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Statistical Inference of Chen Distribution Based on Type I Progressive Hybrid Censored Samples

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Abstract In this paper we study the problem of estimating unknown parameters of a two-parameter distribution with bathtub shape under the assumption that data are type I progressive hybrid censored. We derive maximum likelihood estimators and then obtain the observed Fisher information matrix. Bayes estimators are also obtained under the squared error loss function and highest posterior density intervals are constructed as well. We perform a simulation study to compare proposed methods and analyzed a real data set for illustration purposes. Finally we establish optimal plans with respect to cost constraints and obtain comments based on a numerical study.

Keywords EM algorithm, Tierney and Kadane method, Metropolis-Hastings algorithm, Optimal plan

AMS 2010 subject classifications 62N01 , 62N02

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1. Introduction

The reliability and life testing studies over the years have traditionally focused on finding procedures and appropriate statistical methods required for the analysis of various lifetime data. The common framework under which such studies are modelled is closely related to a series of censoring methodologies in the context of which inferences for any reliability data are made for further interpretations. In literature, several censoring methodologies have been discussed to study different observable physical phenomena. In this view, we mention that type I and type II are two most common censoring schemes which are widely used in reliability and life testing experiments. In type I censoring a total of n test units are subjected to a life test and the experiment stops when a prescribed time point t reaches. The number of observed failures is random in this censoring. On the other hand in type II censoring a test stops when a prescribed r number of failure times is recorded. Here duration of the test is random in nature. In these two respective censoring it may happen that very few observations are recorded during a fixed time period or that the test has to run for very long duration to obtain a prefixed number of observations. To deal with such situations, Epstein [16] introduced the concept of hybrid censoring as a mixture of type I and type II censoring. In this censoring an experiment stops at random time point $\min\{X_{r:n}, t\}$. It should be further noticed that live units from such experiments can be withdrawn only at the end of a test. We now briefly describe the concept of type II progressive censoring where live units can be removed during the experimentation also. At the time of first failure, the s_1 number of units are randomly removed from the remaining $n-1$ test units. Similarly at the time of second failure, the s_2 number of live units are randomly removed from

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the remaining $n - s_1 - 2$ test units. Finally, when rth ($\leq n$, prefixed) failure occurs the test stops and all the remaining live units are removed from the test. Note that we have $n = r + \sum_{i=1}^{r} s_i$ and $(s_1, s_2, \ldots, s_r)$ is referred to as the progressive censoring scheme with r being a prescribed number. One may refer to Balakrishnan [8] and Balakrishnan and Aggarwala [7] for a comprehensive discussion on progressive censoring. We next describe the concept of type I progressive hybrid censoring. Let n test units be put on a life test and their lifetimes are recorded. Then at the time of first failure the s_1 number of units are randomly removed from the remaining $n - 1$ live units. Similarly when the second failure is observed then s_2 number of units are randomly removed from the remaining $n - s_2 - 2$ units and so on. The process stops at a random time $\min\{X_{r:n}, t\}$. In case $X_{r:n} > t$, let k denote the number of failures before time t then observe data turn out to be $X_{1:r:n}, X_{2:r:n}, \ldots, X_{k:r:n}$. Further the remaining live units $s_k^* = n - k - \sum_{i=1}^{k} s_i$ are withdrawn at time t and experiment stops. Tomer and Panwer [35] studied a one-parameter Maxwell distribution based on type I progressive hybrid censoring scheme. Authors derived point and interval estimation procedures for the unknown parameter using classical and Bayesian methods. They compared obtained procedures using simulations and presented analysis of a real data set in support of proposed methods. Kayal et al. [20] obtained classical and Bayes estimates of unknown parameters of a Burr XII distribution using this censoring scheme. The problem of one-sample Bayesian prediction is also discussed. Recently, Kundu and Joarder [22] studied another generalized form of progressive censoring, namely the type II progressive hybrid censoring and studied a one-parameter exponential distribution. Lin et al. [25] discussed an adaptive type I progressive hybrid censoring and analyzed a Weibull distribution based on this censoring. Koley and Kundu [5] studied generalized progressive hybrid censoring. Also, a few more articles can be be referred in the context of progressive hybrid censoring (see, Górny and Cramer [3], Almarashi et al. [1], Lodhi et al. [2], , Basu et al. [4]).

In many lifetime analysis reliability data are often studied based on their hazard rate characteristics and usually different censoring methodologies are then applied to obtain inference for unknown quantities of interest. The most common hazard rate functions that are used to analyze various physical phenomena for a wide variety of problems are constant, increasing or decreasing in nature. This is the case where generalized exponential, gamma, Weibull and lognormal distributions have found wide applications, among others. It is however quite difficult to develop adequate inference using these models if data sets exhibit bathtub-shaped hazard rate function because of their specified characteristics. Nevertheless these models have been applied extensively in such studies, see for instance Kundu [23], Pradhan and Kundu [29], Singh et al. [33] and references cited there-in. Davis [14] analyzed breakdown behavior for a number of motor buses and concluded that initial few failures of such buses cannot be model appropriately with exponential distribution. Later on Smith and Bain [32] studied this data using an exponential power distribution. Hjorth [19] showed that a bathtub model provides good fit to the failure data observed on hydromechanical devices that drive electric generators in aircraft with constant revolutions per unit time. In fact, bathtub shaped distributions are derived as an important class of models that can be used to study wide variety of problems in reliability and life testing experiments. Block and Savits [12] mentioned that failure process of various electronic components such as electric switches and lamps, circuits often lead to the study of bathtub shaped models. One may also refer to Bebbington, Lai and Zitikis [10, 11] for some further applications of such distributions in life testing experiments. In this paper, we study a two-parameter distribution as proposed by Chen [13] The probability density function (PDF) of this distribution is of the form

$$f(x;\eta,\lambda) = \eta\lambda x^{\lambda-1}\exp\left[\eta\left(1-e^{x^{\lambda}}\right)+x^{\lambda}\right],\ \ 0<x<\infty,\ 0<\eta<\infty,\ 0<\lambda<\infty, \tag{1}$$

and the corresponding cumulative distribution function (CDF) is given by

$$F(x;\eta,\lambda) = 1-\exp\left[\eta\left(1-e^{x^{\lambda}}\right)\right],\ \ x>0, \tag{2}$$

where η and λ denote unknown shape parameters. In sequel and for further use we denote this distribution as $B(\eta, \lambda)$. It is known that hazard rate function of this distribution is bathtub shaped for $0 < \lambda < 1$ and is increasing otherwise. In practical studies many empirical phenomena lead to bathtub shaped hazard rate functions, e.g. Rajarshi and Rajarshi [30]. At recent past, this model has gain some interest among researchers. Wu [36] derived maximum likelihood estimates of unknown parameters η and λ based on progressive censoring and also studied the problem of interval estimation. Author compared proposed methods using a simulation study and obtained comments based on this study. Rastogi et al. [31] considered Bayesian estimation against different symmetric and asymmetric loss functions. Ahmed [6] also studied this estimation problem and obtained Bayes estimates of unknown parameters under balanced squared error loss function. Recently Kayal et al. [21] obtained one- and two-sample Bayes predictive estimates and also constructed prediction intervals of censored observations under progressive censoring. They analyzed real data sets in support of proposed methods. In this paper, we consider estimation of unknown parameters under both classical and Bayesian framework assuming that the samples are type I progressive hybrid censored. The problem of optimal censoring is also discussed.

We have organized this paper as follows. In Section 2, we consider maximum likelihood estimation using an expectation maximization (EM) algorithm and based on it we further compute the observed Fisher information matrix. The asymptotic confidence interval (ACI) of unknown parameters η and λ are also constructed. Next in Section 3, we obtain Bayes estimators of unknown parameters under the square error loss function. Tierney and Kadane method and Metropolis-Hastings algorithm are discussed in this regard. Highest posterior density intervals of unknown parameters are also obtained. We numerically compare proposed methods using Monte Carlo simulations in Section 4 and a real data set is analyzed in Section 5 for illustration purposes. Finally in Section 6, we establish optimal plans under cost constraints and a conclusion is given in Section 7.

2. Maximum likelihood estimation

In this section, we obtain maximum likelihood estimators of unknown parameters based on type I progressive hybrid censored data. In this regard, let n identical units $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be put on a test whose lifetimes follow a $B(\eta,\lambda)$ distribution. Here one of the following cases may arise

$$\begin{cases} \text{Case I: } \{X_{1:r:n}, X_{2:r:n}, \ldots, X_{r:r:n}\}, & \text{if } X_{r:r:n} < t, \\ \text{Case II: } \{X_{1:r:n}, X_{2:r:n}, \ldots, X_{k:r:n}\}, & \text{if } X_{k:r:n} < t < X_{k+1:r:n}. \end{cases}$$

The corresponding likelihood function is then given by

$$L(\eta,\lambda \mid \mathbf{x}) \propto \eta^j \lambda^j \exp[G(\eta,\lambda \mid \mathbf{x})] \prod_{i=1}^{j} x_i^{\lambda-1} \tag{3}$$

where

$$j = \begin{cases} r, & \text{Case I}, \\ k, & \text{Case II}, \end{cases}$$

and $G(\eta,\lambda \mid \mathbf{x}) = \sum_{i=1}^{j}\left[x_i^{\lambda} + \eta(1+s_i)(1-e^{x_i^{\lambda}})\right] + s_k^{*}\eta(1-e^{t^{\lambda}})$. We note that for $j=r$ the equation (3) yields type II progressive censoring which is Case I. We denote $x_{i:r:n}$ by x_i for notational convenience. Note that likelihood equations can be obtained by partially differentiating the equation (3) with respect to unknown parameters η and λ. We observe that it is difficult to solve likelihood equations analytically due to their non-linear nature. However, one can employ some numerical methods to obtain the required estimates of unknown parameters. Here instead we propose to apply an EM algorithm for this purpose. One may refer to Dempster et al. [15] (also, Ng et al. [28]) for further details on this method. Now suppose that $X = \{X_1, X_2, \ldots, X_j\}$ denotes the observed samples and $Y = \{Y_1, Y_2, \ldots, Y_j\}$ denotes the censored samples with each of Y_d representing a $1 \times s_d$ vector such that $Y_d = (Y_{d1}, Y_{d2} \ldots, Y_{ds_d}),\ d = 1, 2, \ldots, j$ and that $Y^* = (Y_1^*, Y_2^*, \ldots, Y_{s_k^*}^*)$. Then the complete sample is given by $W = (X, Y, Y^*)$. We now write the associated log-likelihood function as

$$\begin{aligned} l(\eta,\lambda \mid \mathbf{x}) &= \ln L(\eta,\lambda \mid \mathbf{x}) \propto n' \ln\eta + n' \ln\lambda + (\lambda-1)\sum_{i=1}^{j}\ln x_i + \eta\sum_{i=1}^{j}(1-e^{x_i^{\lambda}}) + \sum_{i=1}^{j} x_i^{\lambda} \\ & (\lambda-1)\sum_{i=1}^{j}\sum_{v=1}^{s_i}\ln y_{iv} + \ln\sum_{i=1}^{j}\sum_{v=1}^{s_i}(1-e^{y_{iv}^{\lambda}}) + \sum_{i=1}^{j}\sum_{v=1}^{s_i} y_{iv}^{\lambda} + s_k^*(\ln\eta + \ln\lambda) \\ & +(\lambda-1)\sum_{v=1}^{s_k^*}\ln y_v^* + \eta\sum_{v=1}^{s_k^*}(1-e^{{y^*}^{\lambda}}) + \sum_{v=1}^{s_k^*} {y^*}^{\lambda}. \end{aligned} \tag{4}$$

In the E-step, we find the pseudo log-likelihood function as

$$\begin{aligned} l_c(\eta,\lambda \mid \mathbf{x}) \propto\ & n' \ln\eta + n' \ln\lambda + (\lambda-1)\sum_{i=1}^{j}\ln x_i + \eta\sum_{i=1}^{j}(1-e^{x_i^{\lambda}}) + \sum_{i=1}^{j} x_i^{\lambda} + (\lambda-1) \\ & \sum_{i=1}^{j} s_i A(x_i,\eta_{(m)},\lambda_{(m)}) + \eta\sum_{i=1}^{j} s_i B(x_i,\eta_{(m)},\lambda_{(m)}) + \sum_{i=1}^{j} s_i C(x_i,\eta_{(m)},\lambda_{(m)}) \\ & + s_k^*[\ln\eta + \ln\lambda + (\lambda-1)A(t,\eta_{(m)},\lambda_{(m)}) + \eta B(t,\eta_{(m)},\lambda_{(m)}) + C(t,\eta_{(m)},\lambda_{(m)})] \end{aligned} \tag{5}$$

where

$$n' = k + \sum_{i=1}^{k} s_i,\ A(c,\eta,\lambda) = E[\ln y_{iv} \mid y_{iv} > c] = \frac{1}{\lambda(1-F(c;\eta,\lambda))}\int_0^{c'} \ln(\ln(1-\frac{1}{\eta}u))\,du,\ \ c' = \exp[-\eta(e^{c^{\lambda}}-1)],$$

$$B(c,\eta,\lambda) = E[1-e^{y_{iv}^{\lambda}} \mid y_{iv} > c] = -\frac{1}{1-F(c;\eta,\lambda)}[c'' e^{-\eta c''} + \frac{e^{-\eta c''}}{\eta}],\ \ c'' = \frac{1}{\eta}\ln\frac{1}{c'}$$

and

$$C(c,\eta,\lambda) = E[y_{iv}^{\lambda} \mid y_{iv} > c] = \frac{1}{1-F(c;\eta,\lambda)}\int_0^{c'} \ln(1-\frac{1}{\eta}u)\,du.$$

Now in the M-step, we maximize expression (5) with respect to η and λ. If $(\eta_{(m)},\lambda_{(m)})$ denotes the mth stage estimate of unknown parameter (η,λ) then updated $(m+1)$th stage estimate of λ is obtained by solving the following non-linear equation

$$\frac{n'+s_k^*}{\lambda} + \sum_{i=1}^{j}\ln x_i - \hat{\eta}(\lambda)\sum_{i=1}^{j} e^{x_i^{\lambda}} x_i^{\lambda}\ln x_i + \sum_{i=1}^{j} x_i^{\lambda} + \sum_{i=1}^{j} s_i A(x_i,\eta_{(m)},\lambda_{(m)}) + s_k^* A(t,\eta_{(m)},\lambda_{(m)}) = 0. \tag{6}$$

The corresponding updated estimate of η is then be derived as

$$\hat{\eta}(\lambda) = -\frac{n'+s_k^*}{\sum_{i=1}^{j}[(1-e^{x_i^{\lambda_{(m+1)}}}) + s_i B(x_i,\eta_{(m)},\lambda_{(m+1)})] + s_k^* B(t,\eta_{(m)},\lambda_{(m+1)})}.$$

We continue this iteration process until the desired convergence is achieved. We next discuss the observed Fisher information matrix. The Fisher information matrix is useful in constructing the asymptotic confidence intervals of unknown parameter $\theta = (\eta,\lambda)$. We here use the method of Louis [26] for this purpose which suggest that

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta) \tag{7}$$

where $I_X(\theta), I_W(\theta)$ and $I_{W|X}(\theta)$ represent the observed, complete and missing information matrix respectively. We also have

$$I_X(\theta) = -E\left[\frac{\partial^2 L_c(W;\theta)}{\partial \theta^2}\right], \quad I_{W|X} = \sum_{i=1}^{j} I^{(i)}_{W|X}(\theta) + s_k^* I_{W|t}(\theta),$$

with

$$I^{(i)}_{W|X}(\theta) = -E_{Y_i|x_{(i)}}\left[\frac{\partial^2 \ln f_{Y_i}(y_i|x_{(i)},\theta)}{\partial \theta^2}\right]$$

and

$$I_{W|t}(\theta) = -E_{Y^*|t}\left[\frac{\partial^2 \ln f_{Y^*}(y^*|t,\theta)}{\partial \theta^2}\right].$$

We compute elements of respective matrices in Appendix A1. Accordingly the corresponding variance-covariance matrix of MLEs can be obtained as $I_X^{-1}(\theta)$. The corresponding $100(1-p)\%,\ 0<p<1$ asymptotic confidence intervals for η and λ can now be constructed using the variance-covariance matrix.

3. Bayesian estimation

In this section, we obtain Bayes estimates of unknown parameters η and λ based on type I progressive hybrid censored data. The squared error loss function is taken into consideration for this purpose. Following Rastogi et al. [31] and Ahmed [6] we assume that unknown parameters η and λ are a priori distributed as gamma $G(a', b')$ and $G(p', q')$ distributions respectively. We further note that when both the parameters of Chen distribution are unknown, the joint conjugate prior distributions do not exist. In such situations it is reasonable to consider independent gamma prior distributions for these unknown parameters. Such priors are quite flexible in nature and include noninformative priors as well. One may refer to Kundu and Pradhan [24] for further details on this topic. Thus the joint prior distribution of η and λ is given by

$$\pi(\eta,\lambda) \propto \eta^{a'-1}\lambda^{b'-1}\exp[-(b'\eta + q'\lambda)], \quad a', b', p', q', \eta, \lambda > 0, \tag{8}$$

where a', b', p', q' denote hyperparameters. After a simple calculation, the posterior distribution of η and λ given type I progressive hybrid censored data turns out to be

$$\pi(\eta,\lambda \mid \mathbf{x}) = D^{-1}\eta^{j+a'-1}\lambda^{j+p'-1}\exp[G(\eta,\lambda \mid \mathbf{x}) - (b'\eta + q'\lambda)]\prod_{i=1}^{j} x_i^{\lambda-1}, \tag{9}$$

where D denotes the normalizing constant. The Bayes estimator of a parametric function is its posterior mean if the loss is squared error. We observe that the corresponding posterior expectation exists in the form of ratio of two integrals which is quite difficult to evaluate analytically due to the reason that $\pi(\eta,\lambda \mid \mathbf{x})$ appears in an intractable form. So we need to employ some approximation method to compute the desired estimates. Here we propose to use the method of Tierney and Kadane [34] which is discussed next.

3.1. TK method

Let the posterior expectation of a parametric function $w(\eta,\lambda)$ be given by

$$Q(x) = \frac{\int_0^\infty \int_0^\infty w(\eta,\lambda) e^{l_p(\eta,\lambda|\mathbf{x}) + \tau(\eta,\lambda)}\, d\eta d\lambda}{\int_0^\infty \int_0^\infty e^{l_p(\eta,\lambda|\mathbf{x}) + \tau(\eta,\lambda)}\, d\eta d\lambda}, \tag{10}$$

where $l_p(\eta,\lambda \mid \mathbf{x})$ denotes the log-likelihood function and $\tau(\eta,\lambda) = \ln \pi(\eta,\lambda)$. We now consider functions $\psi(\eta,\lambda) = \frac{l_p(\eta,\lambda|\mathbf{x}) + \tau(\eta,\lambda)}{n}$ and $\phi(\eta,\lambda) = \psi(\eta,\lambda) + \frac{\ln w(\eta,\lambda)}{n}$ and assume that $\hat{\theta}_\psi$ and $\hat{\theta}_\phi$ respectively maximize $\psi(\theta)$

and $\phi(\theta)$, $\theta = (\eta, \lambda)$. Then using the TK method we can approximate $Q(x)$ into the form

$$Q(x) = \sqrt{\frac{|\Phi|}{|\Psi|}} \exp[n\{\phi(\hat{\eta}_\phi, \hat{\lambda}_\phi) - \psi(\hat{\eta}_\psi, \hat{\lambda}_\psi)\}] \tag{11}$$

where Ψ, Φ denote the negative inverse Hessian of $\psi(\eta, \lambda)$ and $\phi(\eta, \lambda)$ respectively. Now $(\hat{\eta}_\psi, \hat{\lambda}_\psi)$ can be obtained by solving the following non-linear equations:

$$\begin{aligned} \frac{1}{n}\left[\frac{j}{\eta} + G_\eta(\eta, \lambda \mid \mathbf{x}) + \frac{a'-1}{\eta} - b'\right] &= 0, \\ \frac{1}{n}\left[\frac{j}{\lambda} + G_\lambda(\eta, \lambda \mid \mathbf{x}) + \sum_{i=1}^{j} \ln x_i + \frac{p'-1}{\lambda} - q'\right] &= 0. \end{aligned}$$

We further observe that the determinant of the negative inverse of Ψ is $|\Psi| = (\Psi_{11}\Psi_{22} - \Psi_{12}^2)^{-1}$ where

$$\begin{aligned} \Psi_{11} &= \frac{1}{n}\left[\frac{j}{\eta^2} - G_{\eta\eta}(\eta, \lambda \mid \mathbf{x}) + \frac{a'-1}{\eta^2}\right] \\ \Psi_{12} &= -\frac{G_{\eta\lambda}(\eta, \lambda \mid \mathbf{x})}{n} \\ \Psi_{22} &= \frac{1}{n}\left[\frac{j}{\lambda^2} - G_{\lambda\lambda}(\eta, \lambda \mid \mathbf{x}) + \frac{p'-1}{\lambda^2}\right]. \end{aligned}$$

with $G_{\eta\eta}(\eta, \lambda \mid \mathbf{x}) = 0,\ G_{\eta\lambda}(\eta, \lambda \mid \mathbf{x}) = -\sum_{i=1}^{j} e^{x_i^\lambda} x_i^\lambda \ln x_i - s_k^* e^{t^\lambda} t^\lambda \ln t$ and $G_{\lambda\lambda}(\eta, \lambda \mid \mathbf{x}) = \sum_{i=1}^{j}\left[x_i^\lambda (\ln x_i)^2 - \lambda(1+s_i)e^{x_i^\lambda} x_i^\lambda (\ln x_i)^2 (1 + x_i^\lambda)\right] - s_k^* \eta e^{t^\lambda} t^\lambda (\ln t)^2 (1 + t^\lambda)$.
Proceeding in a similar manner, we can obtain elements of the matrix Φ. Finally to derive Bayes estimate of η we take $w(\eta, \lambda) = \eta$ and that for estimating λ we take $w(\eta, \lambda) = \lambda$ in the above calculations. It should be notice that the TK method is not useful in interval estimation. In this regard we next discuss a Metropolis-Hastings algorithm which is not only useful in interval estimation but also can be used to obtain point estimates of unknown parameters.

3.2. MH algorithm

In this section, we obtain Bayes estimates of η and λ and also construct HPD intervals using an MH algorithm which was originally explored by Metropolis et al. [34] and Hasting [18]. This method is quite useful in situation where a prescribed posterior distribution is analytically intractable and posterior samples from it can be generated using arbitrary proposal distributions. The generated samples can be used to make Bayes inference on unknown parameters of interest. The desired samples from the posterior distribution as given in (9) can be generated using the following steps.

1. Choose an initial guess of (η, λ) and call it (η_0, λ_0).
2. Set $i = 1$.
3. Generate ϱ' from $N(\ln \lambda_{i-1}, \sigma^2)$ with i denoting an iterative stage and σ^2 is the variance of λ.
4. Set $\lambda' = \exp(\varrho')$.
5. Generate η' from the gamma $G\left(j + a', b' - \sum_{v=1}^{j}\left[\{1 + s_v\}\{1 - e^{x_v^{\lambda'}}\}\right] - s_k^*\{1 - e^{t^{\lambda'}}\}\right)$ distribution.
6. Compute $\Omega = \min\left\{1, \frac{\pi(\eta', \lambda' | \mathbf{x})\lambda'}{\pi(\eta_{i-1}, \lambda_{i-1} | \mathbf{x})\lambda_{i-1}}\right\}$.
7. Generate a sample u from $U(0, 1)$.
8. if $\Omega \geq u$, set

$$\eta_i \leftarrow \eta', \lambda_i \leftarrow \lambda'$$

otherwise

$$\eta_i \leftarrow \eta_{i-1}, \lambda_i \leftarrow \lambda_{i-1}.$$

9. Set $i \leftarrow i+1$.

10. Repeat the steps 2-9 K times.

For computational purpose we discard the initial K_0 number of samples and then obtain estimates of η and λ as $\frac{1}{K-K_0}\sum_{i=K_0+1}^{K}\eta_i$ and $\frac{1}{K-K_0}\sum_{i=K_0+1}^{K}\lambda_i$ respectively. The corresponding $100(1-p)\%$ HPD intervals of unknown parameters η and λ can be constructed using the method of Chen and Shao [9].

4. Simulation study

In previous sections, we considered point and interval estimation of unknown parameters of a $B(\eta,\lambda)$ distribution based on type I progressive hybrid censored data under classical and Bayesian approaches. In this section we compare different estimators of η and λ using simulations in terms of their bias and mean square error values. We arbitrarily take true value of (η,λ) as (1.5,0.5). We compute MLEs of unknown parameters using an EM algorithm. The Bayes estimates are derived using the TK method and MH algorithm with respect to the squared error loss function. We compute these Bayes estimates with respect to gamma prior distributions with a', b', p', q' being hyperparameters. Selection of legitimate values to these parameters is usually based on a priori information. In this regard, suppose that N number of random sample is available from a $B(\eta,\lambda)$ distribution and that $(\hat{\eta}_{(i)}, \hat{\lambda}_{(i)}),\ i=1,2,\ldots,N$ denote corresponding MLEs of (η,λ). Next suppose that a parameter θ is a priori distributed as gamma with density proportional to $\theta^{g_1-1}e^{-g_2\theta}$, where θ can be either η or λ. Then prior mean and prior variance are respectively given by $\frac{g_1}{g_2}$ and $\frac{g_1}{g_2^2}$. Now equating the sample mean and sample variance of $\hat{\theta}_{(i)}$ with the prior mean and prior variance, we have

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_{(i)} = \frac{g_1}{g_2} \text{ and } \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\theta}_{(i)} - \sum_{v=1}^{N}\hat{\theta}_{(v)}\right)^2 = \frac{g_1}{g_2^2}.$$

From these equations we get

$$g_1 = \frac{\left(\frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_{(i)}\right)^2}{\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\theta}_{(i)} - \sum_{v=1}^{N}\hat{\theta}_{(v)}\right)^2} \text{ and } g_2 = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_{(i)}}{\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\theta}_{(i)} - \sum_{v=1}^{N}\hat{\theta}_{(v)}\right)^2}.$$

Thus if the unknown parameter is η then corresponding hyperparameters are estimated as $g_1 = a',\ g_2 = b'$ and likewise for λ we have $g_1 = p',\ g_2 = q'$. Following this we compute hyperparameters by taking $N = 1000$ and for each N a sample of size 100 is taken into consideration to obtain desired estimates. As a consequence hyperparameters are assigned values as $a' = 82.0408, b' = 53.0227, p' = 127.437, q' = 248.618$. We have computed various Bayesian estimates of η and λ based on these assignments. In Table 1, we present average values of all estimates of η and λ with corresponding mean square errors for two arbitrarily chosen values 0.5, 1 of t. These estimates are computed for different sampling combinations and censoring schemes, namely $h_1 = (10, 0^{*9}),\ h_2 = (7, 1, 0^{*5}, 2, 0^{*2}),\ h_3 = (15, 0^{*14}),\ h_4 = (8, 7, 0^{*13}),\ h_5 = (8, 1, 0^{*17}), 1),\ h_6 = (0^{*6}, 6, 0^{*11}, 4, 0),\ h_7 = (5, 0^{*24}),\ h_8 = (2, 0^{*22}, 3, 0),\ h_9 = (15, 0^{*10}, 5, 0^{*8}),\ h_{10} = (0^{*5}, 9, 3^{*2}, 0^{*11}, 5)$. Further for each scheme in Table 1, the first two row respectively correspond to estimated values and MSE for the parameter η and the last two rows are for the parameter λ. Tabulated values suggest that MLEs of both the parameters compete quite well with respective Bayes estimates. However, Bayes estimates perform well as far as bias and MSEs are concerned. Overall we conclude that MH estimates perform better compared to its competitors. We also observe that with the increase in t efficiency of proposed methods increase under a given sampling situation. This holds for all tabulated schemes computed under different setup.

In Table 2, we have constructed asymptotic confidence intervals and highest posterior density intervals of unknown parameters η and λ for different schemes and for two different values of t. For each scheme respective average interval length for η and λ is given along with corresponding coverage probabilities (CPs). It is seen that length of asymptotic intervals are wider than those of HPD intervals. Also length of both the interval tend to decrease with the increase in sample size. Similar behavior is observed for t also. Coverage probabilities of both the interval remain close to the nominal 95% level.

Table 1. Average values and MSEs of MLEs and Bayes estimates of η and λ

(n,r)	Scheme	$t=0.5$			$t=1$		
		ML	TK	MH	ML	TK	MH
(20,10)	h_1	1.9430	1.9320	1.5380	1.6530	1.6440	1.5380
		(0.237000)	(0.227000)	(0.003648)	(0.097100)	(0.093590)	(0.003968)
		0.5149	0.5131	0.5116	0.5115	0.5097	0.5120
		(0.017080)	(0.016920)	(0.000355)	(0.019480)	(0.019310)	(0.000370)
	h_2	1.7980	1.7880	1.5390	1.5940	1.5850	1.5390
		(0.139300)	(0.132900)	(0.003796)	(0.077710)	(0.075370)	(0.003994)
		0.5134	0.5117	0.5118	0.5017	0.5009	0.5118
		(0.016790)	(0.016630)	(0.000368)	(0.015370)	(0.015260)	(0.000373)
(30,15)	h_3	1.9200	1.9100	1.5370	1.6240	1.6150	1.5380
		(0.202500)	(0.193600)	(0.004292)	(0.058700)	(0.056210)	(0.004745)
		0.5102	0.5085	0.5117	0.4985	0.4969	0.5112
		(0.011440)	(0.011330)	(0.000405)	(0.012100)	(0.012020)	(0.000406)
	h_4	1.9090	1.8980	1.5390	1.6180	1.6090	1.5370
		(0.191700)	(0.183100)	(0.004377)	(0.057560)	(0.055170)	(0.004762)
		0.5068	0.5052	0.5114	0.4994	0.4978	0.5113
		(0.010200)	(0.010120)	(0.000407)	(0.011440)	(0.011370)	(0.000427)
(30,20)	h_5	2.2120	2.2000	1.5380	1.7110	1.7030	1.5370
		(0.560100)	(0.543200)	(0.005119)	(0.11110)	(0.10690)	(0.005762)
		0.5886	0.5868	0.5113	0.5447	0.5430	0.5113
		(0.021270)	(0.020880)	(0.000432)	(0.013750)	(0.013530)	(0.000447)
	h_6	1.9400	1.9300	1.5380	1.6150	1.6070	1.5390
		(0.278900)	(0.269200)	(0.005205)	(0.085270)	(0.082720)	(0.005431)
		0.5626	0.5609	0.5114	0.5194	0.5179	0.5114
		(0.018200)	(0.017910)	(0.000453)	(0.008108)	(0.008002)	(0.000464)
(30,25)	h_7	2.4250	2.4130	1.5380	1.8320	1.8230	1.5360
		(0.947100)	(0.923500)	(0.005324)	(0.200600)	(0.193900)	(0.006002)
		0.6470	0.6451	0.5114	0.5891	0.5873	0.5114
		(0.036100)	(0.035470)	(0.000447)	(0.019240)	(0.018860)	(0.000467)
	h_8	2.4260	2.4140	1.5370	1.7400	1.7310	1.5370
		(0.962000)	(0.938400)	(0.005666)	(0.169400)	(0.164300)	(0.006082)
		0.6651	0.6632	0.5110	0.5710	0.5692	0.5115
		(0.045460)	(0.044730)	(0.000448)	(0.015460)	(0.015150)	(0.000482)
(40,20)	h_9	1.7900	1.7810	1.5380	1.5680	1.5600	1.5380
		(0.104800)	(0.099260)	(0.004908)	(0.033630)	(0.032310)	(0.005424)
		0.4868	0.4985	0.5112	0.4903	0.4888	0.5114
		(0.007433)	(0.007390)	(0.000453)	(0.007591)	(0.007575)	(0.000476)
	h_{10}	1.6700	1.6620	1.5390	1.4530	1.4450	1.5380
		(0.085980)	(0.082510)	(0.005220)	(0.036780)	(0.034160)	(0.005216)
		0.5023	0.5008	0.5110	0.4808	0.4794	0.5111
		(0.0083020)	(0.008252)	(0.000480)	(0.005588)	(0.005213)	(0.000486)

5. Data analysis

In this section, we analyze a real data set originally reported in Hand et al. [17]. We have listed the data set in Appendix A2. The data represent graft survival times in months of 148 renal transplant patients. We first verify

Table 2. Average length and CP values of different intervals for α and β

(n, r)	Scheme	$t = 0.5$				$t = 1$			
		AL ACI	CP	AL HPD	CP	AL ACI	CP	AL HPDI	CP
(20,10)	h_1	3.5130	0.9694	0.4491	0.9875	2.5130	0.9378	0.4452	0.9720
		0.5856	0.9610	0.1649	0.9934	0.5428	0.9410	0.1646	0.9601
	h_2	3.2670	0.9899	0.4481	0.9909	2.5620	0.9853	0.4456	0.9349
		0.5896	0.9716	0.1639	0.9998	0.5441	0.9581	0.1636	0.8920
(30,15)	h_3	2.7890	0.9492	0.4413	0.9938	1.9770	0.9005	0.4360	0.9856
		0.4770	0.9643	0.1611	0.9269	0.4321	0.9350	0.1602	0.9334
	h_4	2.7900	0.9965	0.4420	0.9336	1.9960	0.9639	0.4361	0.9999
		0.4626	0.9667	0.1600	0.9759	0.4238	0.9412	0.1594	0.9851
(30,20)	h_5	2.9170	0.9967	0.4332	0.9999	1.8310	0.9782	0.4260	0.9993
		0.5041	0.9822	0.1583	0.9999	0.4335	0.9722	0.1576	0.9641
	h_6	2.5490	0.9210	0.4317	0.9998	1.9060	0.9997	0.4290	0.9991
		0.4731	0.9765	0.1558	0.9669	0.4251	0.9884	0.1554	0.9954
(30,25)	h_7	3.0770	0.9996	0.4276	0.9990	1.8090	0.9994	0.4184	0.9982
		0.5270	0.9365	0.1570	0.9999	0.4404	0.9630	0.1559	0.9999
	h_8	3.0080	0.9189	0.4242	0.9988	1.7550	0.9116	0.4178	0.9983
		0.5275	0.8868	0.1559	0.9763	0.4335	0.9745	0.1552	0.8756
(40,20)	h_9	2.3180	0.9551	0.4349	0.9995	1.7330	0.9409	0.4290	0.9995
		0.4152	0.9700	0.1556	0.9459	0.3740	0.9477	0.1550	0.9999
	h_{10}	2.0900	0.8890	0.4329	0.9999	1.7370	0.9976	0.4311	0.9995
		0.3898	0.9732	0.1518	0.9999	0.3726	0.9747	0.1517	0.9999

whether a Chen distribution with bathtub shape is appropriate for making inference from the considered data. We fitted three more distributions namely generalized exponential, Weibull and exponential to this data set for comparison purposes. The maximum likelihood estimates of unknown parameters of competing models along with values of different model selection criteria like Kolmogorov-Sirnov (KS) statistics, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are reported in Table 3. For computational convenience, we divided each data point by 10. The tabulated values indicate that proposed model fit the data set really good compared to the other distributions. Also in Figure 1, we have provided density plots for all the competing models using the maximum likelihood method. The corresponding visual analysis suggests that prescribed model can be used to make desired inference on unknown quantities of interest. We obtain estimates of unknown parameters with respect to different censoring schemes, namely $h_1^r = (2^{*3}, 0^{*10}, 3^{*8}, 0^{*10}, 2^{*4}, 0^{*75})$, $h_2^r = (2^{*15}, 0^{*10}, 6, 0^{*83}, 2)$, $h_3^r = (, 5^{*4}, 0^{*45}, 2^{*2}, 0^{*45}, 4, 0^{*23})$, $h_4^r = (0^{*8}, 6, 21, 0^{*50}, 1, 0^{*59})$. The observed data for these scheme are listed below.

Scheme h_1^r:

0.0035, 0.0068, 0.01, 0.0101, 0.0167, 0.0168, 0.0197, 0.0213, 0.0233, 0.0234, 0.0508, 0.0508, 0.0533, 0.0633, 0.0767, 0.0768, 0.077, 0.1066, 0.1267, 0.13, 0.1639, 0.1803, 0.1867, 0.218, 0.2967, 0.3328, 0.37, 0.3803, 0.4867, 0.6233, 0.6367, 0.66, 0.66, 0.718, 0.78, 0.7933, 0.7967, 0.8016, 0.83, 0.841, 0.91, 0.9233, 1.0541, 1.0607, 1.0633, 1.0667, 1.1067, 1.2213, 1.2508, 1.2533, 1.38, 1.4267, 1.4475, 1.45, 1.5213, 1.5333, 1.5525, 1.5533, 1.5541, 1.5934, 1.62, 1.63, 1.6344, 1.66, 1.7033, 1.7067, 1.7475, 1.7667, 1.77, 1.7967, 1.8115, 1.8115, 1.8933, 1.8934, 1.9508, 1.9733, 2.018, 2.09, 2.1167, 2.1233, 2.21, 2.2148, 2.2267, 2.25, 2.2533, 2.3738, 2.4082, 2.418, 2.4705, 2.5213, 2.5705, 3.1934, 3.218, 3.2367, 3.2705, 3.3148, 3.3567, 3.4836, 3.4869, 3.6213, 3.941, 3.9433, 4.0001, 4.1733, 4.1734, 4.2311, 4.2869, 4.3279, 4.3902, 4.4267.

Scheme h_2^r:

0.0035, 0.0068, 0.01, 0.0101, 0.0167, 0.0168, 0.0197, 0.0213, 0.0234, 0.0508, 0.0508, 0.0533, 0.0633, 0.0767, 0.0768, 0.077, 0.1066, 0.1267, 0.13, 0.16, 0.1639, 0.1803, 0.1867, 0.218, 0.2667, 0.2967, 0.37, 0.3803, 0.4311, 0.4867, 0.518, 0.6233, 0.66, 0.7667, 0.7733, 0.7967, 0.83, 0.841, 0.8607, 0.8667, 0.88, 0.91, 0.9233, 1.0541, 1.0667, 1.0869, 1.1067, 1.118, 1.2213, 1.2508, 1.4267, 1.4475, 1.45, 1.5333, 1.5525, 1.5533, 1.5541, 1.5934, 1.62, 1.63, 1.66, 1.67, 1.6933, 1.7475, 1.7667, 1.77, 1.7967, 1.8115, 1.8115, 1.9508, 1.9574, 1.9733, 2.0148, 2.018, 2.09, 2.1167, 2.16, 2.21, 2.2148, 2.218, 2.2267, 2.23, 2.25, 2.2533, 2.3738, 2.4082, 2.5213, 2.5705, 2.9705, 3.1934, 3.218, 3.2367, 3.2672, 3.2705, 3.3567, 3.377, 3.3869, 3.4836, 3.4869, 3.5738, 3.618, 3.941, 4.1733, 4.1734, 4.2311, 4.2869, 4.318, 4.3902, 4.4267, 4.4475.

Scheme h_3^r:

0.0035, 0.0068, 0.01, 0.0101, 0.0167, 0.0168, 0.0213, 0.0233, 0.0234, 0.0508, 0.0533, 0.0633, 0.0767, 0.0768, 0.077, 0.1066, 0.13, 0.16, 0.1639, 0.1803, 0.1867, 0.218, 0.2667, 0.2967, 0.3328, 0.3393, 0.37, 0.3803, 0.4311, 0.4867, 0.518, 0.6233, 0.6367, 0.66, 0.66, 0.718, 0.7667, 0.7933, 0.7967, 0.83, 0.841, 0.8607, 0.8667, 0.88, 0.91, 0.9233, 1.0541, 1.0607, 1.0633, 1.0667, 1.0869, 1.1067, 1.118, 1.2508, 1.3467, 1.38, 1.4267, 1.4475, 1.45, 1.5213, 1.5333, 1.5525, 1.5533, 1.5541, 1.62, 1.6344, 1.67, 1.6933, 1.7033, 1.7067, 1.7475, 1.7667, 1.77, 1.7967, 1.8933, 1.8934, 1.9508, 1.9574, 1.9733, 2.018, 2.09, 2.1167, 2.16, 2.21, 2.2148, 2.218, 2.218, 2.2267, 2.23, 2.25, 2.2533, 2.3738, 2.4082, 2.418, 2.4705, 2.5213, 2.5705, 3.0443, 3.1667, 3.1934, 3.218, 3.2672, 3.3148, 3.3567, 3.377, 3.3869, 3.4836, 3.4934, 3.618, 3.6213, 3.941, 3.9433, 3.9672, 4.1733, 4.1734, 4.318, 4.3279, 4.3902, 4.5148, 4.6451.

Scheme h_4^r:

0.0035, 0.0068, 0.01, 0.0101, 0.0167, 0.0168, 0.0197, 0.0213, 0.0233, 0.0234, 0.0508, 0.0508, 0.0533, 0.0633, 0.0767, 0.0768, 0.1066, 0.1267, 0.13, 0.16, 0.1639, 0.1803, 0.1867, 0.218, 0.2667, 0.2967, 0.3328, 0.3393, 0.37, 0.4311, 0.4867, 0.518, 0.6233, 0.6367, 0.66, 0.718, 0.7667, 0.7733, 0.78, 0.7933, 0.7967, 0.8016, 0.83, 0.841, 0.8667, 0.9233, 1.0541, 1.0607, 1.0633, 1.0667, 1.0869, 1.1067, 1.118, 1.2213, 1.2508, 1.3467, 1.38, 1.4267, 1.4475, 1.5213, 1.5333, 1.5525, 1.5533, 1.5541, 1.5934, 1.62, 1.63, 1.6344, 1.66, 1.67, 1.7067, 1.7475, 1.7667, 1.77, 1.7967, 1.8115, 1.8115, 1.8934, 1.9508, 1.9574, 1.9733, 2.1167, 2.1233, 2.16, 2.21, 2.2148, 2.218, 2.2267, 2.23, 2.25, 2.2867, 2.3738, 2.5213, 2.5705, 2.9705, 3.0443, 3.1667, 3.218, 3.2672, 3.2705, 3.3148, 3.3567, 3.377, 3.3869, 3.4836, 3.4869, 3.5738, 3.618, 3.6213, 3.941, 3.9672, 4.0001, 4.1734, 4.2869, 4.318, 4.3279, 4.3902, 4.4267, 4.5148, 4.6451

In Table 4 we have tabulated maximum likelihood and Bayes estimates of unknown parameters η and λ for two different values of t under the suggested censoring schemes. The Bayes estimates are derived with respect to a noninformative prior distribution. It is seen that a fixed t and a given censoring scheme lead to estimates which remain close to each other. This holds true for estimating both the unknown parameters. We further observe that with the increase in t the estimated values of η tend to increase whereas opposite holds true for the parameter λ. In Table 5 we have constructed asymptotic and noninformative HPD intervals of unknown parameters for the arbitrarily given sampling conditions. From this table we observe that in general HPD intervals are shorter than the asymptotic intervals.

Table 3. Goodness of fit tests for competing models in real data set

Distribution	$\hat{\eta}$	$\hat{\lambda}$	K-S	AIC	BIC
Bathtub	0.2650	0.6358	0.0603	433.8000	439.7000
GE	0.8927	0.5348	0.1334	462.8000	468.8000
Weibull	1.0260	0.5615	0.1189	464.0000	470.0000
Exponential		0.5744	0.1232	464.1000	470.1000

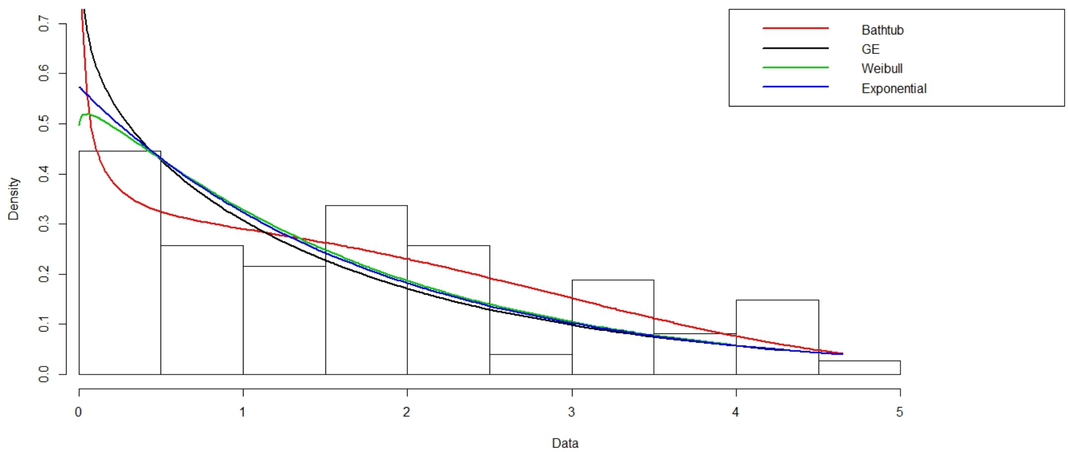

Figure 1. Density plots for competing models

Table 4. Average values of MLEs and Bayes estimates of η and λ for the real data set

(n,r)	Scheme	$t=4.25$			$t=5$		
		ML	T-K	M-H	ML	T-K	M-H
(148,110)	h_1^r	0.3400	0.2961	0.2928	0.3411	0.2909	0.2883
		0.6815	0.6059	0.6092	0.6344	0.6314	0.6338
	h_2^r	0.3452	0.2956	0.2934	0.3470	0.2899	0.2874
		0.6711	0.5742	0.5761	0.6147	0.6017	0.6042
(148,120)	h_3^r	0.3208	0.2803	0.2774	0.3236	0.2731	0.2701
		0.6851	0.6050	0.6079	0.6275	0.6334	0.6363
	h_4^r	0.3432	0.3058	0.3027	0.3466	0.2969	0.2934
		0.6676	0.5730	0.5761	0.5942	0.6112	0.6144

Table 5. Interval estimates of η and λ for the real data set

(n,r)	Scheme	$t=4.25$		$t=5$	
		ACI	HPD	ACI	HPDI
(148,110)	h_1^r	(0.2619,0.4181)	(0.2377,0.3524)	(0.2637,0.4185)	(0.2297,0.3491)
		(0.5901,0.7729)	(0.5332,0.6819)	(0.5599,0.7089)	(0.5560,0.7121)
	h_2^r	(0.2645,0.4259)	(0.2355,0.3551)	(0.2677,0.4262)	(0.2276,0.3476)
		(0.5698,0.7723)	(0.5012,0.6890)	(0.5389,0.6905)	(0.5304,0.6782)
(148,120)	h_3^r	(0.2470,0.3946)	(0.2242,0.3343)	(0.2508,0.3963)	(0.2113,0.3262)
		(0.5904,0.7798)	(0.5306,0.6898)	(0.5545,0.7005)	(0.5595,0.7145)
	h_4^r	(0.2647,0.4217)	(0.2435,0.3647)	(0.2694,0.4237)	(0.2341,0.3530)
		(0.5715,0.7636)	(0.5234,0.6727)	(0.5260,0.6625)	(0.5428,0.6893)

6. Optimal plans

In previous sections, we obtained different point and interval estimates of unknown parameters of a $B(\eta, \lambda)$ distribution under the assumption that samples are type I progressive hybrid censored. Furthermore, these estimates are obtained with respect to prescribed censoring schemes. In many situations, life tests are conducted under cost limits as well. Thus selection of optimal censoring schemes from a class of possible schemes satisfying cost constraints becomes a desirable criterion for an experimenter. It is also desirable that selected schemes should provide efficient procedures for unknown quantities of interest. This problem of comparing possible censoring schemes to obtain optimal plans satisfying certain restrictions has received some attentions in literature, e.g. Kundu [23] and Lin et al. [25]. The proposed criterion is based on efficiently inferring the logarithm of the qth $(0 < q < 1)$ quantile $\ln t_q = \frac{1}{\lambda}\ln[\ln\{1 - \frac{1}{\eta}\ln(1-q)\}]$ of a $B(\eta, \lambda)$ distribution under the given cost limit. To this end, let $\mathfrak{C}_{\mathfrak{b}}$ denote the total available budget associated with a life testing experiment then the subsequent objective is to establish optimum choice (n, r, s, t) so that it provides maximum information for the unknown parameters of a $B(\eta, \lambda)$ distribution. Let $\mathfrak{C}_{\mathfrak{i}}$ denote the inspection cost, $\mathfrak{C}_{\mathfrak{s}}$ denote the salvage cost for a live unit during the inspection and $\mathfrak{C}_{\mathfrak{t}}$ denote the cost per unit time required to run the test. Thus the budget constraint of a life test under the given type I progressive hybrid censoring scheme should satisfy the restriction

$$(\mathfrak{C}_{\mathfrak{i}} - \mathfrak{C}_{\mathfrak{s}})n + \mathfrak{C}_{\mathfrak{s}}E(J) + \mathfrak{C}_{\mathfrak{t}}t \le \mathfrak{C}_{\mathfrak{b}} \tag{12}$$

where $t > 0$ and $E(J) \ge 1$. Note that, $E(J)$ denotes the expected number of failures during a test before its termination at $\min\{X_r, t\}$ (see also, Appendix A3). In sequel, we wish to solve an optimization problem to explore a plan $(n_b, r_b, \mathcal{S}_b, t_b)$ for which the information measure

$$\mathcal{V}(\mathcal{S}) = \left[E_{data}\left(\int_0^1 V_{posterior(\mathcal{S})}(\ln t_q)\, dq\right)\right]^{-1}$$

is maximized where $\mathcal{S} = (s_1, s_2, \ldots, s_r)$ and $V_{posterior(\mathcal{S})}(.)$ denotes the posterior variance. For given n, r and $\mathcal{S}$ we search the optimal solution using the algorithm as discussed by Lin et al. [25]. For the sake of completeness, we provide required steps below:

1. Compute $\bar{n} = \lfloor \frac{\mathfrak{C}_{\mathfrak{b}} - \mathfrak{C}_{\mathfrak{s}})}{\mathfrak{C}_{\mathfrak{i}} - \mathfrak{C}_{\mathfrak{s}}} \rfloor$ where $\bar{n}$ denotes the upper bound of n and $\lfloor \iota \rfloor$ represents the greatest integer less than or equal to ι.
2. Set $n \leftarrow 2$
3. Compute $t' = \frac{\mathfrak{C}_{\mathfrak{b}} - (\mathfrak{C}_{\mathfrak{i}} - \mathfrak{C}_{\mathfrak{s}})n - \mathfrak{C}_{\mathfrak{s}}}{\mathfrak{C}_{\mathfrak{t}}}$ and the upper bound of t satisfying the equation (12).
4. Set $r \leftarrow 1$.
5. We now apply Newton-Raphson method to compute t and then evaluate $\mathcal{V}(\mathcal{S})$ for all possible choices of $\mathcal{S}$ with given n and r using the package `partition` of R software.
6. Set $r \leftarrow r + 1$. If $r \le n - 1$ go to step 5, otherwise go to step 7.
7. Store all the entries of $(n, r, \mathcal{S}, t)$ for which $\mathcal{V}(\mathcal{S})$ has the maximum value among all possible choices of r and $\mathcal{S}$.
8. Set $n \leftarrow n + 1$, if $n \le \bar{n}$ go to step 3, otherwise go to step 9.
9. Tabulate $\{r, t, t', \mathcal{V}(\mathcal{S}), \mathcal{S}\}$ for all n where $2 \le n \le \bar{n}$.
10. Select $\mathcal{V}(\mathcal{S})$ with the largest value and the corresponding $\{n_b, r_b, t_b, t'_b, \mathcal{S}_b\}$.

We note that the expectation $E_{data}\left(\int_0^1 V_{posterior(\mathcal{S})}(\ln t_q)\, dq\right)$ can not be evaluated analytically. We further observe that we have

$$V_{posterior(\mathcal{S})}(\ln t_q) = E_{posterior(\mathcal{S})}(\ln t_q)^2 - (E_{posterior(\mathcal{S})}(\ln t_q))^2.$$

In order to compute the quantity $V_{posterior(\mathcal{S})}(\ln t_q)$, we use the MH algorithm to estimate values of posterior expectations $E_{posterior(\mathcal{S})}(\ln t_q)^2$ and $E_{posterior(\mathcal{S})}(\ln t_q)$. We repeat the algorithm 1000 times to obtain the desired results.

For numerical illustration, suppose that $\mathfrak{C}_{\mathfrak{i}} = 14.5,\ \mathfrak{C}_{\mathfrak{s}} = 5,\ \mathfrak{C}_{\mathfrak{t}} = 6$ and $\mathfrak{C}_{\mathfrak{b}} = 150$ when true values of unknown parameters η and λ are 1.5 and 0.5 respectively. In Table 6, we have reported the maximum values of the optimal criterion along with the corresponding censoring schemes for different values of n satisfying the budget constraint. It is seen that the scheme (13,12,(0^{*11},1),0.08579) provides the maximum information for the prescribed parameters. We also verify whether or not proposed optimal plans are robust in nature. For example we have $\mathcal{V}(\mathcal{S}) = 0.2026$ for the scheme $\mathcal{S}$=(0^{*5},1,0^{*6}) slightly deviated from the optimal scheme $(0^{*11}, 1)$. The corresponding relative efficiency is given by $\frac{0.2026}{0.2083} = 0.9726$, which suggests that both the schemes are nearly optimal. From these observations we conclude that efficiency of optimal plans does not change much under slight departure from these plans.

Table 6. Optimal schemes

n	r	t	Scheme	t'	$\mathcal{V}(\mathcal{S})$
2	1	21	(1)	21	0.1993
3	2	18.58	(0,1)	19.42	0.2006
4	3	16.17	(1,0^{*2})	17.83	0.2049
5	4	13.75	(0^{*3},1)	16.25	0.2017
6	5	11.33	(0^{*4},1)	14.67	0.2077
7	6	8.917	(0^{*5},1)	13.08	0.2016
8	7	6.5	(0^{*6},1)	11.5	0.2074
9	8	4.083	(0^{*7},1)	9.917	0.2030
10	9	1.679	(0^{*8},1)	8.333	0.2023
11	10	0.4871	(0^{*9},1)	6.75	0.2014
12	11	0.2077	(0^{*10},1)	5.167	0.1997
13	12	0.08579	(0^{*11},1)	3.583	**0.2083**
14	13	0.02834	(0^{*12},1)	2	0.2018
15	14	0.004574	(0^{*13},1)	0.4167	0.2004

7. Conclusion

In this paper, we have studied a two-parameter distribution with bathtub shape or increasing hazard rate function under type I progressive hybrid censoring. We computed different estimates for its unknown parameters using maximum likelihood and Bayesian approaches. The asymptotic confidence intervals are constructed from observed Fisher information matrix obtained using an EM algorithm. The Bayes estimates of unknown parameters are developed using TK and MH algorithm with respect to gamma prior distributions. We have also considered highest posterior density intervals based on MH procedure. Simulation results indicated that Bayes estimates perform quite good compared to the corresponding MLEs. In particular, MH method work really good in such situations. However, computational complexity of TK method is quite less compared to the MH procedure. We further observed that TK estimates compete good with MH estimates. We also observed that better estimates are obtained with an increase in duration of the experimentation. Coverage probabilities of both asymptotic and HPD intervals are found to be quite satisfactory with asymptotic intervals being more wider than corresponding HPD intervals. We also analyzed a real data set to illustrate different findings. Finally, we established optimal plans under cost constraints based on a Bayesian criterion. Through a numerical illustration, we found that proposed plans are quite robust in nature in the sense that relative efficiency does not change much with marginal deviations from optimal solutions.

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Appendix

. *A1.*

Let

$$I_W(\theta) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

where $p_{11} = \frac{n'+s_k^*}{\eta^2}$, $p_{12} = p_{21} = \frac{(n'+s_k^*)\eta}{\lambda}\varsigma$, $p_{22} = \frac{n'+s_k^*}{\lambda^2} + \frac{(n'+s_k^*)\eta^2}{\lambda^2}\vartheta - \frac{(n'+s_k^*)\eta}{\lambda^2}\kappa$, and now let $I^{(i)}_{W|X}(\theta)$ denotes the information matrix of a single observation for the considered model truncated at x_i. Then it is observed that

$$I^{(i)}_{W|X}(\theta) = \begin{bmatrix} q_{11}(x_{(i)};\eta,\lambda) & q_{12}(x_{(i)};\eta,\lambda) \\ q_{21}(x_{(i)};\eta,\lambda) & q_{22}(x_{(i)};\eta,\lambda) \end{bmatrix}$$

and

$$I_{W|t}(\theta) = \begin{bmatrix} q_{11}(t;\eta,\lambda) & q_{12}(t;\eta,\lambda) \\ q_{21}(t;\eta,\lambda) & q_{22}(t;\eta,\lambda) \end{bmatrix}$$

where $q_{11}(c;\eta,\lambda) = \frac{1}{\eta^2}$, $q_{12}(c;\eta,\lambda) = q_{21}(c;\eta,\lambda) = \frac{\eta}{\lambda(1-F(c;\eta,\lambda))}\xi(c;\eta,\lambda) - \varphi(c;\eta,\lambda)$ and $q_{22}(c;\eta,\lambda) = \frac{1}{\lambda^2} + \frac{\eta}{\lambda^2(1-F(c;\eta,\lambda))}[\eta\varpi(c;\eta,\lambda) - \delta(c;\eta,\lambda)] - \zeta(c;\eta,\lambda)$. Furthermore,

$$\begin{aligned}
\varsigma &= \int_0^\infty \omega_1(\rho)\, d\rho, \\
\vartheta &= \int_0^\infty \omega_2(\rho)\, d\rho, \\
\kappa &= \int_0^\infty \omega_3(\rho)\, d\rho, \\
\xi(c;\eta,\lambda) &= \int_{e^{c^\lambda}-1}^\infty \omega_1(\rho)\, d\rho, \\
\varphi(c;\eta,\lambda) &= e^{c^\lambda} c^\lambda \ln c, \\
\varpi(c;\eta,\lambda) &= \int_{e^{c^\lambda}-1}^\infty \omega_2(\rho)\, d\rho, \\
\delta(c;\eta,\lambda) &= \int_{e^{c^\lambda}-1}^\infty \omega_3(\rho)\, d\rho, \\
\zeta(c;\eta,\lambda) &= \eta(\ln c)^2 e^{c^\lambda} c^\lambda (1+c^\lambda)
\end{aligned}$$

and $\omega_1(\rho) = (\rho+1)\ln(\rho+1)\ln(1+\rho)e^{-\eta\rho}$, $\omega_2(\rho) = (\rho+1)\ln(\rho+1)\{\ln(\ln(\rho+1))\}^2(1+\ln(\rho+1))e^{-\eta\rho}$ and $\omega_3(\rho) = \ln(\rho+1)\{\ln(\ln(\rho+1))\}^2 e^{-\eta\rho}$.

. ***A2.***

0.035, 0.068, 0.100, 0.101, 0.167, 0.168, 0.197, 0.213, 0.233, 0.234, 0.508, 0.508, 0.533, 0.633, 0.767, 0.768, 0.770, 1.066, 1.267, 1.300, 1.600, 1.639, 1.803, 1.867, 2.180, 2.667, 2.967, 3.328, 3.393, 3.700, 3.803, 4.311, 4.867, 5.180, 6.233, 6.367, 6.600, 6.600, 7.180, 7.667, 7.733, 7.800, 7.933, 7.967, 8.016, 8.300, 8.410, 8.607, 8.667, 8.800, 9.100, 9.233, 10.541, 10.607, 10.633, 10.667, 10.869, 11.067, 11.180, 11.443, 12.213, 12.508, 12.533, 13.467, 13.800, 14.267, 14.475, 14.500, 15.213, 15.333, 15.525, 15.533, 15.541, 15.934, 16.200, 16.300, 16.344, 16.600, 16.700, 16.933, 17.033, 17.067, 17.475, 17.667, 17.700, 17.967, 18.115, 18.115, 18.933, 18.934, 19.508, 19.574, 19.733, 20.148, 20.180, 20.900, 21.167, 21.233, 21.600, 22.100, 22.148, 22.180, 22.180, 22.267, 22.300, 22.500, 22.533, 22.867, 23.738, 24.082, 24.180, 24.705, 25.213, 25.705, 29.705, 30.443, 31.667, 31.934, 32.180, 32.367, 32.672, 32.705, 33.148, 33.567, 33.770, 33.869, 34.836, 34.869, 34.934, 35.738, 36.180, 36.213, 39.410, 39.433, 39.672, 40.001, 41.733, 41.734, 42.311, 42.869, 43.180, 43.279, 43.902, 44.267, 44.475, 44.900, 45.148, 46.451.

. ***A3.***

To obtain the expected number of failures $E(J)$, we need to first compute $P(J=j)$ for a given t. Now for the proposed model $P(J=j)$ is given by

$$P(J=j) = \begin{cases} \exp\left[n\eta(1-e^{t^{\lambda}})\right], & \text{for } j=0, \\ b_{j-1}\exp\left[\eta\nu_{j+1}(1-e^{t^{\lambda}})\right]\sum_{d=1}^{j}\frac{\Upsilon_{d,j}}{\nu_d-\nu_{j+1}}\Delta(d,j), & \text{for } j=1,2,\ldots,r-1, \\ b_{r-1}\sum_{d=1}^{r}\frac{\Upsilon_{d,r}}{\nu_d}\left[1-\exp\left(\eta\nu_d(1-e^{t^{\lambda}})\right)\right], & \text{for } j=m, \end{cases}$$

where $\nu_l = r-l+\sum_{i=l}^{r}s_i$, for $l=1,2,\ldots,r$, $b_{l-1}=\prod_{i=1}^{l}\nu_l$, $\Upsilon_{d,m}=\prod_{v=1,v\neq d}^{m}, \frac{1}{\nu_v-\nu_d}$, for $1\le d\le m\le r$ and $\Upsilon_{1,1}=1$. See Tomer and Panwar [35] for details in this regard.

Furthermore, $\Delta(d,j) = \Big(1-\exp(1-\exp\{(\nu_d-\nu_{j+1})\eta(1-e^{t^{\lambda}})\})\Big)$. Therefore for type I progressive hybrid censoring scheme $E(J)$ can be obtained by computing $E(J)=\sum_{j=1}^{r} jP(J=j)$.

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