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Regression Model for Gamma Lindley Distribution with Application

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Abstract In the current investigation, a unique regression model is presented, in which the response variable is modeled after the gamma Lindley distribution. A approach called as maximum likelihood is utilized in order to estimate the model's parameters that are unknown. An investigation based on simulation is carried out in order to evaluate the accurateness of these estimates. Additionally, in order to determine whether or not the suggested model is adequate, a residual analysis is carried out. There are a number of statistical criteria that are used to evaluate the gamma Lindley model in comparison to other regression models, such as the Weibull model and the gamma regression model. Based on the findings, it can be concluded that the model that was provided offers a better alignment with the data in comparison to the other options. The model is anticipated to have applications in a wide range of sectors, such as economics, biological research, mortality and recovery analysis, health studies, hazard assessment, measurement sciences, medicine, and engineering, among others.

Keywords definition of gamma Lindley distribution, gamma Lindley regression model, maximum likelihood, residual analysis, deviance and martingale residual

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1. Introduction

A variety of probability distributions have been utilized to model data in several domains, including economics, biological research, mortality analysis, recovery rates, health sciences, risk assessment, measurement sciences, medicine, engineering, insurance, and finance. In recent years, numerous initiatives have emerged to provide data modeling methodologies grounded in certain distributions. Cordeiro and Altun in 2020 [5], introduced the unit-improved second-degree Lindley distribution for inference and regression modeling. Ortega et al. in 2011 [20], proposed the log-generalized modified Weibull regression model. Mazucheli et al. 2021 [17], created an innovative quantile regression model for constrained data utilizing the Birnbaum-Saunders distribution. Mallick et al. in 2021 [16], offered the Log-Burr XII regression model, whereas Ortega et al. in 2013 [20], presented the Log Beta Generalized Weibull Regression Model for lifetime data. Salih in 2025 et al [22] introduces a study of lambert topp-Leoni distribution, Salih and Hussain in 2025 [21] introduces the Quasi Lindley regression model.

Korkmaz and Chesneau in 2021 [14], and Khan [13], introduced the Exponentiated Weibull regression model, whereas quantile regression was suggested utilizing the unit Burr-XII distribution, whilst Granzotto et al. in 2018 [11], formulated the Transmuted Weibull Regression Model. Ocloo et al. in 2023 [19], expanded the Burr XII distribution for use in regression analysis. Al-Aqtash et al. in 2021 [2] suggested the Gumbel-Burr XII Regression Model, whereas Altun et al. in 2018 [4], introduced the Zografos–Balakrishnan Burr XII Regression Model. Korkmaz et al. in 2021 [14], formulated the Unit-Chen quantile regression model, whereas Mazucheli et al. 2023

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in [18], proposed the unit generalized half-normal quantile regression model. Benkhelifa in 2022 [6], introduced the log beta power Muth distribution regression model.

The gamma Lindley distribution is a versatile probability distribution employed for modeling non-negative data exhibiting diverse levels of skewness and kurtosis. It is constituted by the amalgamation of the Lindley and gamma distributions, augmenting its capacity to encapsulate intricate data patterns. This distribution has been utilized in multiple domains, including reliability analysis, survival studies, and risk assessment, owing to its versatility in modeling lifespan data and failure rates. The gamma Lindley distribution provides benefits over simpler models by supporting both light and heavy tails, rendering it appropriate for datasets with varied variability. The parameters can be calculated by techniques like as maximum likelihood estimation, and it is frequently juxtaposed with other distributions, including Weibull and gamma models, for performance evaluation.

The organization of this article is delineated as follows: Section 2 Theoretical foundations. Section 3 examines diverse techniques for residual analysis. Section 4 addresses the simulation study. Ultimately, Section 5 presents the conclusions.

2. Theoretical foundations

2.1. Definition Lindley distribution

Zeghdoudi and Nedjar recently introduced the gamma Lindley distribution, which is formed from a combination of the gamma distribution and the one-parameter Lindley distribution. This distribution's features have been examined, encompassing the mean, variance, survival function, hazard rate function, moment-generating function (mgf), probability density function (pdf), cumulative distribution function (cdf), and findings pertaining to stochastic ordering. Furthermore, graphical depictions of the probability density function (pdf) and cumulative distribution function (cdf) for designated parameter values are provided. The probability density function and the associated cumulative distribution function of the gamma Lindley distribution are articulated as follows [3]:

$$f\left(x,a,\lambda\right)=\frac{a^{2}\left[\left(\lambda+\lambda a-a\right)x-1\right]}{\lambda\left(1+a\right)}\exp\left(-ax\right)a,x\geq 0 \tag{1}$$

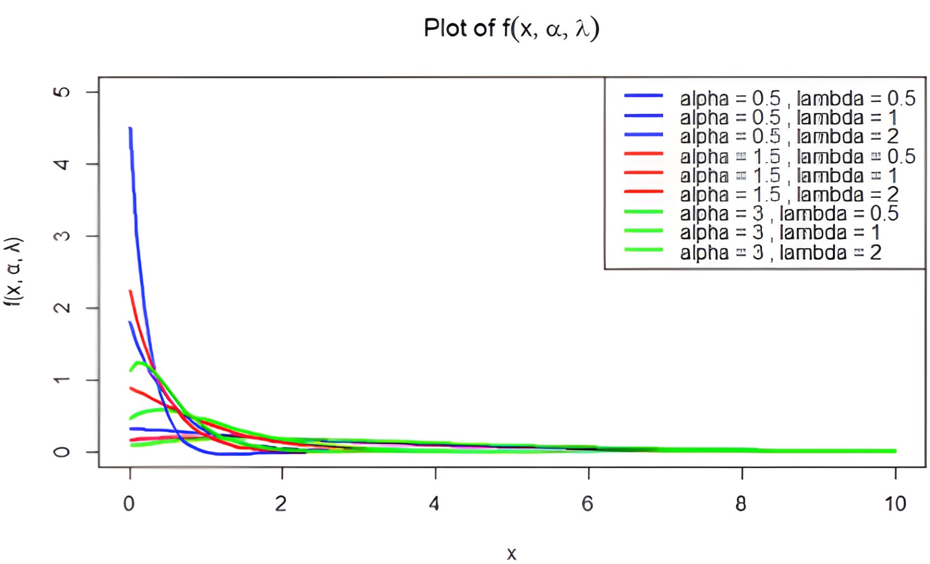

Figure 1. Illustrates the probability density function (PDF) of the gamma Lindley

$$F(x,a,\lambda) = 1 - \frac{((a\lambda + \lambda - a)(ax+1) + a)}{\lambda(1+a)} \exp(-ax)\, a, x \geq 0 \tag{2}$$

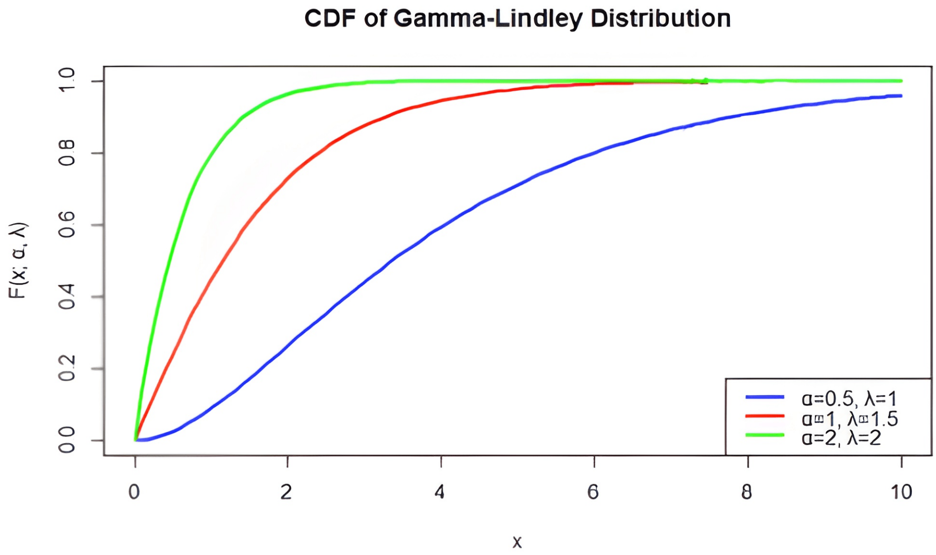

Figure 2. Illustrates Cumulative Distribution Function (CDF) of the gamma Lindley

2.2. *The Moment generating Function*

Assume that the variable X is a random one. If the expected value $[e^{tx}]$ exists and is finite for all real numbers t that fall within a closed interval $[-K, K]$, we defined to the function as the moment-generating function of X as [17].

$$E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)\, dx \tag{3}$$

In the process of substituting Equation (1) into Equation (3), we obtain the following:

$$E\left(e^{tx}\right) = \int_0^\infty \frac{a^2\left[(\lambda + \lambda a - a)x + 1\right]}{\lambda(1+a)} e^{-x(-t+a)} dx \tag{4}$$

By reducing the complexity of equation (4) and performing integration by parts, we arrive at the following:

$$E\left(e^{tx}\right) = \int_0^\infty \frac{a^2\left[(\lambda + \lambda a - a)x + 1\right]}{\lambda(1+a)} e^{-x(-t+a)} + \int_0^\infty \frac{a^2}{\lambda(1+a)} e^{-x(-t+a)} dx$$

$$E\left(e^{tx}\right) = \int_0^\infty \frac{a^2\left[(\lambda + \lambda a - a)x\right]}{\lambda(1+a)} e^{-x(-t+a)} dx$$

$$E\left(e^{tx}\right) = \frac{a^2\left[(\lambda + \lambda a - a)\right]}{\lambda(1+a)} \int_0^\infty x\, e^{-x(-t+a)} dx$$

$$E\left(e^{tx}\right) = \frac{a^2(\lambda + \lambda a - a)}{\lambda(1+a)} \left[x \frac{e^{-x(-t+a)}}{-(a-t)} + \frac{1}{(a-t)} \int_0^\infty e^{-x(-t+a)} dx\right]$$

$$M_1\left(e^{tx}\right) = \frac{a^2(\lambda + \lambda a - a)}{\lambda(1+a)} \left[x \frac{e^{-x(-t+a)}}{-(a-t)} + \frac{e^{-x(-t+a)}}{(a-t)^2}\right]_0^\infty$$

$$M_1\left(e^{tx}\right) = \frac{a^2\left(\lambda+\lambda a-a\right)}{\lambda\left(1+a\right)}\frac{1}{\left(a-t\right)^2} \tag{5}$$

Through the process of differentiating equation (5) and filling in the value of t = 0, we obtain the following:

$$\frac{dM_1\left(t\right)}{dt} = \frac{a^2\left(\lambda+\lambda a-a\right)}{\lambda\left(1+a\right)}\frac{2}{\left(a-t\right)^3}$$

$$E_1\left(x\right) = \frac{dM_1\left(t\right)}{dt} = \frac{2\left(\lambda+\lambda a-a\right)}{\lambda a\left(1+a\right)}$$

By integrating the following function, we get

$$M_2(x) = \int_0^\infty \frac{a^2}{\lambda\left(1+a\right)}\, e^{-x(-t+a)}dx$$

$$E_1\left(x\right) = \frac{dM_1\left(t\right)}{dt} = \frac{2\left(\lambda+\lambda a-a\right)}{\lambda a\left(1+a\right)}$$

$$M_2\left(x\right) = \left[\frac{a^2e^{-x(-t+a)}}{\lambda\left(1+a\right)\left(a-t\right)}\right]_0^\infty \tag{6}$$

$$E_2\left(x\right) = \frac{a^2}{\lambda\left(1+a\right)\left(a\right)^2}$$

$$E_2\left(x\right) = \frac{1}{\lambda\left(1+a\right)} \tag{7}$$

$$E\left(x\right) = E_1\left(x\right) + E_2\left(x\right)$$

By substituting in Equation (7) for the value of the first expectation and the second expectation, we get the following

$$E\left(x\right) = \frac{2(\lambda+\lambda a-a)}{\lambda a\left(1+a\right)} + \frac{1}{\lambda(1+a)}$$

$$E\left(x\right) = \frac{2\lambda\left(1+a\right)-a}{\lambda a\left(1+a\right)} \tag{8}$$

To obtain the variance of the proposed distribution, the second expectation of the distribution must be obtained and then substituted into the general law of variance

$$E\left(x\right) = \frac{2\lambda\left(1+a\right)-a}{\lambda a\left(1+a\right)}$$

$$E_1\left(x^2\right) = \frac{d^2M_1\left(t\right)}{dt^2} = \left[\frac{a^2\left(\lambda+\lambda a-a\right)}{\lambda\left(1+a\right)}\frac{6}{\left(a-t\right)^4}\right]_{t-0} \tag{9}$$

$$E_1\left(x^2\right) = \frac{d^2M_1\left(t\right)}{dt^2} = \left[\frac{\left(\lambda+\lambda a-a\right)}{\lambda\left(1+a\right)}\frac{6}{\left(a\right)^2}\right]_{t-0} \tag{10}$$

$$E_2\left(x^2\right) = \frac{d^2M_1\left(t\right)}{d(t)^2} = \frac{2}{\lambda a\left(1+a\right)} \tag{11}$$

$$E\left(x^2\right) = E_1\left(x^2\right) + E_2\left(x^2\right) \tag{12}$$

$$E\left(x^2\right) = \frac{\left(\lambda+\lambda a-a\right)}{\lambda\left(1+a\right)}\frac{6}{\left(a\right)^2} + \frac{2}{\lambda a(1+a)} \tag{13}$$

$$E\left(x^2\right) = \frac{\left(6\lambda+6\lambda a-4a\right)}{\lambda\left(1+a\right)a^2} + \tag{14}$$

$$V(x) = E\left(x^2\right) + [E(x)]2 \tag{15}$$

Substituting into Equation (15), we get the final form of the proposed distribution variance

$$V(x) = \frac{(6\lambda + 6\lambda a - 4a)}{\lambda(1+a)a^2} + \left[\frac{2\lambda(1+a) - a}{\lambda a(1+a)}\right]^2 \tag{16}$$

2.3. Moments

Moments are quantitative measurements expressed as a function that describes the proposed distribution's characteristic in mathematics. Moments are defined as follows [12]:

Substitute in Equation (17) into the gamma Lindley probability density function

$$E(x^r) = \int_0^\infty x^r f(x)\, dx \tag{17}$$

$$E(x^r) = \frac{a^2\left[(\lambda + \lambda a - a)x + 1\right]}{\lambda(1+a)} \int_0^\infty x^r e^{-ex} f(x)\, dx \tag{18}$$

In simplifying Equation No. (18) and the integration by parts, we get the following

$$E_1(x^r) = \frac{a^2}{\lambda(1+a)} \left[\frac{1}{-a} x^r e^{-ax}\Big|_0^\infty + \frac{r}{a}\int_0^\infty x^{r-1} e^{-ex}\right]$$

$$E_1(x^r) = \frac{a^2}{\lambda(1+a)} \left[r\frac{1}{a}\int_0^\infty x^{r-1} e^{-ex}\right]$$

From the general definition of the gamma function, we get the following

$$E_1(x^r) = \frac{ra^2}{a^{r+1}\lambda(1+a)}\left[\Gamma(r)\right]$$

$$E_2(x^r) = \frac{a^2(\lambda + \lambda a - a)x}{\lambda(1+a)} \int_0^\infty x^{r+1} e^{-ex} dx$$

From the general definition of the gamma function, we get the following

$$E_2(x^r) = \frac{a^2(\lambda + \lambda a - a)}{\lambda(1+a)a^{r+1}}\left[\Gamma(r+2)\right] \tag{19}$$

$$E(x^r) = \frac{a}{\lambda a^r(1+a)}\left[\Gamma r + 1\right] + \frac{a^2(\lambda + \lambda a - a)}{\lambda(1+a)a^{r+1}}\left[\Gamma(r+2)\right]$$

To obtain the first and second moments, we substitute the value of r in the equation by one and 2

$$E(x) = \frac{a}{\lambda a^2(1+a)}\left[\Gamma 2\right] + \frac{a^2(\lambda + \lambda a - a)}{\lambda(1+a)a^3}\left[\Gamma(3)\right] \tag{20}$$

By simplifying Equation (20), we get the expected value for the Lindley gamma distribution

$$E(x) = \frac{2\lambda(1+a) - a}{\lambda a(1+a)}$$

2.4. Survival Function

Let X be a continuous random variable with a cumulative distribution function on the interval $[0, \infty]$ called $F(x)$. The following is its survival or reliability function [1]

$$S(x) = \int_0^\infty f(u)\, du = 1 - F(X) \tag{21}$$

Substituting into Equations (21), we get a function for the gamma -Lindley distribution

$$S(x) = \frac{[(a\lambda + \lambda - a)(ax + 1) + a]}{\lambda(1 + a)} e^{-ax} \tag{22}$$

2.5. Hazard rate function

A continuous random variable X with pdf $f(x)$ and cdf $F(x)$ is defined as having the hazard rate function (also known as the failure rate function) $h(x)$.

$$h(x) = \frac{p(X < x + \Delta x | X > x)}{\Delta x} = \frac{[F(x)]}{1 - F(x)} \tag{23}$$

Substituting into Equation (23), we get a hazard rate function for the gamma-Lindley distribution

$$h(x) = \frac{a^2(\lambda + \lambda a - a)x + 1}{[(a\lambda + \lambda - a)(ax + 1) + a]} \tag{24}$$

2.6. Skewness

Let x be a continuous random variable following a gamma Lindley distribution, then the skewness is defined as

$$\delta_3 = \frac{\mu_3}{\sigma_3} \tag{25}$$

μ_3 the average here is the average around the arithmetic mean we defined μ_3 as follows

$$\mu_3 = \int_0^\infty (x - \mu')\, f(x)\, dx$$
$$\mu_3 = \int_0^\infty \left(x^3 - \mu'^3 - 3x\mu'^2 + 3\mu'^2 x^2\right) f(x)\, dx$$
$$\mu_3 = \int_0^\infty x^3 f(x)\, dx - \mu'^3 \int_0^\infty f(x)\, dx - 3\mu'^2 \int_0^\infty x f(x)\, dx + 3\mu'^2 \int_0^\infty x^2 f(x)\, dx$$
$$\mu_3 = (\mu'_3 + 2\mu'_1 - 3\mu'_1\mu'_2)$$

The variance can be calculated from the following formula

$$\mu^2 = \mu'_2 - [\mu'_1]^2 \tag{26}$$

Where μ'_1 is the expectation about the origin or μ the expectation about the mean. Thus, the skewness becomes

$$\delta_3 = \frac{(\mu'_3 + 2\mu_1'^3 - 3\mu'_1\mu'_2)}{\left[\mu'_2 - [\mu'_1]^2\right]^{3/2}}$$

From Equation (18), by substituting the value of r into one, two, and three, we get the value of the first, second, and third expectation

$$\delta_3 = \frac{(\mu_3^{,} + 2\mu_1^{,3} - 3\mu_1^{,}\mu_2^{,})}{\left[\mu_2^{,} - [\mu_1^{,}]^2\right]^{3/2}}$$

$$\mu_1^{,} = \frac{1}{\lambda(1+a)} + \frac{2(\lambda + \lambda a - a)}{\lambda\,(1+a)\,a} = \frac{2\,(\lambda + \lambda a) - a}{\lambda\,(1+a)\,a}$$

$$\mu_2^{,} = \frac{2}{\lambda a(1+a)} + \frac{6(\lambda + \lambda a - a)}{\lambda\,(1+a)\,a^2} = \frac{6\,(\lambda + \lambda a) - 4a}{\lambda\,(1+a)\,a^2}$$

$$\mu_3^{,} = \frac{6}{\lambda a^2(1+a)} + \frac{24(\lambda + \lambda a - a)}{\lambda\,(1+a)\,a^3} = \frac{24\,(\lambda + \lambda a) - 18a}{\lambda\,(1+a)\,a^3}$$

$$\delta_3 = \frac{\frac{24(\lambda+\lambda a)-18a}{\lambda(1+a)a^3} + 2\left(\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right)^3 - 3\frac{[6(\lambda+\lambda a)-4a][2(\lambda+\lambda a)-a]}{\lambda^2(1+a)^2a^3}}{\left[\frac{[6(\lambda+\lambda a)-4a]}{\lambda(1+a)a^2} - \left[\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right]^2\right]^{3/2}}$$

2.7. kurtosis

If x is a continuous random variable that follows a gamma Lindley distribution, then the kurtosis is defined as

$$\delta_4 = \frac{\mu_4}{\sigma^4} \tag{27}$$

μ_4 the average here is the average around the arithmetic mean we defined μ_4 as follows

$$\mu_4 = \int_0^\infty (x - \mu^{,})^4 f(x)\,dx$$

$$\mu_4 = \int_0^\infty x^4 f(x)\,dx + \int_0^\infty \mu_1^4 f(x)\,dx + \int_0^\infty 4x^2\mu_1^{,2} f(x)\,dx + \int_0^\infty 3x^2\mu_1^{,2} f(x)\,dx - \int_0^\infty 4x^3\mu^{,} f(x)\,dx - \int_0^\infty 4x^3\mu_1^{,3} f(x)\,dx$$

$$\mu_4 = \mu_4^{,} + \mu_1^{,4} + 6\mu_1^{,2}\mu_2^{,} - 4\mu_1^{,}\mu^{,} - 3\mu_1^{,4} \tag{28}$$

The variance can be calculated from the following formula

$$\sigma^2 = \mu_2^{,} + [\mu_1^{,}]^2$$

Where $\mu_1^{,}$ is the expectation about the origin or μ the expectation about the mean. Thus, the kurtosis becomes

$$\delta_4 = \frac{\mu_4^{,} + \mu_1^{,4} + 3\mu_1^{,2}\mu_2^{,} - 4\mu_1^{,}\mu^{,} - 3\mu_1^{,4}}{\left[\mu_2^{,} - [\mu_1^{,}]^2\right]^2} \tag{29}$$

From Equation (18), by substituting the value of r into one, two, three, and four, we get the value of the first, second, and third expectation

$$\mu_1' = \frac{1}{\lambda(1+a)} + \frac{2(\lambda+\lambda a-a)}{\lambda\,(1+a)\,a} = \frac{2\,(\lambda+\lambda a)-a}{\lambda\,(1+a)\,a}$$
$$\mu_2' = \frac{2}{\lambda a(1+a)} + \frac{6(\lambda+\lambda a-a)}{\lambda\,(1+a)\,a^2} = \frac{6\,(\lambda+\lambda a)-4a}{\lambda\,(1+a)\,a^2}$$
$$\mu_3' = \frac{6}{\lambda a^2(1+a)} + \frac{24(\lambda+\lambda a-a)}{\lambda\,(1+a)\,a^3} = \frac{24\,(\lambda+\lambda a)-18a}{\lambda\,(1+a)\,a^3}$$
$$\mu_4' = \frac{24}{\lambda a^3(1+a)} + \frac{120(\lambda+\lambda a-a)}{\lambda\,(1+a)\,a^4}$$

$$\delta_3 = \frac{\left(\frac{24}{\lambda a^3(1+a)} + \frac{120(\lambda+\lambda a-a)}{\lambda(1+a)a^4}\right) + \left(\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right)^4}{\left[\frac{[6(\lambda+\lambda a)-4a]}{\lambda(1+a)a^2} - \left[\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right]^2\right]^2} + \frac{6\left(\frac{2(\lambda+\lambda a-a)}{\lambda(1+a)a}\right)^2\left(\frac{6(\lambda+\lambda a)-4a)}{\lambda(1+a)a^2}\right)}{\left[\frac{[6(\lambda+\lambda a)-4a]}{\lambda(1+a)a^2} - \left[\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right]^2\right]^2}$$
$$- \frac{4\left(\frac{2(\lambda+\lambda a)-a)}{\lambda(1+a)a}\right)\left(\frac{24(\lambda+\lambda a)-18a)}{\lambda(1+a)a^3}\right) - 3\left(\frac{2(\lambda+\lambda a)-a)}{\lambda(1+a)a}\right)^4}{\left[\frac{[6(\lambda+\lambda a)-4a]}{\lambda(1+a)a^2} - \left[\frac{2(\lambda+\lambda a)-a}{\lambda(1+a)a}\right]^2\right]^2}$$

2.8. Regression model for gamma Lindley

For the analysis of censored response variables and covariates, widely used models include the log-location-scale regression model. Over the past decade, researchers have introduced more flexible location-scale regression models to better capture the characteristics of various datasets. Notable works in this area include [15] proposing the Log-Beta Log-Logistic regression model, and Altun et al. [5], introducing a new one-parameter lifetime distribution and its regression model with practical applications.

Let X be a random variable following a gamma Lindley distribution with two parameters. Its probability density function for a continuous random variable can be derived using the following relation:

Parameter $\lambda, \alpha \succ 0$ and is defined by the probability density function of a continuous random variable (y) may be obtained using the relation of

$$f\,(y) = f\left(g^{-1}(y)\right)\left|\frac{dx}{dy}\right| \tag{30}$$

To obtain the Jacobian value, we differentiate the hypothesis with respect to $xJ = \left|\frac{dx}{dy}\right| = \frac{x}{\sigma}$

To obtain value

$$g^{-1}(y)\frac{y-\mu}{\sigma} = \log\,(x) \to x = \exp\left(\frac{y-\mu}{\sigma}\right) \tag{31}$$

$g^{-1}(y)$ Substituting into the probability density function of the gamma Lindley distribution, we get the value of

$$g^{-1} = \frac{a^2\left[(\lambda+\lambda a-a)\exp\left(\frac{v}{\sigma}\right)\right)]}{\lambda(1+a)}\exp\left(-\exp\frac{y-\mu}{\sigma}\right)\ldots x-m \tag{32}$$

Substitute in Equation (28) for the value of $g^{-1}(y)$ we get the following

$$f(y) = \frac{a^2}{\lambda(1+a)\sigma}\exp\left(-\exp\frac{y-\mu}{\sigma}\right)\exp\frac{y-\mu}{\sigma} \tag{33}$$

Where $y \in R, \mu \in R, \sigma \succ 0, \alpha \succ 0, \lambda \succ 0$ where $\mu \in R$ is the location parameter and is the scale parameter, The corresponding commutative function is

$$F\left(x, a, \lambda\right) = \frac{\exp\left(\frac{v-\mu}{\sigma}\right)}{\sigma}\left[1 - \frac{\left[\left(\lambda + \lambda a - a\right) a\ \exp\left(\frac{y-\mu}{\sigma}\right) + 1\ \right) + a\right]}{\lambda(1+a)}\right] \exp\left(-\ a\ \exp\frac{y-\mu}{\sigma}\right) \tag{34}$$

The corresponding survival function is [10]

$$S(y) = \frac{\exp\left(\frac{v-\mu}{\sigma}\right)}{\sigma}\left[\frac{\left[\left(\lambda + \lambda a - a\right) a\ \exp\left(\frac{y-\mu}{\sigma}\right) + 1\ \right) + a\right]}{\lambda(1+a)}\right] \exp\left(-\ a\ \exp\frac{y-\mu}{\sigma}\right) \tag{35}$$

The standardized random variable $z = \left(\frac{y-\mu}{\sigma}\right)$ the density function of the gamma Lindley becomes

$$f(y) = \frac{a^2\left[\left(\lambda + \lambda a - a\right) a \exp\left(z\right) + 1\ \right)\right]}{\lambda\sigma(1+a)} \exp\left(-\ a(z)\ \exp(z)\right. \tag{36}$$

The corresponding standardized random variable commutative function is

$$F\left(x, a, \lambda\right) = \frac{\exp\left(z\right)}{\sigma}\left[1 - \frac{\left[\left(\lambda + \lambda a - a\right) a\ \exp\left(z\right) + 1\ \right) + a\right]}{\lambda(1+a)}\right] \exp\left(-\ a\ \exp(z)\right. \tag{37}$$

The corresponding standardized random variable survival function is

$$S(y) = \frac{\exp\left(z\right)}{\sigma}\left[\frac{\left[\left(\lambda + \lambda a - a\right) a\ \exp\left(z\right) + 1\ \right) + a\right]}{\lambda(1+a)}\right] \exp\left(-\ a\ \exp(z)\right. \tag{38}$$

Parametric regression models are frequently utilized to estimate univariate survival functions for censored data. A parametric model that closely aligns with lifespan data generally produces more precise estimates of the relevant quantities. This estimation is based on the density function of the gamma Lindley distribution. A linear regression model incorporating location-scale parameters is presented, delineating the link between the dependent variable yi and the vector of independent variables, defined as follows:

$$y_i = \mu_i + \sigma z_i$$

$$y_i = x^{`}\beta + \sigma z_i \tag{39}$$

By substituting in Equation (36) for the value of the independent variable, as well as the regression coefficient, we get the following.

$$y_i = (1\ \ x)\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix} + \sigma z_i$$

$$y_i = \beta_0 + \beta_1 + \sigma z_i \tag{40}$$

Where the random error z_i has the density function of the gamma Lindley, $\beta = (\beta_0, \beta_1)'$, The unknown $\mu_i = x'\beta$ is the location of y, the location parameter μ_i, which represents the linear model.

Consider a sample of nnn explanatory observations, where each random dependent variable yi is defined as Equation (39) and follows the gamma Lindley distribution. It is assumed that censoring is non-informative, meaning the observed lifetimes and censoring times are independent of the explanatory variables. Let F and C represent the sets of individuals for whom yi corresponds to log-lifetime and log-censoring, respectively is expressed as follows:

$$l\left(\theta\right)=\sum_{i=f}^{n}\log(f(y_{i})+\sum_{i=f}^{n}\log(f(y_{i} \tag{41}$$

Eqs (31) and (32), respectively, $\log[f(y_i)]$ and $log[S(y_i)]$ is given, where is the unidentified parameter vector. We have the following log-likelihood function for the gamma Lindley regression model after inserting Eqs (31) and (32) into Eq (38). Eqs (31) and (32) into Eq (38).

$$\begin{aligned}l\left(\theta\right)=&\sum_{i=f}^{n}\log\frac{a^{2}\left[\left(\lambda+\lambda a-a\right)\exp\left(\frac{y-\beta_{0}-x\beta_{1}}{\sigma}\right)+1\right]}{\lambda\sigma\left(1+a\right)}\\&+\sum_{i=f}^{n}-a\exp\left(\frac{y-\beta_{0}-x\beta_{1}}{\sigma}\right)+\sum_{i=f}^{n}\left(\frac{y-\beta_{0}-x\beta_{1}}{\sigma}\right)\\&+\sum_{i=f}^{n}\left(\frac{y-\beta_{0}-x\beta_{1}}{\sigma}\right)+\sum_{i=f}^{n}-a\left(\exp\left(\frac{y-\beta_{0}-x\beta_{1}}{\sigma}\right)\right)\\&+\sum_{i=f}^{n}\log(1+a)\left(\frac{(a\lambda+\lambda-a)(a\;(\exp\frac{(y-\beta_{0}-x\beta_{1})+1+a}{\sigma}}{a\lambda(1+a)}\right)\end{aligned} \tag{42}$$

To estimate the parameters of the regression model, we derive the maximum likelihood function, which is represented by Equation (39) with respect to the parameters of the model, and set the differential result to zero.

$$l\left(\theta\right)=K_{1}+K_{2}+K_{3}+K_{4} \tag{43}$$

To estimate β_0 parameter of the regression model, we derive the maximum likelihood function, which is represented by Equation (42) respect to β_0 parameter, and we equate the differential result to zero

$$\frac{\partial l(\theta)}{\partial\beta_{0}}=\frac{\partial K_{1}}{\partial\beta_{0}}+\frac{\partial K_{2}}{\partial\beta_{0}}+\frac{\partial K_{3}}{\partial\beta_{0}}+\frac{\partial K_{4}}{\partial\beta_{0}}=0 \tag{44}$$

$$\begin{aligned}\frac{\partial K_{1}}{\partial\beta_{0}}&=\frac{\lambda\left(1+a\right)\sigma\left[\left(a\lambda+\lambda-a\right)\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}\right]}{\left(\lambda+\lambda a-a\right)\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}+1}\\\frac{\partial K_{2}}{\partial\beta_{0}}&=\sum_{i=1}^{n}\frac{a}{\sigma}\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}-\frac{n}{\sigma}\\\frac{\partial K_{3}}{\partial\beta_{0}}&=-\frac{n}{\sigma}\sum_{i=1}^{n}\frac{a}{\sigma}\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}\\\frac{\partial K_{4}}{\partial\beta_{0}}&=\sum_{i=1}^{n}\frac{\frac{-1}{\sigma}(a\lambda+\lambda-a)(a(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}}{(a\lambda+\lambda-a)(a(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}+1+a}\end{aligned}$$

By substituting in Equation (36) for the value of the previous differentials we get the following

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\frac{-1}{\sigma}(a\lambda+\lambda-a)(a(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}}{(a\lambda+\lambda-a)(a(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}+1+a}+2\sum_{i=1}^{n}\frac{a}{\sigma}(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}))-\frac{n}{\sigma}\\&+\sum_{i=1}^{n}\frac{\lambda(1+a)\sigma(\lambda+\lambda a-a)(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}}{(\lambda+\lambda a-a)(\exp\frac{(y-\beta_{0}-x\beta_{1})}{\sigma}+1}=0\end{aligned} \tag{45}$$

To estimate β_1 parameter of the regression model, we derive the maximum likelihood function, which is represented by Equation (42) respect to β_1 parameter, and we equate the differential result to zero

$$\frac{\partial l(\theta)}{\partial \beta_1} = \frac{\partial K_1}{\partial \beta_1} + \frac{\partial K_2}{\partial \beta_1} + \frac{\partial K_3}{\partial \beta_1} + \frac{\partial K_4}{\partial \beta_1} = 0 \tag{46}$$

$$\frac{\partial K_1}{\partial \beta_1} = \frac{x_i \left[(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}\right]}{\sigma\,(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + 1}$$

$$\frac{\partial K_2}{\partial \beta_1} = \sum_{i=1}^{n} x_i \;\; a \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma})) - \sum_{i=1}^{n} \frac{x_i}{\sigma}$$

$$\frac{\partial K_3}{\partial \beta_1} = \sum_{i=1}^{n} (\frac{x_i}{\sigma}) + \sum_{i=1}^{n} (\frac{x_i}{\sigma}) a \;\; \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}))$$

$$\frac{\partial K_4}{\partial \beta_1} = \sum_{i=1}^{n} \frac{x_i(a\lambda + \lambda - a)(a(\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}}{\sigma(a\lambda + \lambda - a)(a(\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + 1) + a}$$

By substituting in Equation (43) for the value of the previous differentials we get the following

$$-\sum_{i=1}^{n} \frac{x_i \left[(\lambda + \lambda a - a)(\exp \frac{(y - \beta_0 - x\beta_1)}{\sigma}\right]}{\sigma \left[(\lambda + \lambda a - a)(\exp \frac{(y - \beta_0 - x\beta_1)}{\sigma} + 1\right]} + \sum_{i=1}^{n} 2(x_i \; a \; (\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}))$$
$$-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right) - \sum_{i=1}^{n} \frac{(a\lambda + \lambda - a)(\; a(\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}}{\sigma \left[(a\lambda + \lambda - a) \left(a \left(\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}\right)\right) + 1) + a\right]} = 0$$

To estimate α parameter of the regression model, we derive the maximum likelihood function, which is represented by Equation (38) respect to α parameter, and we equate the differential result to zero

$$\frac{\partial l(\theta)}{\partial a} = \frac{\partial K_1}{\partial a} + \frac{\partial K_2}{\partial a} + \frac{\partial K_3}{\partial a} + \frac{\partial K_4}{\partial a} = 0 \tag{47}$$

$$\frac{\partial K_1}{\partial a} = \sum_{i=1}^{n} \frac{[\lambda\,(1 + a)\,\sigma]\,[2a] \left[(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + a^2\,(\lambda - 1) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} - a^2(\sigma \left[(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}\right] + 1\right]}{[\lambda\,(1 + a)\,\sigma]\,[a^2]\,(\exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + 1}$$

$$\frac{\partial K_2}{\partial a} = \sum_{i=1}^{n} - \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}$$

$$\frac{\partial K_3}{\partial a} = \sum_{i=1}^{n} - \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}$$

By substituting in Equation (47) for the value of the previous differentials we get the following

$$\frac{\partial K_4}{\partial a} = \sum_{i=1}^{n} \frac{[\lambda\,(1 + a)\,\sigma]\,[2a] \left[(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + a^2\,(\lambda - 1) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} - a^2\lambda\sigma \left[(\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma}\right] + 1\right]}{[\lambda\,(1 + a)\,\sigma]\,[a^2]\,((\lambda + \lambda a - a) \exp \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} + 1}$$

$$\sum_{i=1}^{n}\frac{[\lambda(1+a)\sigma][2a]\left[((\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1)+a^2(\lambda-1)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}-a^2(\sigma\left[(\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right]+1\right]}{[\lambda(1+a)\sigma]^2[a^2]((\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1}$$
$$-2\sum_{i=1}^{n}\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}$$
$$+\sum_{i=1}^{n}\frac{[\lambda(1+a)\sigma][2a]\left[((\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+a)+a^2(\lambda-1)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}-a^2(\sigma\left[(\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right]+1\right]}{[\lambda(1+a)\sigma]^2[a^2]((\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+a}$$

To estimate λ parameter of the regression model, we derive the maximum likelihood function, which is represented by Equation No(40) respect to λ parameter, and we equate the differential result to zero

$$\frac{\partial l(\theta)}{\partial\lambda}=\frac{\partial K_1}{\partial\lambda}+\frac{\partial K_2}{\partial\lambda}+\frac{\partial K_3}{\partial\lambda}+\frac{\partial K_4}{\partial\lambda}=0 \tag{48}$$

$$\frac{\partial K_1}{\partial a}=\sum_{i=1}^{n}\frac{\left[\lambda(1+a)a^2\right]\left[(1+a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}-a^2\left[(\lambda+\lambda a-a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right]+1](1+a)\sigma\right]}{a^2(\lambda+\lambda a-a)(\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1(\lambda[1+a]\sigma)^2}$$

$$\frac{\partial K_4}{\partial a}=\sum_{i=1}^{n}\frac{\left[\left[(a+1)(a)\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1\right)[(a\lambda+\lambda-a)\ a\ \ \exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1+a\right]}{(a\lambda+\lambda-a)\ a\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}+1}$$

$$\frac{\partial K_4}{\partial a}=\sum_{i=1}^{n}\frac{\left[\left[(a+1)\left(a\left(\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right)+1\right)\right]-[(a\lambda+\lambda-a)\left(a\ \left(\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right)+1\right)+a\right]\sigma(1+\alpha)}{\left[(a\lambda+\lambda-a)\ \left(a\ \left(\exp\frac{(y_i-\beta_0-x_i\beta_1)}{\sigma}\right)+1\right)+\alpha\right][\sigma\lambda(1+\alpha)]}$$

3. Residuals analysis

Residual analysis plays a crucial role in regression analysis to identify deviations from model assumptions and detect outliers. It is important to test whether the residuals are adequate to satisfy the five different types of residuals. Commonly used residuals include those introduced by [8], random quantile residuals defined by [9], and Pearson residuals, along with randomized quantile residuals and martingale residuals, and deviance residuals.

3.1. Cox and Snell residuals (1968)

After fitting the regression model, it is necessary to assess whether the model appropriately fits the data. Residual analysis is key in this validation process. Cox and Snell in 1968 [8], proposed residuals that are approximately distributed as a standard exponential distribution. Residuals are defined by:

$$e_i=-lin\left[1-F\left(Y_i,\widehat{B}\right)\right] \tag{49}$$

Where $F\left(y_i,\widehat{B}\right)$ is the gamma - Lindley Cdf. If the fitted model is correct

By

$$e_i = -lin\left[1 - [1 - \frac{[(a\lambda + \lambda - a)\,(a\,(expu) + 1) + a}{\lambda\,(1+a)} - a(\exp(-a(\exp{(u)})\right] \tag{50}$$

$$u = \frac{(y_i - \beta_0 - x_i\beta_1)}{\sigma} \tag{51}$$

Where

3.2. Pearson Residuals

The Pearson residual is generally used to detect possible outliers in data. The Pearson residual is based on the idea of subtracting the mean and dividing by the standard deviation. For the gamma Lindley regression model, the Pearson residuals can be expressed as.

$$V = \left[\frac{Y_i - \hat{\mu}_i}{\sqrt{v(y_i)}}\right] \tag{52}$$

$$\sigma^2 = \frac{6\,(\lambda + \lambda a) - 4a}{\lambda(1+a)a^2}[\frac{2\,(\lambda + \lambda a) - a}{\lambda\,(1+a)\,a}]2 \tag{53}$$

The plot of these residuals against the index of the observations should reveal no detectable pattern. If the fitted model is correct, the Pearson

4. Results and discussion

4.1. Simulation Study

In this simulation, the objective is to evaluate how the sample size affects the estimation of four model parameters (alpha = 1.5 , lambda = 0.5 , mu = 0, and sigma = 1) using synthetic data. Initially, true values for these parameters are set. The function f(y, alpha, lambda, mu, sigma) is then defined to generate synthetic observed data based on the true values of the parameters. The simulation tests several sample sizes (20, 30, 50, 100, and 500) to assess how the estimates of the parameters change as the sample size increases. adding noise to create observed data (y_{obs}). Linear regression is used to estimate the parameters, and the bias and Mean Squared Error (MSE) for each parameter are computed by comparing the estimated values to the true values. Bias reflects how far the estimates are from the true values, while MSE measures the average squared difference between the estimated and true values For each sample size, synthetic data is generated by sampling from a gamma lindley distribution for y and The results, including the estimated parameters, bias, and MSE for each sample size, are stored and displayed in a table. This process allows for an examination of how larger sample sizes lead to more accurate parameter estimates, with smaller biases and MSE values. The simulation highlights the importance of sample size in obtaining more reliable and precise estimates of model parameters

Table 1. Bias of parameter at different Sample_Size

Sample_Size	Bias_Alpha	Bias_Lambd	Bias_Mu	Bias_Sigma
20	-2.67754	0.1103698	0.0283248	-0.0054669
30	-2.90415	0.2796253	0.0224036	-0.0131686
50	-1.01599	-0.7938920	-0.0507801	-0.0021332
100	-1.98359	0.2263158	0.0240930	-0.0100242
500	-1.499211	-0.5280586	0.0045348	0.0011201

The Table 1 presents the biases observed in the estimation of four statistical parameters-Alpha, Lambda, Mu, and Sigma-across different sample sizes. As the sample size increases from 20 to 500, the biases generally decrease, reflecting the typical statistical behavior where larger samples provide more accurate parameter estimates.. Conversely, larger samples, such as 100 and 500, show reduced biases, with values closer to zero.

Table 2. MSE for parameter at different Sample_Size

Sample_Size	MSE_Alpha	MSE_Lambda	MSE_Mu	MSE_Sigma
20	9.169074	0.0121815	0.0008023	0.0000299
30	8.934116	0.0781903	0.0005019	0.0001734
50	8.432239	0.6302646	0.0025786	0.0000046
100	8.001808	0.0512188	0.0005805	0.0001005
500	2.247633	0.2788459	0.0000206	0.0000013

The table suggests that as the sample size increases, the MSE values decrease, leading to more reliable and precise estimates of model parameters. This highlights the importance of using larger sample sizes for better model performance and accuracy.

4.2. Real Data

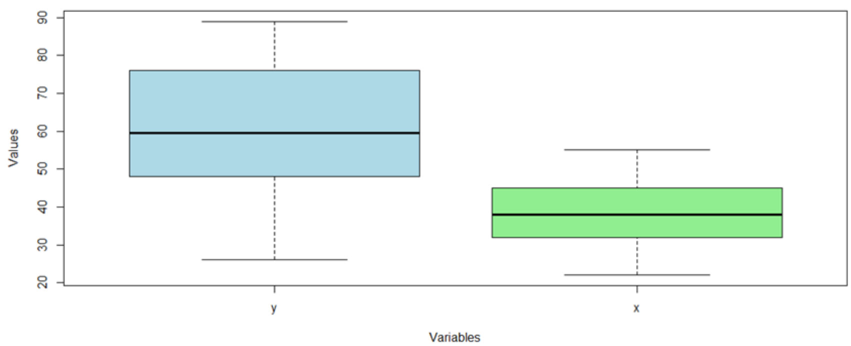

Figure 3. Boxplot for real data

The boxplot compares two variables, y and x, showing their distribution, spread, and central tendency. The y variable (blue) has a higher median (around 60) and a wider interquartile range (IQR) from approximately 45 to 75, indicating greater variability, with whiskers extending from about 25 to 90. In contrast, the x variable (green) has a lower median (around 35) and a more compact IQR (30 to 45), showing less variability, with whiskers extending from about 25 to 50. Overall, y has a broader spread and larger values, while x is more concentrated within a smaller range. The histogram represents the distribution of the dataset, with three probability density functions (PDFs) overlaid for comparison. The Gamma-Lindley distribution (red, solid line) has a sharp peak near zero and a moderately long tail, making it suitable for modeling skewed data. The Lindley distribution (blue, dashed line) also peaks near zero but declines more sharply, showing a faster decay than Gamma-Lindley. The Weibull distribution (green, dot-dash line) behaves differently, with a lower peak and a longer tail, making it more flexible in capturing various data patterns. Overall, all three distributions effectively model right-skewed data, with Gamma-Lindley providing additional flexibility due to its extra parameters.

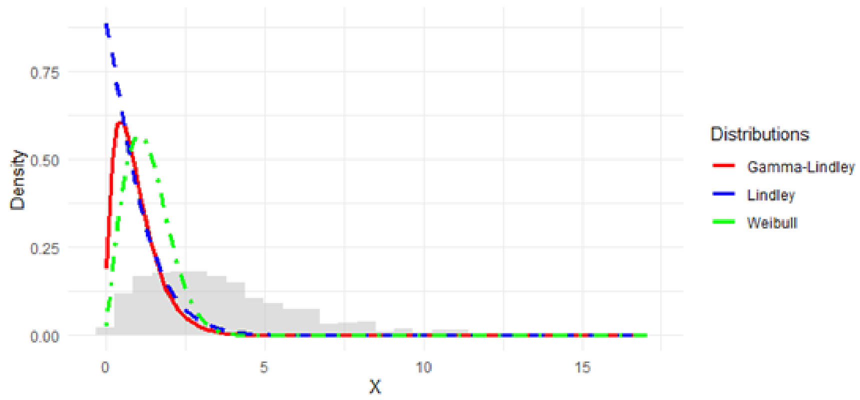

Figure 4. Histogram of Gamma-Lindley, Lindley and Weibull distributions

Table 3. Compare between gamma Lindley regression ,weibull regression ,Lindley regression model

Type	AIC	BIC	HQIC
Gamma Lindley regression	20.2238712	13.211068	13.211068
Weibull regression	40.3879588	43.375156	40.387958
Lindley regression	61.6921177	64.679315	64.679315

The Gamma Lindley regression emerges as the best model, outperforming the Weibull and Lindley regressions in terms of all three criteria-AIC, BIC, and HQIC-due to its significantly lower values across the board. This indicates that it fits the data most efficiently, achieving an optimal balance between model complexity and goodness of fit. In contrast, the Weibull and Lindley regressions exhibit much higher AIC, BIC, and HQIC values, suggesting that they are less effective for this data set. Among the two, the Weibull regression performs better than the Lindley regression, as reflected in its lower AIC and BIC scores. Overall, the Gamma Lindley regression is clearly the most suitable model for this analysis.

4.3. Residual analysis for gamma Lindley Regression Model

Residual analysis is a fundamental diagnostic tool in statistical modeling and regression analysis, used to assess the adequacy of a fitted model. Residuals, which represent the differences between observed and predicted values, provide critical insights into the model's performance and potential shortcomings. By examining residual patterns, analysts can detect issues such as non-linearity, heteroscedasticity, outliers, or violations of model assumptions. This analysis often involves visual techniques, such as residual plots, as well as statistical tests to ensure the model's validity and reliability. Proper residual analysis is essential for refining models, improving predictions, and ensuring robust conclusions in data-driven decision-making. The plot shows residuals versus fitted values from a regression model. Residuals, which are the differences between observed and predicted values, should ideally be randomly scattered around the red dashed zero line with no discernible pattern. However, in this plot, there appears to be a trend or structure in the residuals, suggesting potential issues like non-linearity or heteroscedasticity (changing variance). This indicates that the model may not fully capture the relationship between the predictors and the response variable, warranting further investigation or model adjustments. The QQ plot (quantile-quantile plot) of residuals compares the distribution of residuals to a theoretical normal distribution. If the residuals follow a normal distribution, the points should align closely with the red diagonal line. In this plot, most points lie along the line, but there are deviations at the tails, particularly at the upper end. This suggests potential departures from normality, such as the presence of outliers or skewness. If normality of residuals is a key assumption for the model, further investigation or transformation of the data may be necessary.

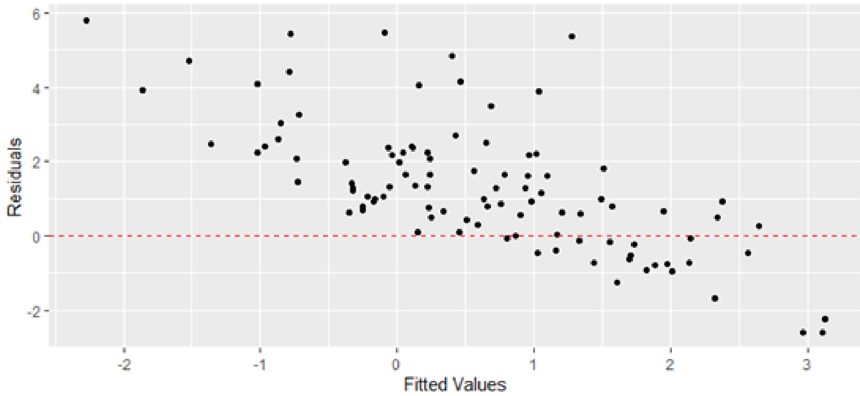

Figure 5. Residuals vs Fitted: Checks for non-linearity or heteroscedasticity

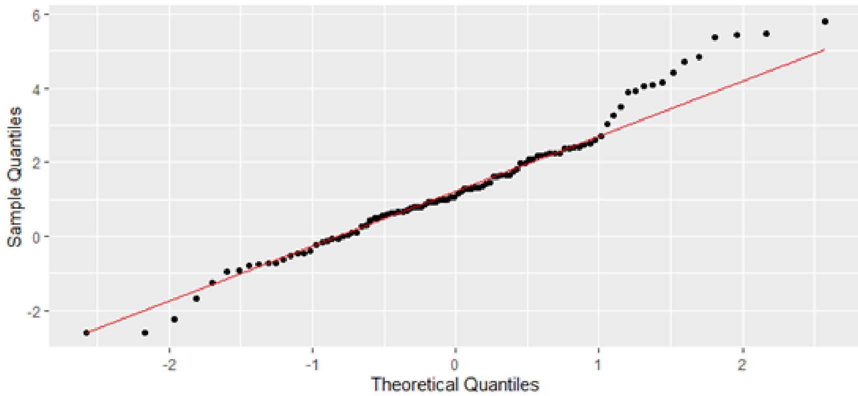

Figure 6. QQ-Plot: Assesses the normality of residuals

5. Conclusions

The maximum likelihood estimation method was utilized in order to carry out the estimation of the model parameters. A comparison was made between the suggested model and its submodels, which included Weibull regression, LGW regression, and Lindley regression, using the AIC and BIC criteria, as well as the likelihood ratio test, in order to evaluate the performance of the proposed model. In addition, the findings demonstrated that the suggested regression model performed better than the submodels in terms of the accuracy of its predictions.

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