to $\widetilde{\nu}_{m,n}(A)$, proving (1).

(2) Let $A \in \mathcal{P}^*_m(U)$ and choose arbitrary

$$u' \in \widetilde{\mu}_{m,n}(A), \qquad v' \in \widetilde{\nu}_{m,n}(A).$$

By definition, there exist

$$u \in \widetilde{\mu}^{O}_{m,n}(A), \qquad v \in \widetilde{\nu}^{O}_{m,n}(A)$$

such that

$$u' = T_n(u), \qquad v' = T_n(v).$$

By (LD2) for $\widetilde{D}^{O}$, the slice

$$F_{m,n}(A; u, v) = \Big\{(s,t) \in \flat_n(u) \times \flat_n(v) \;\Big|\; 0 \le \alpha s + \beta t \le 1\Big\}$$

is nonempty, so choose a pair $(s,t) \in F_{m,n}(A; u, v)$. Then

$$0 \;\le\; \alpha s + \beta t \;\le\; 1. \tag{3}$$

Set

$$s' := \varphi(s), \qquad t' := \varphi(t),$$

so $s', t' \in [0,1]$. By Lemma 3.29, we have

$$\flat_n(u') = \flat_n\big(T_n(u)\big) = \{\varphi(r) \mid r \in \flat_n(u)\},$$

and similarly

$$\flat_n(v') = \flat_n\big(T_n(v)\big) = \{\varphi(r) \mid r \in \flat_n(v)\}.$$

Since $s \in \flat_n(u)$ and $t \in \flat_n(v)$, it follows that

$$s' = \varphi(s) \in \flat_n(u'), \qquad t' = \varphi(t) \in \flat_n(v').$$

Next we verify the Diophantine inequality for s' and t'. By definition of φ,

$$0 \le s' \le s, \qquad 0 \le t' \le t$$

because $\varphi(x) = \min\{x, 1\} \le x$ for all $x \in I_L$ and $\varphi(x) \ge 0$. Since $\alpha, \beta \ge 0$, we obtain

$$\alpha s' + \beta t' \;\le\; \alpha s + \beta t.$$

Combining this with (3) yields

$$0 \;\le\; \alpha s' + \beta t' \;\le\; \alpha s + \beta t \;\le\; 1.$$

Hence $(s', t') \in \flat_n(u') \times \flat_n(v')$ satisfies $0 \le \alpha s' + \beta t' \le 1$, so

$$(s', t') \in F_{m,n}(A; u', v').$$

Therefore $F_{m,n}(A; u', v')$ is nonempty, which proves (2).

(3) Let $A \in \mathcal{P}^{*}_{m}(U)$. By (2), for every pair (u', v') as above, the set $F_{m,n}(A; u', v')$ is nonempty, so there exist $(s', t') \in F_{m,n}(A; u', v')$. By definition of $F_{m,n}(A; u', v')$ we have

$$0 \;\le\; \alpha s' + \beta t' \;\le\; 1.$$

Thus

$$0 \;\le\; 1 - (\alpha s' + \beta t') \;\le\; 1,$$

Algorithm 1: Flattening an n-level membership object

Function `Flat` (z, k):
 `// Input:` $z \in \mathcal{P}^k([0,1])$, `integer` $k \geq 0$.
 `// Output:` $\mathrm{flat}_k(z) \subseteq [0,1]$.
 if $k = 0$ **then**
  return $\{z\}$
 $F \leftarrow \emptyset$
 foreach $T \in z$ **do**
  $F \leftarrow F \cup$ `Flat`$(T, k-1)$
 return F

so every element of

$$\widetilde{\pi}_{m,n}(A) = \Big\{1 - (\alpha s' + \beta t') \;\Big|\; u' \in \widetilde{\mu}_{m,n}(A),\ v' \in \widetilde{\nu}_{m,n}(A),\ (s', t') \in F_{m,n}(A; u', v')\Big\}$$

belongs to $[0,1]$. Moreover, because each $F_{m,n}(A; u', v')$ is nonempty and $\widetilde{\mu}_{m,n}(A), \widetilde{\nu}_{m,n}(A)$ are nonempty, the set $\widetilde{\pi}_{m,n}(A)$ is nonempty. This proves (3).

Putting (1)–(3) together, we see that $\widetilde{D} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$, with the hesitation part $\widetilde{\pi}_{m,n}$, satisfies the same axioms (LD1)–(LD3) as $\widetilde{D}^O$ but with $L = 1$ and all nested values contained in $[0,1]$. Therefore $\widetilde{D}$ is a Linear Diophantine (m,n)-SuperHyperfuzzy Set, and the truncation construction defines a well-defined transformation from Linear Diophantine (m,n)-SuperHyperfuzzy Oversets to Linear Diophantine (m,n)-SuperHyperfuzzy Sets. □

4. Algorithms for $(m,n)-$Linear Diophantine SuperHyperFuzzy Set

In this section we describe a basic validation and construction procedure for (m,n)-Linear Diophantine SuperHyperFuzzy Sets, and we state and prove its correctness and complexity. We assume that the flattening operator

$$\mathrm{flat}_n : \mathcal{P}^n([0,1]) \to \mathcal{P}([0,1])$$

is defined as in the previous subsection. The algorithm for flattening an n-level membership object is given below in Algorithm 1.

Algorithm 1 is the direct operational counterpart of the recursive definition of flat_n.

Next we formalize a validation and construction algorithm for (m,n)-Linear Diophantine SuperHyperFuzzy Sets. We work with a finite index family $\mathcal{A} \subseteq \mathcal{P}_m^*(U)$ of "active" m-level subsets, together with given hyper-membership and hyper-nonmembership mappings

$$\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n} : \mathcal{A} \longrightarrow \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\}.$$

Algorithm 2 presents the procedure for validation and hesitation construction for (m,n)-LD SuperHyperFuzzy Sets.

Remark 4.1. If $\mathcal{A}$ is chosen to be the full family $\mathcal{P}_m^*(U)$, then Algorithm 2 validates the complete (m,n)-Linear Diophantine SuperHyperFuzzy structure on U. In many applications, only a finite subfamily of "active" m-level subsets is needed, so a finite index set $\mathcal{A}$ is natural.

Algorithm 2: Validation and hesitation construction for (m, n)-LD SuperHyperFuzzy Sets

Data: Nonempty finite universe U; integers $m, n \geq 0$; finite $\mathcal{A} \subseteq \mathcal{P}_m^*(U)$; weights $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$; mappings $\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n}$ on $\mathcal{A}$.

Result: Boolean flag `isValid`; hyper-hesitation mapping $\widetilde{\pi}_{m,n}$ on $\mathcal{A}$ if `isValid` = true.

```
isValid ← true
foreach A ∈ 𝒜 do
    // Check nonemptiness of hyper-membership and hyper-nonmembership sets
    if μ̃_{m,n}(A) = ∅ or ν̃_{m,n}(A) = ∅ then
        isValid ← false
        break
    // Precompute flattened sets of degrees
    S^μ_A ← ∅
    S^ν_A ← ∅
    foreach u ∈ μ̃_{m,n}(A) do
        F_u ← Flat(u, n)
        foreach s ∈ F_u do
            if s < 0 or s > 1 then
                isValid ← false
                break
            S^μ_A ← S^μ_A ∪ {s}
        if isValid = false then
            break
    if isValid = false then
        break
    foreach v ∈ ν̃_{m,n}(A) do
        G_v ← Flat(v, n)
        foreach t ∈ G_v do
            if t < 0 or t > 1 then
                isValid ← false
                break
            S^ν_A ← S^ν_A ∪ {t}
        if isValid = false then
            break
    if isValid = false then
        break
    // Check Diophantine constraint and construct hesitation degrees
    π̃_{m,n}(A) ← ∅
    foreach s ∈ S^μ_A do
        foreach t ∈ S^ν_A do
            v_{α,β} ← α · s + β · t
            if v_{α,β} < 0 or v_{α,β} > 1 then
                isValid ← false
                break
            π̃_{m,n}(A) ← π̃_{m,n}(A) ∪ {1 − v_{α,β}}
        if isValid = false then
            break
    if isValid = false then
        break
if isValid = false then
    // Optionally clear π̃_{m,n} if invalid
    foreach A ∈ 𝒜 do
        π̃_{m,n}(A) ← ∅
return (isValid, π̃_{m,n})
```

Theorems concerning these algorithms are presented below.

Theorem 4.2 (Correctness of Algorithm 2)
Let U, m, n, α, β, $\mathcal{A}$, $\widetilde{\mu}_{m,n}$, and $\widetilde{\nu}_{m,n}$ be as in Algorithm 2, and suppose all sets involved are finite. Then the following hold.

1. If `isValid`= true on termination, then $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ satisfies the definition of an (m, n)-Linear Diophantine SuperHyperFuzzy Set on $\mathcal{A}$, and for every $A \in \mathcal{A}$ we have

$$\widetilde{\pi}_{m,n}(A) = \big\{1 - (\alpha s + \beta t) \;\big|\; u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\}.$$

2. Conversely, assume that $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperFuzzy Set on $\mathcal{A}$ in the sense of Definition 3.5. Then Algorithm 2 terminates with `isValid`= true and the returned $\widetilde{\pi}_{m,n}$ coincides with the one defined in Definition 3.5.

Proof
We prove each item separately.

(1) Assume that Algorithm 2 terminates with `isValid` = true.

First, for every $A \in \mathcal{A}$ the algorithm explicitly tests whether $\widetilde{\mu}_{m,n}(A)$ and $\widetilde{\nu}_{m,n}(A)$ are empty. If either were empty, the flag `isValid` would be set to `false`, and the outer loop would break. Since the final value is `true`, it follows that

$$\widetilde{\mu}_{m,n}(A) \neq \emptyset, \qquad \widetilde{\nu}_{m,n}(A) \neq \emptyset \quad \text{for all } A \in \mathcal{A}.$$

Hence the codomain constraint $\mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$ is satisfied.

Next, for every $u \in \widetilde{\mu}_{m,n}(A)$ the algorithm computes $F_u = \mathtt{Flat}(u, n)$, and for every $s \in F_u$ it checks the inequalities $s \geq 0$ and $s \leq 1$. If some s violated $0 \leq s \leq 1$, then `isValid` would be set to `false`. Because the final flag is `true`, all degrees s in all sets F_u satisfy $0 \leq s \leq 1$. The same argument applies to all $t \in G_v = \mathtt{Flat}(v, n)$ for all $v \in \widetilde{\nu}_{m,n}(A)$. Therefore, for every $A \in \mathcal{A}$ and for all

$$u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v),$$

we have $s, t \in [0, 1]$.

Then, in the nested loop over $s \in \mathsf{S}^{\mu}_A$ and $t \in \mathsf{S}^{\nu}_A$, the algorithm computes

$$v_{\alpha,\beta} = \alpha s + \beta t$$

and checks whether $v_{\alpha,\beta} \in [0, 1]$. If some pair (s, t) produced a value $v_{\alpha,\beta} < 0$ or $v_{\alpha,\beta} > 1$, then `isValid` would be set to `false`. Hence, because the final flag is `true`, we obtain

$$0 \leq \alpha s + \beta t \leq 1$$

for all admissible (s, t); this is exactly the linear Diophantine condition required in Definition 3.5.

Finally, for each $A \in \mathcal{A}$ the algorithm constructs $\widetilde{\pi}_{m,n}(A)$ by the update rule

$$\widetilde{\pi}_{m,n}(A) \leftarrow \widetilde{\pi}_{m,n}(A) \cup \big\{1 - (\alpha s + \beta t)\big\}$$

for all $s \in \mathsf{S}_A^\mu$ and $t \in \mathsf{S}_A^\nu$. By the definition of S_A^μ and S_A^ν, we have

$$\mathsf{S}_A^\mu = \bigcup_{u \in \widetilde{\mu}_{m,n}(A)} \mathrm{flat}_n(u), \qquad \mathsf{S}_A^\nu = \bigcup_{v \in \widetilde{\nu}_{m,n}(A)} \mathrm{flat}_n(v).$$

Therefore, the set of all values $1 - (\alpha s + \beta t)$ added to $\widetilde{\pi}_{m,n}(A)$ coincides with

$$\big\{1 - (\alpha s + \beta t) \;\big|\; u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\},$$

as required. This proves Item (1).

(2) Conversely, assume that $(\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ is an (m, n)-Linear Diophantine SuperHyperFuzzy Set on $\mathcal{A}$ in the sense of Definition 3.5. Then, by the definition, for every $A \in \mathcal{A}$ we have $\widetilde{\mu}_{m,n}(A) \neq \emptyset$ and $\widetilde{\nu}_{m,n}(A) \neq \emptyset$, so the initial emptiness checks in Algorithm 2 never fail.

Moreover, for every $A \in \mathcal{A}$, every $u \in \widetilde{\mu}_{m,n}(A)$, every $v \in \widetilde{\nu}_{m,n}(A)$, and every $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$, Definition 3.5 ensures that

$$0 \le s \le 1, \quad 0 \le t \le 1, \quad 0 \le \alpha s + \beta t \le 1.$$

Hence none of the inequalities checked in the inner loops can fail, and the flag `isValid` remains `true` throughout the execution.

By construction of S_A^μ and S_A^ν and by the same reasoning as in Item (1), the sets $\widetilde{\pi}_{m,n}(A)$ computed by the algorithm are identical to those prescribed in Definition 3.5. Therefore the algorithm terminates with `isValid` $= \mathrm{true}$ and with the correct hyper-hesitation mapping. This proves Item (2) and completes the proof. □

Theorem 4.3 (Time complexity of Algorithm 2)
Let $\mathcal{A}$ be finite. For each $A \in \mathcal{A}$, denote

$$M_A := \big|\widetilde{\mu}_{m,n}(A)\big|, \qquad N_A := \big|\widetilde{\nu}_{m,n}(A)\big|,$$

and for each $u \in \widetilde{\mu}_{m,n}(A)$, $v \in \widetilde{\nu}_{m,n}(A)$ write

$$L_u := \big|\mathrm{flat}_n(u)\big|, \qquad L_v := \big|\mathrm{flat}_n(v)\big|.$$

Then the total number of elementary loop iterations in Algorithm 2 is bounded by

$$T \;\le\; \sum_{A \in \mathcal{A}} \left(\sum_{u \in \widetilde{\mu}_{m,n}(A)} L_u \;+\; \sum_{v \in \widetilde{\nu}_{m,n}(A)} L_v \;+\; \sum_{u \in \widetilde{\mu}_{m,n}(A)} \sum_{v \in \widetilde{\nu}_{m,n}(A)} L_u L_v \right).$$

In particular, if we set

$$M := \max_{A \in \mathcal{A}} M_A, \quad N := \max_{A \in \mathcal{A}} N_A, \quad L := \max\big\{L_u, L_v \;\big|\; A \in \mathcal{A},\ u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A)\big\},$$

and write $|\mathcal{A}| = K$, then

$$T = O\big(K(M + N)L + KMNL^2\big) = O\big(KMNL^2\big).$$

Proof
Fix $A \in \mathcal{A}$. The cost of the block

$$\forall\, u \in \widetilde{\mu}_{m,n}(A)\ \forall\, s \in \mathtt{Flat}(u, n)\ :\ (\cdots)$$

is $\sum_{u\in\widetilde{\mu}_{m,n}(A)} L_u$. Similarly, the cost of processing all $v \in \widetilde{\nu}_{m,n}(A)$ and their flattened degrees $t \in \texttt{Flat}(v,n)$ is $\sum_{v\in\widetilde{\nu}_{m,n}(A)} L_v$.

For the nested loops over $s \in \mathsf{S}_A^{\mu}$ and $t \in \mathsf{S}_A^{\nu}$, observe that

$$\mathsf{S}_A^{\mu} = \bigcup_{u\in\widetilde{\mu}_{m,n}(A)} \mathrm{flat}_n(u), \quad \mathsf{S}_A^{\nu} = \bigcup_{v\in\widetilde{\nu}_{m,n}(A)} \mathrm{flat}_n(v).$$

Thus the total number of pairs (s,t) visited is at most

$$\Bigg(\sum_{u\in\widetilde{\mu}_{m,n}(A)} L_u\Bigg)\cdot\Bigg(\sum_{v\in\widetilde{\nu}_{m,n}(A)} L_v\Bigg) \le \sum_{u\in\widetilde{\mu}_{m,n}(A)}\sum_{v\in\widetilde{\nu}_{m,n}(A)} L_u L_v,$$

where we used the inequality $(\sum_i a_i)(\sum_j b_j) \le \sum_{i,j} a_i b_j$ for nonnegative a_i, b_j.

Summing these contributions over all $A \in \mathcal{A}$ yields the stated upper bound on T:

$$T \le \sum_{A\in\mathcal{A}}\Bigg(\sum_{u\in\widetilde{\mu}_{m,n}(A)} L_u + \sum_{v\in\widetilde{\nu}_{m,n}(A)} L_v + \sum_{u\in\widetilde{\mu}_{m,n}(A)}\sum_{v\in\widetilde{\nu}_{m,n}(A)} L_u L_v\Bigg).$$

For the simplified Big-O estimate, note that for each $A \in \mathcal{A}$ we have

$$\sum_{u\in\widetilde{\mu}_{m,n}(A)} L_u \le M_A L \le ML, \qquad \sum_{v\in\widetilde{\nu}_{m,n}(A)} L_v \le N_A L \le NL,$$

and

$$\sum_{u\in\widetilde{\mu}_{m,n}(A)}\sum_{v\in\widetilde{\nu}_{m,n}(A)} L_u L_v \le M_A N_A L^2 \le MNL^2.$$

Therefore

$$T \le \sum_{A\in\mathcal{A}} \big(ML + NL + MNL^2\big) = K(M+N)L + KMNL^2.$$

In Big-O notation this is $T = O\big(K(M+N)L + KMNL^2\big)$, and in particular $T = O(KMNL^2)$ as claimed. □

5. Conclusion

In this paper, we introduced two novel frameworks—the *Linear Diophantine Hyperfuzzy Set* and the *Linear Diophantine SuperHyperfuzzy Set*—by embedding linear Diophantine constraints into the hyperfuzzy and superhyperfuzzy paradigms. We also examined their fundamental concrete examples and the associated algorithms. On that basis, we evaluated the algorithmic complexity and the validity of the proposed methods. For reference, Table 5 presents the comparison of the HyperFuzzy Set and the Linear Diophantine HyperFuzzy Set, and Table 6 provides the comparison of the SuperHyperFuzzy Set and the (m,n)-Linear Diophantine SuperHyperFuzzy Set. From these observations, we believe that such constructions offer a promising framework for clearly representing real-world concepts that require hierarchical and uncertain Linear Diophantine conditions.

Table 5. Brief comparison of HyperFuzzy Set and Linear Diophantine HyperFuzzy Set

Aspect	HyperFuzzy Set	Linear Diophantine HyperFuzzy Set
Membership representation	Each x has $\widetilde{\mu}(x) \subseteq [0,1]$ nonempty.	Each x has $\widetilde{\mu}(x), \widetilde{\nu}(x) \subseteq [0,1]$ nonempty.
Additional parameters	No global weights.	Uses fixed (α, β) with $0 \leq \alpha + \beta \leq 1$.
Core constraint	Only $0 \leq u \leq 1$ for $u \in \widetilde{\mu}(x)$.	Requires $0 \leq \alpha u + \beta v \leq 1$ for all $u \in \widetilde{\mu}(x)$, $v \in \widetilde{\nu}(x)$.
Hesitation / indeterminacy	Not canonically specified; may be derived externally.	$\widetilde{\pi}(x) = \{1 - (\alpha u + \beta v) \mid u \in \widetilde{\mu}(x),\ v \in \widetilde{\nu}(x)\}$.
Modeling focus	Multiple plausible membership grades per element.	Coupled sets of support and opposition under a linear constraint.

Table 6. Brief comparison of SuperHyperFuzzy Set and (m, n)-Linear Diophantine SuperHyperFuzzy Set

Aspect	SuperHyperFuzzy Set	(m, n)-Linear Diophantine SuperHyperFuzzy Set
Membership representation	$\widetilde{\mu}_{m,n} : \mathcal{P}_m^*(U) \to \mathcal{P}(\mathcal{P}_n([0,1])) \setminus \{\emptyset\}$.	$\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n} : \mathcal{P}_m^*(U) \to \mathcal{P}(\mathcal{P}_n([0,1])) \setminus \{\emptyset\}$.
Additional parameters	No global weights.	Uses fixed (α, β) with $0 \leq \alpha + \beta \leq 1$.
Core constraint	Flattened grades $s \in \mathrm{flat}_n(u)$ lie in $[0, 1]$.	$0 \leq \alpha s + \beta t \leq 1$ for all $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$.
Hesitation / indeterminacy	May be introduced ad hoc from $\widetilde{\mu}_{m,n}$.	$\widetilde{\pi}_{m,n}(A) = \{1 - (\alpha s + \beta t) \mid u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\}$.
Modeling focus	Hierarchical membership uncertainty on iterated powersets.	Hierarchical balance of membership and nonmembership with tunable trade-offs.

Looking ahead, we plan to extend these models to additional uncertainty frameworks, including Neutrosophic Sets [68, 15], Shadowed Sets[69], Soft Sets[70, 71], Plithogenic Sets [72, 14], Hesitant Fuzzy Sets [73], Z-Numbers [74, 75], and other related frameworks. We will also investigate algorithmic strategies and computational implementations to validate and apply these constructs in real-world decision-making scenarios. Furthermore, we intend to explore graph-based generalizations by integrating the Linear Diophantine Hyperfuzzy Set concept into HyperGraphs [76] and SuperHyperGraphs [77]. Furthermore, we believe that the algorithms presented in this paper still have room for improvement. We hope that, in the future, experts in the field will further refine and enhance their computational complexity.

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Author's Contributions

Conceptualization, All authors; Investigation, All authors; Methodology, All authors; Writing – original draft, All authors; Writing – review & editing, All authors.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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