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On F-implicit minimal vector variational inequalities

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Abstract In this paper, by introducing some new concepts in minimal spaces, we prove a generalized form of the Fan-KKM theorem in minimal vector spaces. A new class of minimal generalized vector F-implicit variational inequality problems and, as an application of Fan-KKM theorem is investigated. Moreover, an existence theorem for this kind of problems under some suitable assumptions in minimal vector spaces is given.

Keywords Minimal vector space, Fan-KKM theorem, *co*-compact property, Vector F-implicit variational inequality problem.

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1. Introduction

Nonlinear analysis presents many problems that can be resolved by the nonemptyness of the intersection of a certain family of subsets of a underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, a solution point for complementarity problem, a solution point for variational problem, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in [10], which is concerned with certain types of multimaps called the KKM maps. The KKM theory is the study of KKM maps and their applications. Generalized form of the KKM theorem namely Fan-KKM principle provides a foundation for many of the modern essential results in diverse areas of mathematical sciences (for more details see [15]). However the Fan-KKM theorem has essential role in solving all kinds of implicit complementarity problems or variational inequality problems, particularly in generalized vector F-implicit variational inequality problems [17]. The vector variational inequalities and vector complementarity problems have found many of their applications in vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problems.

Huang and Li [7] introduced and studied a class of scalar F-implicit variational inequality problems in Banach spaces. In 2006, Li and Huang [12] generalized the results from the scalar case in [7] to the vector case. The vector variational inequalities and vector complementarity problems have found many of its applications in vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problems. Recently, Lee, Khan, and Salahuddin [11] introduced the class of generalized vector F-implicit variational inequality problems in Banach spaces, which generalized some results of [7] and [12] to a

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more generalized vector case. Some new existence theorems of solutions for generalized F-implicit variational inequality problems were also proved in [11].

In this paper, motivated by the above mentioned works, we obtain some new results to prove a generalized form of the Fan-KKM theorem in minimal vector spaces. Also we introduce a new class of vector F-implicit variational inequality problems and as an application of our obtained Fan-KKM theorem we derive an existence theorem for this kind of problems.

2. Preliminaries

The concepts of minimal structures and minimal spaces, as generalization of topology and topological spaces were introduced in [14]. For easy understanding of the material incorporated in this paper, we recall some basic definitions and results. Also some new concepts are introduced in minimal spaces. Further results about minimal spaces can be found in [1, 2, 3, 4, 13], [16] and some references cited therein.

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a *minimal space*. For example, let (X, τ) be a topological space, then τ, $SO(X)$, $PO(X)$, $\alpha O(X)$ and $\beta O(X)$ are minimal structures on X [13]. In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. For any set $A \subseteq X$, define $m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\text{-Cl}(A) = \bigcap\{F : A \subseteq F,\ F^c \in \mathcal{M}\}$. Note that for $A \subseteq X$, m-Cl(A) (resp. m-Int(A)) is not necessarily m-closed (resp. m-open). The following lemma may be useful to apply in a minimal space.

Lemma 1
[13] For any two sets A and B in a minimal space X,
(a) $A \subseteq m\text{-Cl}(A)$ and $A = m\text{-Cl}(A)$ if A is an m-closed set.
(b) $m\text{-Cl}(A \cap B) \subseteq (m\text{-Cl}(A)) \cap (m\text{-Cl}(B))$.
(c) $m\text{-Cl}(m\text{-Cl}(B)) = m\text{-Cl}(B)$.
(d) $(m\text{-Cl}(A))^c = m\text{-Int}(A^c)$.

Definition 1
[16] Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. A function $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is called *minimal continuous* (briefly, m-continuous) if $f^{-1}(U) \in \mathcal{M}$ for any $U \in \mathcal{N}$.

Definition 2
[1] For two minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, we define *minimal product structure* for $X \times Y$ as follows:

$$\mathcal{M} \times \mathcal{N} = \{A \subseteq X \times Y : \forall\, (x, y) \in A,\ \exists\, U \in \mathcal{M}, \exists\, V \in \mathcal{N}; (x, y) \in U \times V \subseteq A\}.$$

Definition 3
[1] A *linear minimal structure* on a vector space X over the complex field $\mathbb{F}$ is a minimal structure $\mathcal{M}$ on X such that the two mappings

$$+ : X \times X \to X,\ (x, y) \mapsto x + y,$$
$$. : \mathbb{F} \times X \to X,\ (t, x) \mapsto tx,$$

are m-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F} \times X$ and $X \times X$ have the corresponding product minimal structures. A *linear minimal space* (or *minimal vector space*) is a vector space together with a linear minimal structure.

Obviously, any topological vector space is a minimal vector space but the converse is not true generally. In the following, it is shown that there is a linear minimal space which is not a topological vector space.

Example 1
Consider the real field $\mathbb{R}$ and let $\mathcal{M} = \{(a, b) : a, b \in \mathbb{R} \cup \{\pm\infty\}\}$. Clearly, $\mathcal{M}$ is a minimal structure on $\mathbb{R}$. We

claim that $\mathcal{M}$ is a linear minimal structure on $\mathbb{R}$. For this, we must prove that, two operations $+$ and $\cdot$ are m-continuous. Suppose $(x_0, y_0) \in +^{-1}(a,b)$ and so $x_0 + y_0 \in (a,b)$. Put $\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}$ and so $x_0 \in (x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2})$ and $y_0 \in (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})$. Hence

$$x_0 + y_0 \in ((x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}) + (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})) \subseteq (a,b);$$

which implies that $+^{-1}(a,b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$; that is $+$ is m-continuous. Also, suppose $(\alpha_0, x_0) \in \cdot^{-1}(a,b)$. Since $\alpha_0 x_0 \in (a,b)$ and $\lim_{s,t \to 0}(\alpha_0 - s)(x_0 - t) = \alpha_0 x_0$, one can find some $0 < \delta$ for which $\mid \alpha_0 - s \mid < \delta$ and $\mid x_0 - t \mid < \delta$ imply that $a < (\alpha_0 - s)(x_0 - t) < b$. Therefore,

$$(\alpha_0, x_0) \in (\alpha_0 - \delta, \alpha_0 + \delta) \cdot (x_0 - \delta, x_0 + \delta) \subseteq (a,b);$$

i.e., $\cdot^{-1}(a,b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$, which implies that the operation $\cdot$ is m-continuous.

Definition 4
[1] Consider a minimal space $(X, \mathcal{M})$ and a nonempty subset Y of X. The family $\mathcal{M}|_Y = \{U \cap Y : U \in \mathcal{M}\}$ is called *induced minimal structure* by $\mathcal{M}$ on Y. $(Y, \mathcal{M}|_Y)$ is called a *minimal subspace* of $(X, \mathcal{M})$. For any subset A of X,

$$m\text{-Int}_Y(A) = \bigcup\{V : V \in \mathcal{M}|_Y \text{ and } V \subseteq A\}$$

and

$$m\text{-Cl}_Y(A) = \bigcap\{F : F^c \in \mathcal{M}|_Y \text{ and } A \subseteq F\}.$$

Definition 5
[16] For a minimal space $(X, \mathcal{M})$,

(a) a family of m-open sets $\mathcal{A} = \{A_j : j \in J\}$ in X is called an *m-open cover* of K if $K \subseteq \bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also an m-open cover of K is called a *subcover* of $\mathcal{A}$ for K;
(b) a subset K of X is *m-compact* whenever given any m-open cover of K has a finite subcover.

Theorem 1
[16] Suppose that X and Y are two minimal spaces and $f : X \to Y$ is an m-continuous function. For any m-compact subset $K \subseteq X$, $f(K)$ is m-compact in Y.

Lemma 2
Suppose that X is an m-compact minimal space and $Y \subseteq X$. Then m-Cl(Y) is m-compact.

Proof
Suppose that $\{U_\alpha\}_{\alpha \in I}$ is an m-open cover of m-Cl(Y). So

$$X = m\text{-}\,\mathrm{Cl}(Y) \cup m\text{-}\,\mathrm{Int}(Y^c) \subseteq \bigcup_{\alpha \in I} U_\alpha \cup \bigcup_{\beta \in J} G_\beta,$$

where G_β's are m-open subsets of Y^c. Since X is an m-compact minimal space we have:

$$X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \cup \bigcup_{j=1}^{m} G_{\beta_j} \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \cup m\text{-Int}(Y^c).$$

Then m-Cl$(Y) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. □

Lemma 3
[3] Suppose $(X, \mathcal{M})$ is an m-compact minimal space, $\{A_i : i \in I\}$ is a family of subsets of X. If $\{m\text{-Cl}(A_i) : i \in I\}$ has the finite intersection property, then

$$\bigcap_{i \in I} m\text{-}\,\mathrm{C}\,l(A_i) \neq \emptyset.$$

In the following result, we prove the minimal version of Tychonoff theorem

Theorem 2
The minimal product space $\big(\prod_{\alpha\in I} X_\alpha, \prod_{\alpha\in I} \mathcal{M}_\alpha\big)$ is m-compact if and only if $(X_\alpha, \mathcal{M}_\alpha)$ is an m-compact minimal space, for any $\alpha \in I$.

Proof
One direction is an immediate consequence of Theorem 1. For the converse, on the contrary, assume that $\mathcal{A} \subseteq \prod_{\alpha\in I} \mathcal{M}_\alpha$ is an m-open cover of $\prod_{\alpha\in I} X_\alpha$ without any finite subcover for $\prod_{\alpha\in I} X_\alpha$. For any $\alpha \in I$, set $\mathcal{U}_\alpha = \{V \in \mathcal{M}_\alpha : \pi_\alpha^{-1}(V) \in \mathcal{A}\}$. Since $\mathcal{A}$ has no finite subcover for $\prod_{\alpha\in I} X_\alpha$, no finite subcover of $\mathcal{U}_\alpha$ can cover X_α, for any $\alpha \in I$. Now, m-compactness of X_α implies that $\mathcal{U}_\alpha$ can not cover X_α. Therefore there exists $x_\alpha \in X_\alpha \setminus \bigcup\{V : V \in \mathcal{U}_\alpha\}$, for any $\alpha \in I$. Set $x = (x_\alpha)_{\alpha\in I}$. Then $x \in \prod_{\alpha\in I} X_\alpha \setminus \bigcup\{A : A \in \mathcal{A}\}$, which implies that $\mathcal{A}$ is not an m-open cover for $\prod_{\alpha\in I} X_\alpha$, a contradiction. $\square$

3. Generalized Fan-KKM Theorem

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y. Given $A \subseteq X$, set $F(A) = \bigcup_{x\in A} F(x)$. A multimap $F : X \multimap Y$ is said to be *minimal transfer closed* if for any $x \in X$ and $y \notin F(x)$, there exists $x_0 \in X$ for which $y \notin m\text{-Cl}(F(x_0))$. Clearly, $F : X \multimap Y$ is minimal transfer closed if and only if $\bigcap_{x\in X} F(x) = \bigcap_{x\in X} m\text{-Cl}(F(x))$. It is obvious that an m-closed valued multimap is a minimal transfer closed multimap.

A subset A of a vector space X is *convex* if we have $ty + (1-t)z \in A$, whenever $y, z \in A$ and $t \in [0, 1]$. Also the *convex hull* of A, denoted by $co(A)$, is the smallest convex set that contains A, that is, the intersection of all convex sets containing A.

Definition 6
Suppose that D is a convex subset of a minimal vector space X. A multimap $F : D \multimap X$ is called a KKM *map* if $co(A) \subseteq F(A)$ for any $A \in \langle D\rangle$, where the notation $\langle D\rangle$ means the set of all finite subsets of D.

The following theorem is a generalized form of the Fan-KKM theorem, as a special case of Theorem 4.7 in [3], related to the minimal vector spaces.

Theorem 3
Suppose that X is a minimal vector space. Consider two nonempty valued multimaps $F, G : X \multimap X$ satisfying

(a) $F(x) \subseteq G(x)$ for all $x \in X$,
(b) F is a KKM map,
(c) $\bigcap_{x\in M} m\text{-Cl}(G(x))$ is m-compact for some $M \in \langle D\rangle$,
(d) for all $A \in \langle X\rangle$, G is minimal transfer closed on $co(A)$,
(e) for all $A \in \langle X\rangle$, $m\text{-Cl}\Big(\bigcap_{x\in co(A)} G(x)\Big) \cap co(A) \subseteq \bigcap_{x\in co(A)} G(x) \cap co(A)$.

Then $\bigcap_{x\in X} G(x) \neq \emptyset$.

Now, we introduce a new concept in minimal vector spaces with some examples which is useful for the proof of our results.

Definition 7
Suppose that X is a minimal (topological) vector space. A nonempty set $A \subseteq X$ has the *minimal* (*topological*) *finitely adjoint co-compact property* if $co(A \cup B)$ is an m-compact (compact) subset of X, for any $B \in \langle X\rangle$.

Example 2 (a) Consider $\mathbb{R}$ with it's usual topology. Then every finite subset of $\mathbb{R}$ has the topological finitely adjoint co-compact property.

(b) Since every topological space is a minimal space, any set with the topological finitely adjoint co-compact property in a topological vector space has the minimal finitely adjoint co-compact property if we consider the topological structure as minimal structure, but the converse is not true. Consider the minimal structure $\mathcal{M} = \{\mathbb{R}, \emptyset, (a, b)\}$ on $\mathbb{R}$ where $a, b \in \mathbb{R}^-$ and let $c \in \mathbb{R}^+$. For any $A \in \langle\mathbb{R}\rangle$,

$$co((c, +\infty) \cup A) = \begin{cases} (c, +\infty), & \text{c} < \min A, \\ [\min A, +\infty), & \text{c} \geq \min A, \end{cases}$$

which is not compact in $\mathbb{R}$ when we consider the usual topology. But it is an m-compact subset of $\mathbb{R}$ via minimal structure $\mathcal{M}$ (just $\mathbb{R}$ is it's cover). So there exists a set with the minimal finitely adjoint co-compact property which does not have the topological finitely adjoint co-compact property.

(c) Suppose that X is a minimal vector space. Any m-compact convex subset of X has the minimal finitely adjoint co-compact property. To see this, suppose that A is an m-compact convex subset of X and $B = \{b_1, b_2, \ldots, b_n\} \in \langle X\rangle$. Define the function $\varphi : A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1} \to co(A \cup B)$ by

$$\varphi(a, b_1, b_2, \ldots, b_n, t_1, t_2, \ldots, t_{n+1}) = \sum_{i=1}^{n} t_i b_i + t_{n+1} a,$$

with $\Delta_n = \left\{ \sum\limits_{i=1}^{n} t_i e_i; \sum\limits_{i=1}^{n} t_i = 1, t_i \geq 0 \right\}$, where (e_i) is the standard base of $\mathbb{R}^n$. It is not hard to see that φ is an onto m-continuous function by Definition 3. Since we can consider Δ_{n+1} as an m-compact subset of $\mathbb{R}^n$, according to the Theorem 2, $A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1}$ is m-compact. Now, from Theorem 1, $\varphi(A \times \{b_1\} \times \cdots \times \{b_n\} \times \Delta_{n+1}) = co(A \cup B)$ is m-compact.

From Theorem 3 along with Definition 7 we can achieve another version of the Fan-KKM theorem which is our main task in this section.

Theorem 4
Suppose that in Theorem 3 condition (c) is replaced with the following condition:

(c') there exists a nonempty subset $B \subseteq X$ with the minimal finitely adjoint co-compact property such that m-Cl$\big(\bigcap\limits_{x \in B} G(x)\big)$ is m-compact.

Then $\bigcap\limits_{x \in X} G(x) \neq \emptyset$.

Proof
For $A \in \langle X\rangle$, set $L_A := co(A \cup B)$. Define multimaps $F_A, G_A : L_A \multimap L_A$ by

$$G_A(x) = G(x) \cap L_A \;\; and \;\; F_A(x) = F(x) \cap L_A.$$

For any $x \in L_A$, $F_A(x) \subseteq G_A(x)$. It is not hard to see that F_A is a KKM multimap because the multimap F is KKM. According to the definition of the multimap G_A, we have $m\text{-Cl}_{L_A}(G_A(x)) \subseteq m\text{-Cl}_{L_A}(L_A) = L_A$ for any $x \in L_A$. Since B has the minimal finitely adjoint co-compact property then L_A is m-compact and Lemma 2 implies that $m\text{-Cl}_{L_A}(G_A(x))$ is m-compact for any $x \in L_A$. All conditions of Theorem 3 are satisfied for F_A, G_A and L_A. Hence $\bigcap\limits_{x \in L_A} G_A(x) \neq \emptyset$ for $A \in \langle X\rangle$. Now, suppose that $M_A = \bigcap\limits_{x \in L_A} G(x)$, for each $A \in \langle X\rangle$. The family $\mathcal{R} = \{M_A : A \in \langle X\rangle\}$ has the finite intersection property. To see this, assume that $A_1, A_2, \ldots, A_n \in \langle X\rangle$. Since $\bigcup\limits_{i=1}^{n} A_i \in \langle X\rangle$, we get $M_{\cup_{i=1}^n A_i} = \bigcap\limits_{x \in L_{\cup_{i=1}^n A_i}} G(x) \neq \emptyset$. Then

$$\emptyset \neq M_{\cup_{i=1}^n A_i} \subseteq \bigcap_{i=1}^{n} M_{A_i},$$

which gives the result.

It follows that $M_A \subseteq \bigcap_{x\in B} G(x)$ and so $m\text{-Cl}(M_A) \subseteq m\text{-Cl}\big(\bigcap_{x\in B} G(x)\big)$. According to the assertion (c'), m-Cl$\big(\bigcap_{x\in B} G(x)\big)$ is m-compact and Lemma 3 implies that $\bigcap_{A\in\langle X\rangle} m\text{-Cl}(M_A) \neq \emptyset$. Now, choose $\bar{x} \in \bigcap_{A\in\langle X\rangle} m$-Cl$(M_A)$ and arbitrary $x \in X$. For $A_0 = \{\bar{x}, x\}$, we have:

$$\bar{x} \in m\text{- Cl}(M_{A_0}) = m\text{- Cl}\Big(\bigcap_{y\in L_{A_0}} G(y)\Big) \subseteq m\text{- Cl}\Big(\bigcap_{y\in co(A_0)} G(y)\Big).$$

By assertion (e),

$$\bar{x} \in m\text{- Cl}\Big(\bigcap_{y\in co(A_0)} G(y)\Big) \cap co(A_0) \subseteq \bigcap_{y\in co(A_0)} G(y) \cap co(A_0) \subseteq G(x).$$

Then $\bar{x} \in \bigcap_{x\in X} G(x)$. □

Remark 1
It should be noticed that

(a) according to Example 2(c), when B is an m-compact convex subset of X, the first assertion in condition (c') of Theorem 4 is fullfilled.
(b) Theorem 4 generally goes back to the Fan-KKM theorem discussed in [5].

4. Minimal Generalized Vector F-implicit Variational Inequality Problem

In this section, as an application of the generalized Fan-KKM theorem, we give sufficient conditions to solve a minimal generalized vector F-implicit variational inequality problem in the minimal vector spaces.

A nonempty subset P of a vector space X is called a *convex cone* if $P + P = P$ and $\alpha P \subseteq P$ for all $\alpha \geq 0$, where $P + P = \{x + y : x, y \in P\}$ and $\alpha P = \{\alpha x : x \in P\}$. A cone is said to be *pointed* if $P \cap -P = \{0\}$.

The following "*generalized vector F-implicit variational inequality problem (GVF-IVIP)*" problem has been considered and solved in [11]:

Let X be a real Banach space, $K \subseteq X$ be a nonempty closed convex cone and (Y, P) be an ordered Banach space induced by the pointed closed convex cone P. Denote the space of all continuous linear mappings from X into Y by $L(X, Y)$ and the value of a linear continuous mapping $t \in L(X, Y)$ at x by $\langle t, x\rangle$. Let $A, T : K \to L(X, Y)$, $g : K \to K$, $F : K \to Y$ and $N : L(X, Y) \times L(X, Y) \to L(X, Y)$ be mappings.
Find $x \in K$ such that

$$\langle N(Ax^*, Tx^*), g(y) - g(x^*)\rangle + F(g(y)) - F(g(x^*)) \geq 0,\ for\ all\ y \in K.$$

Theorem 5
Suppose that

(a) five mappings N, g, A, T and F are continuous,
(b) there exists a mapping $h : K \times K \to Y$ such that
  (i) $h(x, x) \geq 0$ for all $x \in K$;
  (ii) $\langle N(Ax, Tx), g(y) - g(x)\rangle + F(g(y)) - F(g(x)) - h(x, y) \geq 0$ for all $x, y \in K$;
  (iii) the set $\{y \in K : h(x, y) \ngeq 0\}$ is convex for all $x \in K$;
(c) there exists a nonempty compact convex subset C of K such that for all $x \in K \setminus C$ there exists $y \in C$ such that $\langle N(Ax, Tx), g(y) - g(x)\rangle + F(g(y)) - F(g(x)) \geq 0$.

Then GVF-IVIP has a solution. Furthermore, the solution set of (GVF-IVIP) is closed.

Now, suppose that X and Y are two minimal vector spaces and M is a nonempty convex subset of X. Set $L_m(X,Y)$ as the set of all m-continuous linear mappings from X into Y. Let $\langle T, x\rangle$ be the value of $T \in L_m(X,Y)$ at point x. In addition, let $A, B : M \multimap L_m(X,Y)$, $g : M \to M$, $F : M \times M \to Y$, $S : L_m(X,Y) \times L_m(X,Y) \to L_m(X,Y)$ and let $C : M \multimap Y$ has nonempty convex pointed cone values; that is for all $y \in M$, $C(y)$ is a nonempty and convex pointed cone in Y.

We consider the following "*minimal generalized vector F-implicit variational inequality problem (MGVF-IVIP)*" which is generalization of (GVF-IVIP):

Find $y \in M$ such that for any $x \in M$,

$$\langle S(Ay, By), g(x) - g(y)\rangle + F(g(x), g(y)) - F(g(y), g(x)) \in C(y).$$

The following theorem gives a solution for MGVF-IVIP, in the minimal vector spaces.

Theorem 6
With the above notations, suppose that

(a) for all $A \in \langle M\rangle$ the multimap $G_A : co(A) \multimap M$ defined by

$$G_A(x)=\{y \in M; \langle S(Ay, By), g(x)-g(y)\rangle+F(g(x), g(y))-F(g(y), g(x)) \in C(y)\},$$

is a minimal transfer closed multimap,

(b) there is a nonempty m-compact subset $B = m\text{-Cl}(B) \subseteq M$ and a nonempty set $D \subseteq M$ with the minimal finitely adjoint co-compact property such that for any $x \in M \setminus B$ there is $y \in D$ such that

$$\langle S(Ax, Bx), g(y) - g(x)\rangle + F(g(y), g(x)) - F(g(x), g(y)) \notin C(x),$$

(c) there is a mapping $f : M \times M \to Y$ such that
   (i) $f(g(x), g(x)) \in C(x)$ for all $x \in M$,
   (ii) $\langle S(Ay, By), g(x) - g(y)\rangle + F(g(x), g(y)) - F(g(y), g(x)) - f(g(y), g(x)) \in C(y)$ for all $x, y \in M$,
   (iii) the set $\{y \in M : f(g(x), g(y)) \notin C(x)\}$ is convex for all $x \in M$.

Then the MGVF-IVIP has a solution.

Proof
Define two multimaps $G, H : M \multimap M$ as follows:

$$G(x)=\{y \in M : f(g(y), g(x)) \in C(y)\},$$
$$H(x)=\{y \in M : \langle S(Ay, By), g(x)-g(y)\rangle+F(g(x), g(y))-F(g(y), g(x)) \in C(y)\}.$$

From $(c)(ii)$ and the fact that the multimap C is nonempty convex pointed cone valued, we have $G(x) \subseteq H(x)$ for all $x \in M$. Assume that $A = \{x_1, x_2, \ldots, x_n\} \subseteq M$ and $z \in co(A)$ be such that $z \notin \bigcup\limits_{i=1}^{n} G(x_i)$. So $f(g(z), g(x_i)) \notin C(z)$ for each $i = 1, 2, \ldots, n$. Now condition $(c)(iii)$ implies that $f(g(z), g(z)) \notin C(z)$ which contradicts $(c)(i)$. Then G is a KKM multimap.

From assertion (b), for each $x \in M \setminus B$, there exists $y \in D$ such that $x \notin H(y)$. Hence $x \in M \setminus H(y)$ which implies that $M \setminus B \subseteq \bigcup\limits_{y \in D} M \setminus H(y)$ or equivalently $\bigcap\limits_{y \in D} H(y) \subseteq B$. It follows that $m\text{-Cl}\left(\bigcap\limits_{y\in D} H(y)\right) \subseteq m\text{-Cl}(B) = B$. From Lemma 2, it is easy to see that $m\text{-Cl}\left(\bigcap\limits_{y\in D} H(y)\right)$ is m-compact. By assertion (a), we can derive conditions (d) and (e) of Theorem 4. Condition (d) is obvious since H, when is restricted to the $co(A)$, coincides with the multimap G_A for each $A \in \langle M\rangle$. For condition (e), since

$$\bigcap_{x \in co(A)} H(x) = \bigcap_{x \in co(A)} m\text{-}\mathrm{Cl}(H(x)),$$

then we get

$$\begin{aligned} m\text{-}\operatorname{Cl}\Big(\bigcap_{x\in co(A)} H(x)\Big)\cap co(A) &= m\text{-}\operatorname{Cl}\Big(\bigcap_{x\in co(A)} m\text{-}\operatorname{Cl}(H(x))\Big)\cap co(A)\\ &\subseteq \bigcap_{x\in co(A)} m\text{-}\operatorname{Cl}(m\text{-}\operatorname{Cl}(H(x)))\cap co(A)\\ &= \bigcap_{x\in co(A)} m\text{-}\operatorname{Cl}(H(x))\cap co(A)\\ &= \bigcap_{x\in co(A)} H(x)\cap co(A). \end{aligned}$$

Now all conditions of Theorem 4 are satisfied and so $\bigcap_{x\in M} H(x)\neq\emptyset$. Then the solution set of the MGVF-IVIP is nonempty. □

Example 3
Consider $\mathbb{R}$ with the minimal structure $\mathcal{M}=\{\mathbb{R},\emptyset\}\cup\{(a,b):a,b\in\mathbb{R}^-\}\cup\{\mathbb{R}\setminus[a,b]:a,b\in[0,\infty)\}$. In Theorem 6, set $X=Y=\mathbb{R}$, $M=[0,+\infty)$, $D=(1,+\infty)$ and $B=[0,1]$. It is clear that B is an m-closed m-compact set and D has the minimal finitely adjoint co-compact property. Also $[0,+\infty)$ is a convex pointed cone. For any $x,y\in M$ define $g(x)=x/2$, $F(x,y)=x$, $f(x,y)=2(y-x)$, $A(y)=B(y)=y$, $C(x)=[0,+\infty)$ and $S(Ay,By)=Ay=y$. So for each $A\in\langle M\rangle$, G_A is m-closed valued and hence it is a minimal transfer closed multimap. All conditions are satisfied and then $\bigcap_{x\in M} H(x)=\bigcap_{x\in[0,+\infty)}[0,x]=\{0\}$ is a solution of the MGVF-IVIP.

Remark 2 (a) Theorem 6 reduces to Theorem 5 if we consider all details in a topological vector space instead of minimal vector space, $F:M\to Y$ and $C\equiv[0,+\infty)$, a pointed convex cone.
(b) Theorem 6 is a generalization of Theorem 2.2 in [8] and Theorem 3.2 in [12, 18]. Also Theorem 6, extends and improves the corresponding results of [7].

5. Conclusion

Vector variational inequality (VVI) is a base for vector equilibrium problems, which can be applied to traffic networks and migration equilibrium problems (see [6]). Meanwhile, Fan-KKM principle can be utilized to solve some special problems related to vector variational inequalities. Additionally, many new problems in nonlinear analysis can be solved by introducing generalized concepts and extending spaces to provide a proper solution. The above description motivated us to introduce the concepts of generalized Fan-KKM multimaps and generalized vector F-implicit variational inequality problems (See, also [9]). In this work, we introduced a generalized version of the Fan-KKM theorem for vector F-implicit variational inequality problems in minimal vector spaces that covers many previous results.

REFERENCES

1. M. Alimohammady and M. Roohi, *Linear minimal space*, Chaos, Solitons & Fractals, vol. 33, no. 4, pp. 1348–1354, 2007.
2. M. Alimohammady, M. Roohi and M. Rostamian Delavar, *Transfer closed and transfer open multimaps in minimal spaces*, Chaos, Solitons & Fractals, vol. 40, no. 3, pp. 1162–1168, 2009.
3. M. Alimohammady, M. Roohi and M. Rostamian Delavar, *Transfer closed multimaps and Fan-*KKM *principle*, Nonlinear Funct. Anal. Appl. vol. 13, no. 4, pp. 597–611, 2008.
4. M. Eshaghi Gordji , M. Rostamian Delavar and M. Roohi, *Applications of Fan's matching theorem in MKKM spaces*, Math. Inequal. Appl. vol. 17, no. 1, pp. 233–240, 2014.
5. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. vol. 266, pp. 519–537, 1984.
6. F. Giannessi, Vector Variational Inequalities and Vector Equilibria, vol. 38, Kluwer Academic, Dordrecht, The Netherlands, 2000.
7. N. J. Huang and J. Li, *F-implicit complementarity problems in Banach spaces*, Z. Anal. Anwendungen. vol. 23, pp. 293–302, 2004.

8. M. K. Kang, *Some Vector Implicit Complementarity Problems with Corresponding Variatuonal Inequality Problems*, Commun. Korean Math. Soc. vol. 22, no. 3, pp. 401–410, 2007.
9. Z. Khan, S. S. Irfan, M. Firdosh Khan and P. Shukla, *An itertive algorithm with error terms for solving a system of implicit n-variational inclusions*, Statistics, Optimization & Information Computing, vol. 8, pp. 242-253, 2020.
10. B. Knaster, K. Kuratowski and S. Mazurkiewicz, *Ein Beweis des fixpunktsatzes fur n-dimensionale simplexe*, Fund. Math. vol. 14, pp. 132–137, 1929.
11. B. S. Lee, M. F. Khan and Salahuddin, *Vector F-implicit complementarity problems with corresponding variational inequality problems*, Appl. Math. Lett. vol. 20, pp. 433–438, 2007.
12. J. Li and N. J. Huang, *Vector F-implicit complementarity problems in Banach spaces*, Appl. Math. Lett. vol. 19, pp. 464–471, 2006.
13. H. Maki, *On generalizing semi-open sets and preopen sets*, Meeting on Topological Spaces Theory and its Application, pp. 13–18, Augoust 1996.
14. H. Maki, J. Umehara and T. Noiri, *Every topological space is pre $T_{\frac{1}{2}}$*, Mem. Fac. Sci. Kochi Univ. Ser A. Math. vol. 17, pp. 33–42, 1996.
15. S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. vol. 27, pp. 193–232, 1999.
16. V. Popa and T. Noiri, *On M-continuous functions*, Anal. Univ. Dunarea Jos-Galati, Ser. Mat. Fiz. Mec. Teor. Fasc. II vol. 18, no. 23, pp. 31–41, 2000.
17. M. M. Wong, Q. H. Ansari and J. C Yao, *Existence of solutions of generalized variational inequalities in reflexive Banach spaces*, Appl. Math. Lett. vol. 22, n. 2, pp. 197–201, 2009.
18. K. Q. Wu and N. J. Huang, *General mixed vector F-implicit complementarity problems in Banach spaces*, J. Appl. Anal. vol. 14, no. 1, pp. 73–88, 2008.