



Vector-valued nonuniform multiresolution analysis related to Walsh function

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Abstract In this paper, we introduce vector-valued nonuniform multiresolution analysis on positive half-line related to Walsh functions. We obtain the necessary and sufficient condition for the existence of associated wavelets.

Keywords Vector-valued Wavelets, Multiresolution analysis, Scaling function, Wash-Fourier transform.

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1. Introduction

The idea of multiresolution analysis has become one of the main tools for constructing wavelets. In his paper, Mallat [15] first formulated the remarkable idea of multiresolution analysis (MRA) that deals with a general formism for construction of an orthogonal basis of wavelet bases. A multiresolution analysis consists of a sequence of embedded closed subspaces $\{V_j : j \in \mathbb{Z}\}$ for approximating $L^2(\mathbb{R})$ functions such that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there exist an element $\varphi \in V_0$ such that the collection of integer translates of φ , $\{\varphi(x - n)\}_{n \in \mathbb{Z}}$ is a complete orthonormal system for V_0 . Any compactly supported wavelet must come from a MRA. The function $\varphi \in V_0$ is called a scaling function.

The idea of wavelet and multiresolution analysis has been extended to many different setups. One can replace the dilation factor 2 by an integer $N \geq 2$ and one needs to construct $N - 1$ wavelets to generate the whole space $L^2(\mathbb{R})$. In general, in higher dimensions, it can be replaced by a dilation matrix A , in which case the number of wavelets required is $|\det A| - 1$. But in all these cases, the translation set is always a group. In the two papers [7], [8], Gabardo and Nashed considered a generalization of Mallat's [15] celebrated theory of MRA, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of \mathbb{Z} and a translate of \mathbb{Z} . More precisely, this set is of the form $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer relatively prime to N . They call this a nonuniform multiresolution analysis (NUMRA) and is based on the theory of spectral pairs.

The idea of dyadic multiresolution analysis has been studied on $L^2(\mathbb{R}^+)$, where \mathbb{R}^+ denotes the positive half-real line by Protasov and Farkov. Farkov [5] has given a general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [6] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. Protasov and Farkov constructed examples of refinable functions that generate MRA on half-line and Vilenkin groups with the help of Cohen conditions in [18]. F. A. Shah [19], studied the construction

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of p -wavelet packets associated with the multiresolution analysis defined by Farkov [5], for $L^2(\mathbb{R}^+)$. Recently, Meenakshi et al. [17] studied NUMRA on positive half line and they also derived the analogue of Cohen’s condition for the nonuniform multiresolution analysis on the positive half-line.

In [24], Xia and Suter, introduced vector-valued multiresolution analysis and vector-valued wavelets for vector-valued signal spaces. Sun and Cheng, introduced vector-valued multiresolution analysis with dilation factor $\alpha \geq 2$ and orthogonal vector-valued wavelets in [23]. The properties of vector-valued wavelets packets are presented in [3]. In [1], Abdullah, introduced vector-valued multiresolution analysis on local field of positive characteristic and obtained the necessary and sufficient condition for the existence of associated wavelets.

In the present paper, we study vector-valued nonuniform multiresolution analysis on positive half-line and we obtain the necessary and sufficient condition for the existence of associated wavelets. The paper is organized as follows. In Sec. 2, we discuss some preliminary facts about NUMRA on positive half-line. We introduce the notion of VNUMRA on positive half-line and find a necessary and sufficient condition for the existence of associated wavelets in Sec. 3.

2. Nonuniform multiresolution analysis on positive half-line

Let \mathbb{Z}^+ and \mathbb{R}^+ denote the set of positive integers and positive real numbers, respectively. For the basic results related to Walsh-Fourier analysis and definition of multiresolution p -analysis on $L^2(\mathbb{R}^+)$, we refer to ([5], [6], [17], [19], [20], [21], [22]).

Let N be an integer, $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N - 1$, r and N are relatively prime, we consider the translation set $\Lambda_{r,N}^+$ as

$$\Lambda_{r,N}^+ = \left\{ 0, \frac{r}{N} \right\} + \mathbb{Z}^+ = \left\{ \frac{rk}{N} + n : n \in \mathbb{Z}^+, k = 0, 1 \right\}.$$

Let $[x]$ denotes integer part of x . For $x \in \mathbb{R}^+$ and for any positive integer j , we set

$$x_j = [N^j x](\text{mod } N), \quad x_{-j} = [N^{1-j} x](\text{mod } N), \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, N - 1\}$.

Non-uniform multiresolution analysis in $L^2(\mathbb{R}^+)$ defined by Meenakshi et. al. is as follows:

Definition 2.1. A nonuniform multiresolution analysis on positive half-line is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R}^+)$, $j \in \mathbb{Z}$ such that the following properties hold:

- (1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
- (2) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
- (3) $f(\cdot) \in V_j$ if and only if $f(N \cdot) \in V_{j+1}$,
- (4) There exists a function φ in V_0 such that $\{\varphi(\cdot \ominus \lambda) : \lambda \in \Lambda_{r,N}^+\}$ where $\Lambda_{r,N}^+ = \{0, r/N\} + \mathbb{Z}^+$, is a complete orthonormal system for V_0 .

The function φ whose existence is asserted in (4) is called a scaling function of the given NUMRA.

When, $N > 1$, the dilation factor of N ensures that $N\Lambda_{r,N}^+ \subset \mathbb{Z}^+ \subset \Lambda_{r,N}^+$. Condition (3) and (4) implies that

$$\varphi_{1,\lambda}(x) = N^{1/2} \varphi(Nx \ominus \lambda), \quad \lambda \in \Lambda_{r,N}^+$$

constitute and orthonormal basis in V_1 .

Since $V_0 \subset V_1$, the function $\varphi \in V_1$ and has the Fourier expansion

$$\varphi(x) = \sum_{\lambda \in \Lambda^+} h_\lambda(N)^{1/2} \varphi(Nx \ominus \lambda),$$

where $h_\lambda = \int_{\mathbb{R}^+} \varphi(x) \overline{\varphi_{1,\lambda}(x)} dx$.

This implies that

$$\varphi(x) = N \sum_{\lambda \in \Lambda^+} a_\lambda \varphi(Nx \ominus \lambda), \quad \sum_{\lambda \in \Lambda^+} |a_\lambda|^2 < \infty \tag{2.2}$$

where $a_\lambda = N^{-1/2} h_\lambda$.

Now, we consider W_0 the orthogonal complement of V_0 on V_1 , i.e.,

$$V_1 = V_0 \oplus W_0.$$

If $\psi_1, \psi_2, \dots, \psi_{N-1}$ are the functions in W_0 , then for $\ell = 0, 1, \dots, N - 1$, there exists sequences $\{a_\lambda^\ell\}_{\lambda \in \Lambda^+}$ satisfying $\sum_{\lambda \in \Lambda^+} |a_\lambda^\ell|^2 < \infty$ such that

$$\psi_\ell(x) = N \sum_{\lambda \in \Lambda^+} a_\lambda^\ell \varphi(Nx \ominus \lambda). \tag{2.3}$$

On taking the Walsh-Fourier transform, we have

$$\hat{\psi}_\ell(\xi) = m_\ell \left(\frac{\xi}{N} \right) \hat{\varphi} \left(\frac{\xi}{N} \right) \tag{2.4}$$

where the functions $m_\ell(\xi) = \sum_{\lambda \in \Lambda^+} a_\lambda^\ell \overline{\chi(\lambda, \xi)}$ and $\chi(\lambda, \xi) = \exp(\frac{2\pi i}{N} \sum_{j=1}^\infty (\lambda_j \xi_{-j} + \lambda_{-j} \xi_j))$ where λ_j and ξ_j are calculated by (2.1) for a positive integer N .

Since $\Lambda_{r,N}^+ = \{0, r/N\} + \mathbb{Z}^+$, we can write that

$$m_\ell(\xi) = m_\ell^1(\xi) + \overline{\chi \left(\frac{r}{N}, \xi \right)} m_\ell^2(\xi), \quad \ell = 0, 1, \dots, N - 1, \tag{2.5}$$

where m_ℓ^1 and m_ℓ^2 are locally L^2 functions.

In this case $\{\psi_1, \psi_2, \dots, \psi_{N-1}\}$ is a set of basic wavelets associated with a scaling function φ . It is easy to show that $\{\psi_\ell(x \ominus \lambda) : 1 \leq \ell \leq N - 1\}$ is an orthonormal basis of W_0 . An obvious rescaling shows that

$$\left\{ \psi_{\ell,j,\lambda} = (N)^{j/2} \psi_\ell((N)^j x \ominus \lambda) : 1 \leq \ell \leq N - 1, \lambda \in \Lambda_{r,N}^+ \right\}$$

is an orthonormal basis of W_j . Since $\cup_{j \in \mathbb{Z}} W_j$ is dense in $L^2(\mathbb{R}^+)$, the collection $\{\psi_{\ell,j,\lambda} : j \in \mathbb{Z}, \lambda \in \Lambda_{r,N}^+, 1 \leq \ell \leq N - 1, \}$ is an orthonormal basis of $L^2(\mathbb{R}^+)$.

3. Vector-valued nonuniform multiresolution analysis on positive half-line

We denote the set of all vector-valued functions $\mathbf{f}(x)$ by $L^2(\mathbb{R}^+, \mathbb{C}^d)$, i.e.

$$L^2(\mathbb{R}^+, \mathbb{C}^d) = \{ \mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_d(x))^T : x \in \mathbb{R}^+, f_k(x) \in L^2(\mathbb{R}^+), k = 1, 2, \dots, d \},$$

where T denotes the transpose and \mathbb{C}^d denotes the d -dimensional complex Euclidean space.

For $\mathbf{f} \in L^2(\mathbb{R}^+, \mathbb{C}^d)$, $\|\mathbf{f}\|$ denotes the norm of the vector-valued function \mathbf{f} and is defined as:

$$\|\mathbf{f}\| = \left(\sum_{s=1}^d \int_{\mathbb{R}^+} |f_s(x)|^2 dx \right)^{\frac{1}{2}}.$$

For a vector-valued function $\mathbf{f} \in L^2(\mathbb{R}^+, \mathbb{C}^d)$ the integration of $\mathbf{f}(x)$ is defined as:

$$\int_{\mathbb{R}^+} \mathbf{f}(x) dx = \left(\int_{\mathbb{R}^+} f_1(x) dx, \int_{\mathbb{R}^+} f_2(x) dx, \dots, \int_{\mathbb{R}^+} f_d(x) dx \right)^T.$$

The Walsh-Fourier transform of $\mathbf{f}(x)$ is defined by:

$$\tilde{\mathbf{f}}(\xi) = \int_{\mathbb{R}^+} \mathbf{f}(x) \overline{\chi(x, \xi)} dx.$$

For any two vector-valued functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^+, \mathbb{C}^d)$, $\langle \mathbf{f}, \mathbf{g} \rangle$ denotes the integration of the matrix product $\mathbf{f}(x) \cdot \mathbf{g}^*(x)$ as:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^+} \mathbf{f}(x) \mathbf{g}^*(x) dx,$$

where $*$ denotes the transpose and the complex conjugate and $\langle \mathbf{f}, \mathbf{g} \rangle$ is matrix valued not scalar valued but satisfied the inner product properties: (i)linearity $\langle \mathbf{f}, a_1 g_1 + a_2 g_2 \rangle = a_1^* \langle \mathbf{f}, g_1 \rangle + a_2^* \langle \mathbf{f}, g_2 \rangle$ and (ii) commutativity $\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle^*$.

Therefore, the operation $\langle \dots \rangle$ is called an inner product.

Definition 3.1. Let N be an integer, $N \geq 1$, and $\Lambda_{r,N}^+ = \{0, r/N\} + \mathbb{Z}^+$, where r is an odd integer relatively prime to N with $1 \leq r \leq 2N - 1$. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^+, \mathbb{C}^d)$ is called a vector-valued nonuniform multiresolution analysis (VNUMRA) on positive half-line if the following conditions are satisfied:

- (1) $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$,
- (2) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^+, \mathbb{C}^d)$,
- (3) $\bigcap_{j \in \mathbb{Z}} V_j = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2(\mathbb{R}^+, \mathbb{C}^d)$,
- (4) $\Phi(x) \in V_j$ if and only if $\Phi(Nx) \in V_{j+1}$.
- (5) There exists a function $\Phi(x)$ in V_0 such that $\{\Phi(x \ominus \lambda) : \lambda \in \Lambda_{r,N}^+\}$ is an orthonormal basis for V_0 .

The vector-valued function $\Phi(x)$ whose existence is asserted in (5) is called a scaling function in $L^2(\mathbb{R}^+, \mathbb{C}^d)$.

Since $\Phi(x) = (\phi_1(x), \phi_2(x), \dots, \phi_d(x))^T \in L^2(\mathbb{R}^+, \mathbb{C}^d)$, satisfy the following refinement equation

$$\Phi(x) = \sum_{\lambda \in \Lambda_{r,N}^+} H_\lambda \Phi(Nx \ominus \lambda), \tag{3.1}$$

where $\{H_\lambda\}_{\lambda \in \Lambda_{r,N}^+}$ is $d \times d$ constant matrix sequence that has only finite number of terms.

Define a closed subspace $V_j \subset L^2(\mathbb{R}^+, \mathbb{C}^d)$ by

$$V_j = \text{clos}_{L^2(\mathbb{R}^+, \mathbb{C}^d)}(\text{span}\{\Phi(N^j x \ominus \lambda) : \lambda \in \Lambda_{r,N}^+\}), \quad j \in \mathbb{Z}. \tag{3.2}$$

Given a VNUMRA on positive half-line, we denote W_m the orthogonal complement of V_m in V_{m+1} . Then

$$L^2(\mathbb{R}^+, \mathbb{C}^d) = \bigoplus_{m \in \mathbb{Z}} W_m$$

Definition 3.2. A collection of $N - 1$ vector-valued functions $\Psi_k(t) \in V_1, k = 1, 2, \dots, N - 1$ is called a vector-valued wavelet system associated with a given VNUMRA on positive half-line if the family of functions $\{\Psi_k(x \ominus \lambda)\}_{k=1,2,\dots,N-1, \lambda \in \Lambda_{r,N}^+}$ form an orthonormal basis of W_0 .

Taking the Walsh-Fourier transform on both side of Equation (3.1), we have

$$\hat{\Phi}(N\eta) = H(\eta)\hat{\Phi}(\eta), \quad \eta \in \mathbb{R}^+, \tag{3.3}$$

where

$$H(\eta) = \frac{1}{N} \sum_{\lambda \in \Lambda_{r,N}^+} H_\lambda \overline{\chi(\lambda, \eta)} \tag{3.4}$$

and

$$\chi(\lambda, \eta) = \exp\left(\frac{2\pi i}{N} \sum_{j=1}^{\infty} (\lambda_j \xi_{-j} + \lambda_{-j} \xi_j)\right)$$

where λ_j and ξ_j are calculated by (2.1).

Since $\Lambda_{r,N}^+ = \{0, r/N\} + \mathbb{Z}^+$, we can write that

$$H(\eta) = H_\lambda^1 + \chi\left(\frac{r}{N}, \eta\right) \overline{H_\lambda^2} \tag{3.5}$$

where $\{H_\lambda^1\}$ and $\{H_\lambda^2\}$ are $d \times d$ constant symmetric matrix sequences. Then

$$\begin{aligned} \hat{\Phi}(\eta) &= H\left(\frac{\eta}{N}\right) \hat{\Phi}\left(\frac{\eta}{N}\right) \\ &= H\left(\frac{\eta}{N}\right) H\left(\frac{\eta}{N^2}\right) H\left(\frac{\eta}{N^3}\right) \dots \hat{\Phi}(0) \\ &= \prod_{k=1}^{\infty} H\left(\frac{\eta}{N^k}\right) \hat{\Phi}(0) \end{aligned} \tag{3.6}$$

Eq. (3.6) implies that

$$H(0) = I_d \quad \text{or} \quad \sum_{\lambda \in \Lambda_{r,N}^+} H_\lambda = I_d \tag{3.7}$$

where I_d , denotes the identity matrix of order $d \times d$

$$W_j = \text{clos}_{L^2(\mathbb{R}^+, \mathbb{C}^d)}(\text{span}\{\Psi_k(N^j x \ominus \lambda) : \lambda \in \Lambda_{r,N}^+, k = 1, 2, \dots, N - 1\}), \quad j \in \mathbb{Z}.$$

Since $\Psi_k(x) \in W_0 \subset V_1$, for each k , there exists a uniquely supported sequence $\{G_{\lambda,k}\}_{\lambda \in \Lambda_{r,N}^+, k=1,2,\dots,N-1}$ of $d \times d$ constant matrices such that

$$\Psi_k(x) = \sum_{\lambda \in \Lambda_{r,N}^+} G_{\lambda,k} \Phi(Nx \ominus \lambda) \tag{3.8}$$

On taking Walsh-Fourier transform on both sides of Eq. (3.8), we have

$$\tilde{\Psi}_k(N\eta) = G_k(\eta)\tilde{\Phi}(\eta),$$

where

$$G_k(\eta) = \frac{1}{N} \sum_{\lambda \in \Lambda_{r,N}^+} G_{\lambda,k} \overline{\chi(\lambda, \eta)} \tag{3.9}$$

and

$$\chi(\lambda, \eta) = \exp\left(\frac{2\pi i}{N} \sum_{j=1}^{\infty} (\lambda_j \xi_{-j} + \lambda_{-j} \xi_j)\right) \tag{3.10}$$

where λ_j and ξ_j are given by (2.1) for a positive integer N .

Since $\Lambda_{r,N}^+ = \{0, r/N\} + \mathbb{Z}^+$, we can write that

$$G_k(\eta) = G_{\lambda,k}^1 + \chi\left(\frac{r}{N}, \eta\right) G_{\lambda,k}^2, \quad k = 0, 1, \dots, N - 1, \tag{3.11}$$

where $\{G_{\lambda,k}^1\}$ and $\{G_{\lambda,k}^2\}$ are $d \times d$ constant symmetric matrix sequences.

Lemma 3.3. If $\Phi(t) \in L^2(\mathbb{R}^+, \mathbb{C}^d)$ defined by Eq. (3.1) is an orthogonal vector-valued scaling function then we have

$$\sum_{m \in \mathbb{Z}^+} H_m H_{N(\lambda \ominus \lambda') \oplus m}^* = N \delta_{\lambda, \lambda'} I_d, \quad \forall \lambda, \lambda' \in \Lambda_{r,N}^+, \tag{3.12}$$

where $\delta_{\lambda, \lambda'}$ denotes the Kronecker's delta.

Proof. Since $\Phi(t) \in L^2(\mathbb{R}^+, \mathbb{C}^d)$ defined by Eq. (3.1) are orthogonal vector-valued scaling functions, therefore for $\lambda, \lambda' \in \Lambda_{r,N}^+$, we have

$$\begin{aligned} \delta_{\lambda, \lambda'} I_d &= \int_{\mathbb{R}^+} \Phi(x \ominus \lambda) \Phi^*(x \ominus \lambda') dx \\ &= \sum_{u \in \Lambda_{r,N}^+} \int_{\mathbb{R}^+} H_u \Phi(Nx \ominus N\lambda \ominus u) \sum_{v \in \Lambda_{r,N}^+} \Phi^*(Nx \ominus N\lambda' \ominus v) H_v^* dx \\ &= \sum_{u \in \Lambda_{r,N}^+} \sum_{v \in \Lambda_{r,N}^+} H_u \left(\int_{\mathbb{R}^+} \Phi(Nx \ominus N\lambda \ominus u) \Phi^*(Nx \ominus N\lambda' \ominus v) dx \right) H_v^* \\ &= \frac{1}{N} \sum_{u \in \Lambda_{r,N}^+} \sum_{v \in \Lambda_{r,N}^+} H_u \left(\int_{\mathbb{R}^+} \Phi(x \ominus N\lambda \ominus u) \Phi^*(x \ominus N\lambda' \ominus v) dx \right) H_v^* \end{aligned}$$

Taking $u = m, v = n$, where $m, n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
 \delta_{\lambda, \lambda'} I_d &= \int_{\mathbb{R}^+} \Phi(x \ominus \lambda) \Phi^*(x \ominus \lambda') dx \\
 &= \frac{1}{N} \sum_{u \in \Lambda_{r,N}^+} \sum_{v \in \Lambda_{r,N}^+} H_u \langle \Phi(x \ominus N\lambda \ominus u), \Phi(x \ominus N\lambda' \ominus v) \rangle H_v^* \\
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} H_m \langle \Phi(x \ominus N\lambda \ominus m), \Phi(x \ominus N\lambda' \ominus n) \rangle H_n^* \\
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^+} H_m H_{N(\lambda \ominus \lambda') \oplus m}^*.
 \end{aligned}$$

Therefore,

$$\sum_{m \in \mathbb{Z}^+} H_m H_{N(\lambda \ominus \lambda') \oplus m}^* = N \delta_{\lambda, \lambda'} I_d, \quad \forall \lambda, \lambda' \in \Lambda_{r,N}^+.$$

Taking $u = \frac{r}{N} + m, v = n$, where $m, n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
 \delta_{\lambda, \lambda'} I_d &= \frac{1}{N} \sum_{u \in \Lambda_{r,N}^+} \sum_{v \in \Lambda_{r,N}^+} H_u \langle \Phi(x \ominus N\lambda \ominus u), \Phi(x \ominus N\lambda' \ominus v) \rangle H_v^* \\
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} H_{\frac{r}{N} + m} \langle \Phi(x \ominus N\lambda \ominus (\frac{r}{N} + m)), \Phi(x \ominus N\lambda' \ominus n) \rangle H_n^* \\
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^+} H_m H_{N(\lambda \ominus \lambda') \oplus m}^*.
 \end{aligned}$$

Therefore in both the cases, we get

$$\sum_{m \in \mathbb{Z}^+} H_m H_{N(\lambda \ominus \lambda') \oplus m}^* = N \delta_{\lambda, \lambda'} I_d, \quad \forall \lambda, \lambda' \in \Lambda_{r,N}^+.$$

Lemma 3.4. Let $\Psi_0 = \Phi, G_0 = H$ and suppose that $\Psi_k(t)$, where $k \in \mathcal{D}_1 = \{1, 2, \dots, N - 1\}$ be the vector-valued functions in V_1 . Then the family of functions $\{\Psi_k(x \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1}$ will form an orthonormal system for V_1 if and only if for $k, l \in \{0, 1, 2, \dots, N - 1\}$

$$\sum_{p=0}^{N-1} G_k \left(\frac{\xi + p}{N} \right) G_l^* \left(\frac{\xi + p}{N} \right) = \delta_{k,l} I_d. \tag{3.13}$$

Proof. First, we will prove the necessary condition.

By orthonormality of $\Psi_k(x) \in L^2(\mathbb{R}^+, \mathbb{C}^d), k = 0, 1, 2, \dots, N - 1$, we have

$$\langle \Psi_k(x), \Psi_l^*(x \ominus \lambda) \rangle = \int_{\mathbb{R}^+} \Psi_k(x) \Psi_l^*(x \ominus \lambda) dt = \delta_{k,l} \delta_{0,\lambda} I_d$$

where $\lambda \in \Lambda_{r,N}^+$ and $k, l \in \{0, 1, 2, \dots, N - 1\}$.

Equivalently, in the frequency domain, we have

$$\delta_{k,l} \delta_{0,\lambda} I_d = \int_{\mathbb{R}^+} \tilde{\Psi}_k(\xi) \tilde{\Psi}_l^*(\xi) \overline{\chi(\lambda, \xi)} d\xi.$$

Define

$$F_{k,l}(\xi) = \sum_{j \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + Nj) \tilde{\Psi}_l^*(\xi + Nj).$$

On taking $\lambda = n$, where $n \in \mathbb{Z}^+$, $0 \leq k, l \leq N - 1$, we have

$$\begin{aligned} \delta_{k,l} \delta_{0,n} I_d &= \int_{\mathbb{R}^+} \tilde{\Psi}_k(\xi) \tilde{\Psi}_l^*(\xi) \overline{\chi(n, \xi)} d\xi \\ &= \int_{[0,N)} \sum_{j \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + Nj) \tilde{\Psi}_l^*(\xi + Nj) \overline{\chi(n, \xi)} d\xi \\ &= \int_{[0,N)} F_{k,l}(\xi) \overline{\chi(n, \xi)} d\xi \\ &= \int_{[0,1)} \left[\sum_{p=0}^{N-1} F_{k,l}(\xi + p) \right] \overline{\chi(n, \xi)} d\xi. \end{aligned}$$

Therefore, we obtain

$$\sum_{p=0}^{N-1} F_{k,l}(\xi + p) = \delta_{k,l} I_d. \tag{3.14}$$

On taking $\lambda = \frac{r}{N} + n$, where $n \in \mathbb{Z}^+$, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^+} \tilde{\Psi}_k(\xi) \tilde{\Psi}_l^*(\xi) \overline{\chi\left(\frac{r}{N} + n, \xi\right)} d\xi \\ &= \int_{[0,N)} \sum_{j \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + Nj) \tilde{\Psi}_l^*(\xi + Nj) \overline{\chi(n, \xi)} \overline{\chi\left(\frac{r}{N}, \xi\right)} d\xi \\ &= \int_{[0,N)} F_{k,l}(\xi) \overline{\chi(n, \xi)} \overline{\chi\left(\frac{r}{N}, \xi\right)} d\xi \\ &= \int_{[0,1)} \left[\sum_{p=0}^{N-1} \overline{\chi\left(\frac{r}{N}, p\right)} F_{k,l}(\xi + p) \right] \overline{\chi(n, \xi)} \overline{\chi\left(\frac{r}{N}, \xi\right)} d\xi. \end{aligned}$$

We conclude that

$$\sum_{p=0}^{N-1} \overline{\chi\left(\frac{r}{N}, p\right)} F_{k,l}(\xi + p) = 0$$

i.e.

$$\sum_{p=0}^{N-1} \alpha^p F_{k,l}(\xi + p) = 0, \tag{3.15}$$

where $\alpha = \overline{\chi\left(\frac{r}{N}, 1\right)}$, since $\overline{\chi\left(\frac{r}{N}, p\right)} = \left[\overline{\chi\left(\frac{r}{N}, 1\right)}\right]^p$ for $p = 0, 1, 2, \dots, N - 1$.

Also, we have

$$\sum_{p=0}^{N-1} F_{k,l}(\xi + p) = \sum_{p \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + p) \tilde{\Psi}_l^*(\xi + p).$$

Therefore, Eq.(3.14) reduces to

$$\sum_{p \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + p)\tilde{\Psi}_l^*(\xi + p) = \delta_{k,l}I_d. \tag{3.16}$$

Also,

$$\begin{aligned} F_{k,l}(N\xi) &= \sum_{j \in \mathbb{Z}^+} \tilde{\Psi}_k(N\xi + Nj)\tilde{\Psi}_l^*(N\xi + Nj) \\ &= \sum_{j \in \mathbb{Z}^+} G_k(\xi + j)\tilde{\Phi}(\xi + j)\tilde{\Phi}^*(\xi + j)G_l^*(\xi + j) \\ &= \sum_{j=nN} G_k(\xi + nN)\tilde{\Phi}(\xi + nN)\tilde{\Phi}^*(\xi + nN)G_l^*(\xi + nN) \\ &\quad + \sum_{j=nN+1} G_k(\xi + nN + 1)\tilde{\Phi}(\xi + nN + 1)\tilde{\Phi}^*(\xi + nN + 1) \\ &\quad \times G_l^*(\xi + nN + 1) + \sum_{j=nN+2} G_k(\xi + nN + 2)\tilde{\Phi}(\xi + nN + 2) \\ &\quad \times \tilde{\Phi}^*(\xi + nN + 2)G_l^*(\xi + nN + 2) + \dots + \\ &\quad \sum_{j=nN+(N-1)} G_k(\xi + nN + (N - 1))\tilde{\Phi}(\xi + nN + (N - 1)) \\ &\quad \times \tilde{\Phi}^*(\xi + nN + (N - 1))G_l^*(\xi + nN + (N - 1)) \\ &= G_k(\xi) \left[\sum_{j=nN} \tilde{\Phi}(\xi + nN)\tilde{\Phi}^*(\xi + nN) \right] G_l^*(\xi) + G_k(\xi + 1) \\ &\quad \times \left[\sum_{j=nN+1} \tilde{\Phi}(\xi + nN + 1)\tilde{\Phi}^*(\xi + nN + 1) \right] G_l^*(\xi + 1) \\ &\quad + G_k(\xi + 2) \left[\sum_{j=nN+2} \tilde{\Phi}(\xi + nN + 2)\tilde{\Phi}^*(\xi + nN + 2) \right] \\ &\quad \times G_l^*(\xi + 2) + \dots + G_k(\xi + (N - 1)) \\ &\quad \times \left[\sum_{j=nN+(N-1)} \tilde{\Phi}(\xi + nN + (N - 1))\tilde{\Phi}^*(\xi + nN + (N - 1)) \right] \\ &\quad \times G_l^*(\xi + (N - 1)) \\ &= G_k(\xi)G_l^*(\xi) + G_k(\xi + 1)G_l^*(\xi + 1) + G_k(\xi + 2)G_l^*(\xi + 2) + \dots \\ &\quad + G_k(\xi + (N - 1))G_l^*(\xi + (N - 1)) \\ &= \sum_{j=0}^{N-1} G_k(\xi + j)G_l^*(\xi + j). \end{aligned}$$

Therefore, we have

$$\sum_{p \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi + p)\tilde{\Psi}_l^*(\xi + p) = \sum_{p=0}^{N-1} G_k\left(\frac{\xi + p}{N}\right)G_l^*\left(\frac{\xi + p}{N}\right).$$

By using (3.16), we conclude that

$$\sum_{p=0}^{N-1} G_k \left(\frac{\xi+p}{N} \right) G_l^* \left(\frac{\xi+p}{N} \right) = \delta_{k,l} I_d,$$

i.e.

$$G_k \left(\frac{\xi}{N} \right) G_l^* \left(\frac{\xi}{N} \right) + G_k \left(\frac{\xi+1}{N} \right) G_l^* \left(\frac{\xi+1}{N} \right) + G_k \left(\frac{\xi+2}{N} \right) G_l^* \left(\frac{\xi+2}{N} \right) + \dots + G_k \left(\frac{\xi+N-1}{N} \right) G_l^* \left(\frac{\xi+N-1}{N} \right) = \delta_{k,l} I_d.$$

Now, we will prove the sufficiency.

By Eq.(3.9), we have

$$\begin{aligned} & \sum_{j \in \mathbb{Z}^+} \tilde{\Psi}_k(\xi+j) \tilde{\Psi}_l^*(\xi+j) \\ &= \sum_{j \in \mathbb{Z}^+} G_k \left(\frac{\xi+j}{N} \right) \tilde{\Phi} \left(\frac{\xi+j}{N} \right) \tilde{\Phi}^* \left(\frac{\xi+j}{N} \right) G_l^* \left(\frac{\xi+j}{N} \right) \\ &= G_k \left(\frac{\xi}{N} + n \right) \left[\sum_{j=nN} \tilde{\Phi} \left(\frac{\xi}{N} + n \right) \tilde{\Phi}^* \left(\frac{\xi}{N} + n \right) \right] \\ & \quad \times G_l^* \left(\frac{\xi}{N} + n \right) + G_k \left(\frac{\xi}{N} + n + \frac{1}{N} \right) \\ & \quad \times \left[\sum_{j=nN+1} \tilde{\Phi} \left(\frac{\xi}{N} + n + \frac{1}{N} \right) \tilde{\Phi}^* \left(\frac{\xi}{N} + n + \frac{1}{N} \right) \right] G_l^* \left(\frac{\xi}{N} + n + \frac{1}{N} \right) \\ & \quad + \dots + G_k \left(\frac{\xi}{N} + n + \frac{N-1}{N} \right) \left[\sum_{j=nN+(N-1)} \tilde{\Phi} \left(\frac{\xi}{N} + n + \frac{N-1}{N} \right) \right. \\ & \quad \left. \times \tilde{\Phi}^* \left(\frac{\xi}{N} + n + \frac{N-1}{N} \right) \right] G_l^* \left(\frac{\xi}{N} + n + \frac{N-1}{N} \right) \\ &= G_k \left(\frac{\xi}{N} \right) G_l^* \left(\frac{\xi}{N} \right) + G_k \left(\frac{\xi+1}{N} \right) G_l^* \left(\frac{\xi+1}{N} \right) + G_k \left(\frac{\xi+2}{N} \right) G_l^* \left(\frac{\xi+2}{N} \right) \\ & \quad + \dots + G_k \left(\frac{\xi+N-1}{N} \right) G_l^* \left(\frac{\xi+N-1}{N} \right) \\ &= \delta_{k,l} I_d. \end{aligned}$$

It proves the orthonormality of the system $\{\Psi_k(x \ominus \lambda) : \lambda \in \Lambda_{r,N}^+, k = 0, 1, 2, \dots, N-1\}$.

Now, we have the following result on the existence of a vector-valued wavelet function

Theorem 3.5. Suppose $\{\Psi_k(x \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1}$ is an orthonormal system in V_1 . Then this system is complete in $W_0 \equiv V_1 \ominus V_0$.

Proof. Since the system $\{\Psi_k(x \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1}$ is an orthonormal system in V_1 . By Lemma 3.4, we have

$$G_k \left(\frac{\xi}{N} \right) G_l^* \left(\frac{\xi}{N} \right) + G_k \left(\frac{\xi+1}{N} \right) G_l^* \left(\frac{\xi+1}{N} \right) + G_k \left(\frac{\xi+2}{N} \right) G_l^* \left(\frac{\xi+2}{N} \right) + \dots + G_k \left(\frac{\xi+N-1}{N} \right) G_l^* \left(\frac{\xi+N-1}{N} \right) = \delta_{k,l} I_d.$$

$$G_l^* \left(\frac{\xi + 2}{N} \right) + \dots + G_k \left(\frac{\xi + (N - 1)}{N} \right) G_l^* \left(\frac{\xi + (N - 1)}{N} \right) = \delta_{k,l} I_d.$$

We will now prove its completeness.

For $\mathbf{f}_k \in W_0$, there exists constant matrices $\{P_{\lambda,k}\}$ such that

$$\mathbf{f}_k(t) = \sum_{\lambda \in \Lambda_{r,N}^+} P_{\lambda,k} \Phi(Nt \ominus \lambda), k = 0, 1, 2, \dots, N - 1.$$

Thus

$$\tilde{\mathbf{f}}_k(\xi) = P_k \left(\frac{\xi}{N} \right) \tilde{\Phi} \left(\frac{\xi}{N} \right) \tag{3.17}$$

where

$$P_k \xi = \frac{1}{N} \sum_{\lambda \in \Lambda_{r,N}^+} P_{\lambda,k} \overline{\chi(\lambda, \xi)}.$$

On the other hand $\mathbf{f}_k \notin V_0$ and $\mathbf{f}_k \in W_0$ implies

$$\int_{\mathbb{R}^+} \mathbf{f}_k(t) \Phi^*(t \ominus \lambda) dt = 0, \quad \lambda \in \Lambda_{r,N}^+.$$

This condition is equivalent to

$$\sum_{j \in \mathbb{Z}^+} \tilde{\mathbf{f}}_k(\xi + j) \tilde{\Phi}^*(\xi + j) = 0, \quad \xi \in \mathbb{R}^+.$$

Therefore, from Eqs. (3.3) and (3.17), we have

$$\sum_{j \in \mathbb{Z}^+} P_k \left(\frac{\xi + j}{N} \right) \tilde{\Phi} \left(\frac{\xi + j}{N} \right) \tilde{\Phi}^* \left(\frac{\xi + j}{N} \right) H^* \left(\frac{\xi + j}{N} \right) = 0, \quad \xi \in \mathbb{R}^+.$$

As similar to the identity (3.13) in Lemma 3.4, we have

$$\begin{aligned} &P_k \left(\frac{\xi}{N} \right) H^* \left(\frac{\xi}{N} \right) + P_k \left(\frac{\xi + 1}{N} \right) H^* \left(\frac{\xi + 1}{N} \right) + \dots + \\ &P_k \left(\frac{\xi + (N - 1)}{N} \right) H^* \left(\frac{\xi + (N - 1)}{N} \right) = 0. \end{aligned} \tag{3.18}$$

Let

$$\begin{aligned} P_{k'} \left(\frac{\xi}{N} \right) &= \left(P_k \left(\frac{\xi}{N} \right), P_k \left(\frac{\xi + 1}{N} \right), \dots, P_k \left(\frac{\xi + (N - 1)}{N} \right) \right)^*, \\ H' \left(\frac{\xi}{N} \right) &= \left(H \left(\frac{\xi}{N} \right), H \left(\frac{\xi + 1}{N} \right), \dots, H \left(\frac{\xi + (N - 1)}{N} \right) \right)^*, \\ G_{k'} \left(\frac{\xi}{N} \right) &= \left(G_k \left(\frac{\xi}{N} \right), G_k \left(\frac{\xi + 1}{N} \right), \dots, G_k \left(\frac{\xi + (N - 1)}{N} \right) \right)^*. \end{aligned}$$

Then, the identity (3.13) implies that for any $\xi \in \mathbb{R}^+$, the column vectors in the $Nd \times d$ matrix $H'(\xi)$ and the column vectors in the $Nd \times d$ matrix $G'_k(\xi)$ are orthogonal for $l = 0, 1, 2, \dots, N - 1$ and these vectors form orthogonal basis of Nd -dimensional complex Euclidian space \mathbb{C}^{Nd} .

The identity (3.18) implies that the column vectors in $Nd \times d$ matrix $P'_k(\xi)$ and the column vectors in $Nd \times d$ matrix $H'(\xi)$ are orthogonal. Thus \exists an $d \times d$ matrix $Q_k(\xi)$ such that

$$P_k(\xi) = Q_k(\xi)G_k(\xi), \quad \xi \in \mathbb{R}^+, \quad k = 0, 1, 2, \dots, N - 1.$$

Therefore, from Eqs. (3.9) and (3.17), we have

$$\begin{aligned} \hat{\mathbf{f}}_k(\xi) &= P_k\left(\frac{\xi}{N}\right)\tilde{\Phi}\left(\frac{\xi}{N}\right) \\ &= P_k\left(\frac{\xi}{N}\right)G_k\left(\frac{\xi}{N}\right)\tilde{\Phi}\left(\frac{\xi}{N}\right) \\ &= Q_k\left(\frac{\xi}{N}\right)\tilde{\Psi}_k(\xi). \end{aligned}$$

By using the orthonormality of $\{\Psi_k(x \ominus \lambda) : \lambda \in \Lambda_{r,N}^+\}$, we have

$$\int_{\mathbb{R}^+} \tilde{\mathbf{f}}_k(N\xi)\tilde{\mathbf{f}}_k^*(N\xi)d\xi = \int_{\mathbb{R}^+} Q_k(\xi)\tilde{\Psi}_k(N\xi)\tilde{\Psi}_k^*(N\xi)Q_k^*(\xi)d\xi.$$

Therefore, we have

$$\int_{\mathbb{R}^+} \tilde{\mathbf{f}}_k(N\xi)\tilde{\mathbf{f}}_k^*(N\xi)d\xi = \int_0^1 Q_k(\xi)Q_k^*(\xi)d\xi.$$

This proves that $P_k(\xi)$ has Fourier series expansion and let the constant $d \times d$ matrices $\{A_{\lambda,k}\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1}$ be its Fourier coefficients. Therefore,

$$\mathbf{f}_k(x) = \sum_{\lambda \in \Lambda_{r,N}^+} A_{\lambda,k}\Psi_k(x \ominus \lambda).$$

This proves the completeness of $\{\Psi_k(x \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1}$ in W_0 .

If $\Psi_0, \Psi_1, \Psi_2, \dots, \Psi_{N-1} \in V_1$ are as in Theorem 3.5, one can obtain from them as orthonormal basis for $L^2(\mathbb{R}^+, \mathbb{C}^d)$ by procedure for construction of wavelet from given MRA. It can be easily checked that for every $j \in \mathbb{Z}$ the collection

$$\left\{N^{\frac{m}{2}}\Psi_k(N^m t \ominus \lambda)\right\}_{\lambda \in \Lambda_{r,N}^+, k=0,1,2,\dots,N-1},$$

is a complete orthogonal system for V_{m+1} .

For every $m \in \mathbb{Z}$, define W_m the orthogonal complement of V_m in V_{m+1} . Then we have

$$V_{m+1} = V_m \oplus W_m \text{ and } W_k \perp W_l \text{ if } k \neq l$$

and

$$L^2(\mathbb{R}^+, \mathbb{C}^d) = \bigoplus_{m \in \mathbb{Z}} W_m$$

a decomposition of $L^2(\mathbb{R}^+, \mathbb{C}^d)$ into mutually orthogonal subspaces.

The collection $\{N^{\frac{m}{2}}\Psi_k(N^m t \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+, m \in \mathbb{Z}, k=0,1,2,\dots,N-1}$ forms a complete orthonormal system for $L^2(\mathbb{R}^+, \mathbb{C}^d)$.

4. Conclusion

In this study, we introduced a new concept called vector-valued nonuniform multiresolution analysis on positive half-line related to Walsh functions where associated subspace V_0 of $L^2(\mathbb{R}^+, \mathbb{C}^d)$ has, an orthonormal basis, a collection of translates of vector-valued function Φ of the form $\{\Phi(x \ominus \lambda)\}_{\lambda \in \Lambda_{r,N}^+}$. We Also obtained the necessary and sufficient condition for the existence of associated wavelets. Our results will be mostly used by that part of mathematical society who works in wavelet analysis and their applications.

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