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Performance of the Generalized Shrinkage Estimator in Zero-Inflated Bell Regression Model

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Abstract The Poisson regression model is a vital analytical instrument that should be employed in the modeling of count data. When the excess dispersion of variables is present, the model is inappropriate to employ if the mean value does not equal the variance of the Poisson distribution. The results are compatible with data when there is the use of Bell regression model. The number of zeros in the count data that is seen is very high. In this case, the Zero-Inflated Bell regression model is an alternative to the Bell regression model. Parameters of the Zero-Inflated Bell regression model are estimated mostly through the approach of maximum likelihood. In an extended linear model, in which the response variable is modeled by two or more explanatory variables, as in the Zero-Inflated Bell regression model, linear dependence is a threat in a real-life analysis. It reduced the maximum likelihood estimator in its effectiveness. In a bid to solve this issue, this paper explores the performance of the generalized shrinkage estimator in the zero-inflated Bell regression model. The superiority of the proposed approaches over the traditional maximum likelihood estimator is validated by results of the simulations and implementations.

Keywords Over-dispersion, Poisson regression, Shrinkage estimator, Zero-Inflated Bell Regression

AMS 2010 subject classifications 62Jxx

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1. Introduction

Poisson regression should be used especially where the data involves counts of events as the response variable, such as how many customer complaints there were on a given day, how many hospital admissions there were last month, or how many accidents there were at an intersection. Poisson regression, in contrast to linear regression, is able to model predictions that are non-negative integers, unlike linear regression, which can predict unrealistic negative values of counts (Aldoori et al. [3], 2025, Amin et al.[11], 2020, Salih and Hmood[32], 2020,Salih and Hmood[33], 2021, Hamad and Algamal[22], 2021, Rashad and Algamal[30], 2019, Yahya Algamal[39], 2019, Salih and Hussein[34]). The Poisson regression model is unquestionably the most widely used model for count data in practice (Algamal[5], 2012, Alanaz and Algamal[2], 2018). The assumption that the variance and the mean of the distribution are the same is often made in the distribution. A major weakness in the Poisson regression model which normally occurs in count data is over-dispersion or variation that is higher than the mean. On the Bell distribution and the count data Bell regression model, (Castellares et al[16]., 2018) proposed another count data discrete distribution model called the Bell regression model in order to model count data with over-dispersion. Recently, the ridge estimator and the Liu estimator were introduced to the estimation of the parameter of the Bell

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regression model with multicollinearity by (Amin et al.[10], 2023, Majid et al.[27], 2022, Algamal et al.[7], 2022, Algamal et al.[8], 2023, Seifollahi et al.[37], 2025, Bulut et al.[15], 2024, Seifollahi et al[36]., 2024, Ertan et al.[19], 2025, Ertan et al.[20], 2023). Another weakness of Bell regression model is the excess zeros in the count data. Many applications such as medicine, public health, environmental sciences, agriculture, and manufacturing applications are prone to the number being much larger than positive (or having many zeros) So a more appropriate model to the count data is the Zero-Inflated Bell regression model. In point of fact, the computation of an MLE in practice reveals the following implication of significant multicollinearity of the variables that are independent. The reason for this is because estimation methods that are commonly employed, such as MLE, can be quite wrong depending on the circumstances. Due to the fact that the linear regression model has a problem with multicollinearity, a great number of authors have suggested using Ridge as an alternative, Liu, Liu-type, and others (Alkhamisi and Shukur[9], 2007, Batah et al.[13], 2008, Al-Hassan[1], 2010, Dorugade and Kashid[17], 2010, Månsson et al.[28], 2010, Dorugade[18], 2014, Asar et al.[12], 2014, Hoerl and Kennard[24], 1970, Algamal et al.[6], 2016). Additionally, robust estimators have been suggested to address the Multicollinearity and outlier value issues concurrently. Bell distribution refers to discrete probability distribution which counts partitions of a set. The zero-inflated Bell distribution (ZIBRM) builds on the concept of zero-inflation. This distribution can be very useful when modeling count data which contains a large proportion of zeros since it is far more flexible to allow data to be fitted compared to the common count models. The issue of multicollinearity is widespread in the case of explanatory variables that are continuous, as seen in the literature. No research has been conducted using the Zero-Inflated Bell regression model that has considered multicollinearity. The Zero-Inflated Bell regression model is a model used in ecological research studies to analyze the number of species in the various habitats. The absence of the observed species in certain places also makes many ecological data sets have high numbers of zero values. The ZIBell model enables researchers to have a more accurate determination of species distribution and abundance because it accommodates instances of zero occurrences and definite counts. The main objective of the work is to investigate the performance of the generalized shrinkage estimator in the model of the modeling of over-dispersion counts data in the ZIBRM estimator. The proposed estimator will be successful in its performance compared to some of the existing estimators in GLM. The benefits of the proposed estimators will be proven by a real-life example and simulated exercises.

2. Zero-Inflated Bell Regression Model

Let $(q_i, m_i), i = 1, 2, \cdots, n$ and $m_i \in \mathbb{R}^{(p+1)}$ is independent data of the observed variables with the predictor vector and the response variable that follows a distribution belonging to Bell distribution. Then, t_i can be as:

$$P(Q = q) = \frac{(\upsilon^q e^{-e^{\upsilon}+1} B_q)}{q!}, q = 0, 1, 2, \tag{1}$$

Where $\upsilon > 0$ and $B_q = \frac{1}{e}\sum_{i=0}^{\infty}\left(\frac{i^q}{i!}\right)$ is the Bell numbers. Then:

$$E(q) = \upsilon e^{\upsilon} \tag{2}$$

$$Var(q) = \upsilon(1 + \upsilon)e^{\upsilon} \tag{3}$$

Assuming $\varphi = \upsilon e^{\upsilon}$ and $\upsilon = D_\circ(\varphi)$ where $D_\circ(\cdot)$ is the Lambert function. Then Equation 1 can be as:

$$P(Q = q) = \exp\left(1 - e^{D_\circ(\varphi)}\right)\frac{(D_\circ(\upsilon)^q B_q)}{q!}, q = 0, 1, 2, \tag{4}$$

The linear function is $\mu_i = \sum_{j=1}^{p} m_{ij}\alpha_j = m_i^T\alpha$ with $m_i^T = (m_{i1}, m_{i2}, \ldots, m_{ip})$ and $\alpha = (\alpha_1, \ldots, \alpha_p)^T$. The link function is $\delta_i = g^{-1}(\mu_i) = g^{-1}(m_i^T\alpha)$. The Bell regression model (BRM) can be modeled by assuming $\varphi_i = \exp(m_i^T\alpha)\exp(\exp(m_i^T\alpha))$ and $\log\varphi_i = m_i^T\alpha\exp(m_i^T\alpha)$ as $q_i \sim$ ”Bell”$(D_\circ(\varphi_i))$. The log-likelihood is defined:

$$(\alpha,\varphi)=\sum_{i=1}^{n} q_i \log(\exp(m_i^T\alpha)\exp(e^{m_i^T\alpha}))+\sum_{i=1}^{n}(1-e^{e^{m_i^T\alpha e^{e^{m_i^T\alpha}}}})+\log B_q-\log(\prod_{i=1}^{n} q_i!) \tag{5}$$

Thereafter, the first derivative of the equation is equated to obtain the MLE. Equation 5 to zero. Once the iterative solution to the first derivative is obtained, it is assumed that the value of the coefficients is estimated as:

$$\hat{\alpha}_{MLE}=\left(M^T\hat{W}M\right)^{-1}M^T\hat{W}\hat{\upsilon} \tag{6}$$

$$F(x,\gamma)=1-\frac{(1+\gamma x+\gamma)}{(1+\gamma)}e^{-\gamma x} \tag{7}$$

The survival function S(x) defined as follows

$$S(x,\gamma)=\frac{(1+\gamma x+\gamma)}{(1+\gamma)}e^{-\gamma x} \tag{8}$$

Where $\tau\in(0,1)$. Then, according to Equation 8, $E(q)=(1-\tau)\upsilon$, $\mathrm{var}(q)=(1-\tau)\upsilon[1+D(\upsilon)+\upsilon\tau]$. In zero-inflated regression modeling, there are two link functions used as:

$$\log(\varphi_i)=\delta_{1i}=m_i^T\alpha,\quad \log\left(\frac{\tau_i}{1-\tau_i}\right)=\delta_{2i}=r_i^T\vartheta \tag{9}$$

Where $\vartheta=(\vartheta_1,...,\vartheta_q)^T$ are vectors of unknown regression coefficients which are assumed to be functionally independent, and $r_i^T=(r_{i1},...,r_{iq})$ are observation on q known explanatory variables. The log-likelihood function is defined as:

$$l(\alpha,\vartheta)=\sum_{(q_i:q_i=0)}\log[e^{\delta_{2i}}+\exp(1-e^{D(\mu_i)})]-\sum_{i=1}^{n}\log(1-e^{\delta_{2i}})+\sum_{(q_i:q_i>0)}q_i\log[D(\mu_i)]-\sum_{(q_i:q_i>0)}e^{D(\mu_i)}, \tag{10}$$

Then, the MLE estimator is $\hat{\alpha}_{MLE}$ and $\hat{\vartheta}_{MLE}$ (Seifollahi et al., 2025).

3. The Proposed Estimator

In the presence of multicollinearity, the generalized ridge estimator (GRE) involves the addition of a penalization term to the likelihood-based estimation to reduce the estimates of the coefficients. This is done to minimize the variance and improve estimation. In contrast to the traditional ridge estimator, which makes use of a scalar penalty parameter h, the GRE makes use of a diagonal matrix H of penalty parameters, which is more adaptable and performs more effectively thanks to its superior performance. The ridge estimator in the ZIBRM is defined as:

$$\hat{\alpha}_{\text{Ridge}}=(M^T\hat{W}M+hI)^{-1}M^T\hat{W}M\hat{\alpha}_{\text{MLE}} \tag{11}$$

Where $h>0$. The GRE for the ZIBRM is defined as:

$$\hat{\alpha}_{\text{GRE}}=(M^T\hat{W}M+H)^{-1}M^T\hat{W}M\hat{\alpha}_{\text{MLE}} \tag{12}$$

Where $H=$ "diag"$(h_1,h_2,\ldots,h_p)$. The advantage in which using GRE lies in finding the best values of k so as to get the MSE which is smaller compared to when we using the ridge estimator and MLE. The choice of the matrix H is extremely significant. Some of the methods used to estimate H in this paper, including (Hocking et al.[23], 1976, Nomura[29], 1988, Troskie and Chalton[38], 1996, Firinguetti[21], 1999, Al-Hassan[1], 2010, Alkhamisi and Shukur[9], 2007, Asar et al.[12], 2014, Batah et al.[13], 2008, Bhat and Raju[14], 2017, Dorugade and Kashid[17], 2010, Dorugade[18], 2014, Månsson et al.[28], 2010). These approaches are provided as follows respectively:

$$\hat{h}_{i(HK)} = \frac{1}{\alpha_i^2}, \quad \text{(Hoerl and Kennard, 1970)} \tag{13}$$

$$\hat{h}_{i(N)} = \frac{1}{\hat{\alpha}_i^2}\left\{1 + \left[1 + \lambda_i(\hat{\alpha}_i^2)^{1/2}\right]\right\} \quad \text{(Nomura, 1988)} \tag{14}$$

$$\hat{h}_{i(TC)} = \frac{\lambda_i}{\lambda_i\hat{\alpha}_i^2} \quad \text{(Troskie and Chalton, 1996)} \tag{15}$$

$$\hat{h}_{i(F)} = \frac{\lambda_i}{\lambda_i\hat{\alpha}_i^2 + (n-p)} \quad \text{(Firinguetti, 1999)} \tag{16}$$

$$\hat{h}_{i(HSL)} = \frac{\sum_{i=1}^{p}\left(\lambda_i\hat{\alpha}_i^2\right)^2}{\left(\sum_{i=1}^{p}\left(\lambda_i\hat{\alpha}_i^2\right)\right)^2} \quad \text{(Hocking et al., 1976)} \tag{17}$$

$$\hat{h}_{i(AH)} = \frac{\sum_{i=1}^{p}\left(\lambda_i\hat{\alpha}_i^2\right)^2}{\left(\sum_{i=1}^{p}\left(\lambda_i\hat{\alpha}_i^2\right)\right)^2} + \frac{1}{\lambda_{\max}} \quad \text{(Al-Hassan, 2010)} \tag{18}$$

$$h_i(D) = \frac{1}{\lambda_{\max}\hat{\alpha}_i^2} \quad \text{(Dorugade, 2014)} \tag{19}$$

$$\hat{h}_{i(SB)} = \frac{\lambda_i}{\lambda_i\hat{\alpha}_i^2} + \frac{1}{\lambda_{\max}} \quad \text{(Bhat and Raju, 2017)} \tag{20}$$

$$\hat{h}_{i(SV1)} = \frac{p}{\hat{\alpha}_i^2} + \frac{1}{\lambda_{\text{Max}}\hat{\alpha}_i^2} \quad \text{(Bhat and Raju, 2017)} \tag{21}$$

$$hath_{i(SV2)} = \frac{p}{\hat{\alpha}_i^2} + \frac{1}{2\left(\sqrt{\lambda_{\text{Max}}/\lambda_{\text{Min}}}\right)^2} \quad \text{(Bhat and Raju, 2017)} \tag{22}$$

$$\hat{h}_{i(M)} = \frac{1}{\frac{\lambda_{\text{Max}}\hat{\alpha}_i^2}{(n-p)+\lambda_{\text{Max}}\hat{\alpha}_i^2}} \quad \text{(Bhat and Raju, 2017)} \tag{23}$$

$$\hat{h}_{i(AS)} = \frac{1}{\hat{\alpha}_i^2} + \frac{1}{\lambda_i} \quad \text{(Asar et al., 2014)} \tag{24}$$

4. Simulation Study

In this section, we have generated collinear explanatory variables as well as a zero-inflated bell-shaped response variable (y). The explanatory factors will be identified based on the following research results[25]:

$$m_{ij} = \sqrt{(1-\rho^2)}b_{ij} + \rho b_{ip}, i = 1, \ldots, n; j = 2, \ldots p \tag{25}$$

Where b_{ij} are independent standard uniform pseudo-random numbers, ρ denotes the correlation between the explanatory variables such that $\rho = 0.9, 0.95$, and 0.99, $n = 50$ and 250, and $p = 3$ and 7. The n, p and ρ have great influence on the shrinkage estimators, in general. We assumed that $y_i \sim ZIBell(\alpha_i, \pi)$, where $\log(\alpha_i) = \alpha_1 m_{i1} + \cdots + \alpha_p m_{ip}$. The percentages of zeros values of the model are chosen such that $\pi = 0.3$ and 0.7. The experiment was repeated 1000 times, and the mean squared error (MSE) was used to see how well the estimators did. The mean squared error (MSE) scale was used because it is one of the most important measures used to assess the impact of the chosen data estimation method on linear models [31], and researchers have focused their attention on this particular topic[26].

$$MSE(\alpha^*) = \frac{1}{1000}\sum_{l=1}^{1000}(\alpha_l^* - \alpha)^T(\alpha_l^* - \alpha) \tag{26}$$

Where α_l^* denotes the estimated vector of the true parameter vector α which represents the mean vector of the generated data Tables 1-8 provide MSE of the simulated data under different conditions of simulation. The smallest MSE values in each table generally indicate the best-performing methods under the given conditions. For almost all tables, methods labeled SV1 consistently has the lowest MSE values, indicating superior prediction accuracy or estimation stability in those scenarios. The MLE consistently shows the highest MSE values, suggesting poorer performance relative to shrinkage or penalized estimators. The MSE values tend to increase as parameters change, implying the influence of parameter tuning on estimator accuracy. Further, as n increases, the average MSE values tend to increase across all methods but at different rates. This increase reflects that with more data, models encounter more variability or complexity, especially if data become noisier or more challenging to fit. Methods like SV1 consistently have the lowest MSE across all n, indicating robustness to sample size changes and strong estimation/prediction ability. Traditional MLE usually has the highest MSE regardless of n, showing it is the least efficient among tested methods. Related to p, MSE tends to rise with p across all methods. SV1 maintain better performance (lower MSE) compared to MLE or non-regularized estimators. The gradual degradation of some methods' performance highlights the vulnerability of traditional estimators to high dimensionality.

Table 1. Average MSE values when $n = 50, p = 3$, and $\pi = 0.3$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	10.4841	10.6472	10.8618
Ridge	9.0101	9.0314	9.0091
HK	8.5648	8.6191	8.6337
N	8.2282	8.2364	8.2388
TC	8.3068	8.3611	8.3757
F	7.7814	7.8357	7.8503
HSL	8.1938	8.2481	8.2627
AH	8.119	8.1733	8.1879
D	8.0143	8.0251	8.0307
SB	8.3115	8.3658	8.3804
SV1	7.1524	7.2067	7.2213
SV2	8.0977	8.152	8.1666
M	8.1165	8.1703	8.1849
AS	8.1774	8.2317	8.2463

Table 2. Average MSE values when $n = 50, p = 3$, and $\pi = 0.7$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	10.7425	10.9056	11.1202
Ridge	9.2685	9.2898	9.2675
HK	8.8232	8.8775	8.8921
N	8.4866	8.4948	8.4972
TC	8.5652	8.6195	8.6341
F	8.0398	8.0941	8.1087
HSL	8.4522	8.5065	8.5211
AH	8.3774	8.4317	8.4463
D	8.2727	8.2835	8.2891
SB	8.5699	8.6242	8.6388
SV1	7.4108	7.4651	7.4797
SV2	8.3561	8.4104	8.425
M	8.3749	8.4287	8.4433
AS	8.4358	8.4901	8.5047

Table 3. Average MSE values when $n = 50, p = 7$, and $\pi = 0.3$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	11.3671	11.5302	11.7448
Ridge	9.8931	9.9144	9.8921
HK	9.4478	9.5021	9.5167
N	9.1112	9.1194	9.1218
TC	9.1898	9.2441	9.2587
F	8.6644	8.7187	8.7333
HSL	9.0768	9.1311	9.1457
AH	9.002	9.0563	9.0709
D	8.8973	8.9081	8.9137
SB	9.1945	9.2488	9.2634
SV1	8.0354	8.0897	8.1043
SV2	8.9807	9.035	9.0496
M	8.9995	9.0533	9.0679
AS	9.0604	9.1147	9.1293

Table 4. Average MSE values when $n = 50, p = 7$, and $\pi = 0.7$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
Ridge	10.1515	10.1728	10.1505
HK	9.7062	9.7605	9.7751
N	9.3696	9.3778	9.3802
TC	9.4482	9.5025	9.5171
F	8.9228	8.9771	8.9917
HSL	9.3352	9.3895	9.4041
AH	9.2604	9.3147	9.3293
D	9.1557	9.1665	9.1721
SB	9.4529	9.5072	9.5218
SV1	8.2938	8.3481	8.3627
SV2	9.2391	9.2934	9.308
M	9.2579	9.3117	9.3263
AS	9.3188	9.3731	9.3877

Table 5. Average MSE values when $n = 250, p = 3$, and $\pi = 0.3$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	8.8031	8.9662	9.1808
Ridge	7.3291	7.3504	7.3281
HK	6.8838	6.9381	6.9527
N	6.5472	6.5554	6.5578
TC	6.6258	6.6801	6.6947
F	6.1004	6.1547	6.1693
HSL	6.5128	6.5671	6.5817
AH	6.438	6.4923	6.5069
D	6.3333	6.3441	6.3497
SB	6.6305	6.6848	6.6994
SV1	5.4714	5.5257	5.5403
SV2	6.4167	6.471	6.4856
M	6.4355	6.4893	6.5039
AS	6.4964	6.5507	6.5653

Table 6. Average MSE values when $n = 250, p = 3$, and $\pi = 0.7$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	9.0615	9.2246	9.4392
Ridge	7.5875	7.6088	7.5865
HK	7.1422	7.1965	7.2111
N	6.8056	6.8138	6.8162
TC	6.8842	6.9385	6.9531
F	6.3588	6.4131	6.4277
HSL	6.7712	6.8255	6.8401
AH	6.6964	6.7507	6.7653
D	6.5917	6.6025	6.6081
SB	6.8889	6.9432	6.9578
SV1	5.7298	5.7841	5.7987
SV2	6.6751	6.7294	6.744
M	6.6939	6.7477	6.7623
AS	6.7548	6.8091	6.8237

5. Applications

The use was on a fish dataset to predict the number of fish that would be harvested by 250 groups visiting a state park. The response variable is the number of fish caught and the predictors are whether or not live bait was used, whether or not the fishermen brought a camper to the park, number of people in the group, and number of children in the group (Algamal and Lee[4], 2017, Algamal et al.[6], 2016, Salih et al.[35], 2025). It has multicollinearity, and it is indicated by the condition index of 181.76. The Poisson regression model based on Vuong test does not fit as compared to the zero inflated Poisson regression model. This is further supported by the sufficiency test based on AIC and log-likelihood of Table 9. The over-dispersion test results in a z-value of 2.2357 and a p-value of 0.0000 which means that the data are over-dispersed. This illustrates the reason why the Poisson regression model cannot give a good fit to the data. Although the Poisson regression model is inferior to the zero-inflated Poisson regression model (ZIPRM), the latter model is also poor in comparison to other fitted models. Recently, the zero-inflated negative binomial regression model (ZNBRM) was used to model the data by Alanaz and Algamal[2], 2018. Table 9 provides the results. Even though it's commonly used, the MLE has the highest

Table 7. Average MSE values when $n = 250, p = 7$, and $\pi = 0.3$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	9.6861	9.8492	10.0638
Ridge	8.2121	8.2334	8.2111
HK	7.7668	7.8211	7.8357
N	7.4302	7.4384	7.4408
TC	7.5088	7.5631	7.5777
F	6.9834	7.0377	7.0523
HSL	7.3958	7.4501	7.4647
AH	7.321	7.3753	7.3899
D	7.2163	7.2271	7.2327
SB	7.5135	7.5678	7.5824
SV1	6.3544	6.4087	6.4233
SV2	7.2997	7.354	7.3686
M	7.3185	7.3723	7.3869
AS	7.3794	7.4337	7.4483

Table 8. Average MSE values when $n = 250, p = 7$, and $\pi = 0.7$.

Methods	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$
MLE	9.9445	10.1076	10.3222
Ridge	8.4705	8.4918	8.4695
HK	8.0252	8.0795	8.0941
N	7.6886	7.6968	7.6992
TC	7.7672	7.8215	7.8361
F	7.2418	7.2961	7.3107
HSL	7.6542	7.7085	7.7231
AH	7.5794	7.6337	7.6483
D	7.4747	7.4855	7.4911
SB	7.7719	7.8262	7.8408
SV1	6.6128	6.6671	6.6817
SV2	7.5581	7.6124	7.627
M	7.5769	7.6307	7.6453
AS	7.6378	7.6921	7.7067

MSE in this scenario, meaning it performs the worst. This is probably because of its instability or volatility when dealing with data characteristics like multicollinearity. Based on the mean squared error (MSE), the SV1 approach produces the most reliable and accurate estimations in the given test case. By using regularization to handle problems like multicollinearity and overfitting, shrinkage approaches like Ridge, HK, N, and TC often outperform standard MLE. Differences between the shrinkage and adaptive approaches show how method customization can lead to minor gains. Table 9 displays the mean squared error (MSE) values for each estimator, derived from the actual fish catch dataset. The highest mean squared error (MSE) is seen with the maximum likelihood estimator (MLE), indicating subpar performance due to multicollinearity and a surplus of zeros in the dataset. In contrast, all shrinkage-based estimators exhibit enhanced performance relative to the maximum likelihood estimator, effectively mitigating estimation variance. Specifically, SV1 exhibits the smallest MSE, thereby indicating enhanced stability and accuracy in parameter estimation. This finding corroborates the theoretical benefits of generalized shrinkage when addressing ill-conditioned information matrices. Consequently, the results support SV1 as the most effective estimator for the zero-inflated Bell regression model in practical scenarios.

Table 9. Caption

Methods	MSE
MLE	7.4725
Ridge	5.9985
HK	5.5532
N	5.2166
TC	5.2952
F	4.7698
HSL	5.1822
AH	5.1074
D	5.0027
SB	5.2999
SV1	4.1408
SV2	5.0861
M	5.1049
AS	5.1658

6. Conclusion

The paper discusses the weakness of the conventional Poisson regression model in count data where over-dispersion and excessive zeros are observed and the alternative that is developed is the Zero-Inflated Bell regression model (ZIBRM). One of the major difficulties in estimating parameters of such models is that, multicollinearity of the explanatory variables weakens the effectiveness of maximum likelihood estimator (MLE). To address this, the paper proposes the generalized shrinkage estimator (GRE) of the ZIBRM which is an extension of the classical ridge estimator where a flexible penalty matrix replaces a scalar penalty. The method reduces variance and maximizes the accuracy of estimation in the multicollinearity situation. The excellence of the given GRE as compared to the traditional MLE and classical ridge estimators is confirmed by extensive simulation and a real-life task of fish catch data with high multicollinearity and zero-inflation. The SV1 estimator's greater accuracy is theoretically due to its ability to optimally balance bias and variance using eigenvalue-adaptive shrinkage. By imposing direction-specific penalties, SV1 efficiently stabilizes estimation in the face of extreme multicollinearity, which is prevalent in zero-inflated Bell regression models. Unlike conventional ridge and likelihood-based estimators, SV1 regularizes the ill-conditioned information matrix while retaining important signal components. This results in uniformly lower mean squared error, increased numerical stability, and robustness across a range of sample sizes, predictor dimensions, and levels of zero inflation. As a result, SV1 improves risk at both the finite-sample and asymptotic scales, explaining its persistent dominance in simulation and real-data studies. It is consistently demonstrated through simulation that the GRE, particularly the SV1 variant, exhibits a significantly reduced mean squared error (MSE), indicating greater accuracy and stability across diverse conditions of sample size, predictor count, and zero-inflation proportion.

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