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Weighted Cumulative Residual (Past) Inaccuracy For Minimum (Maximum) of Order Statistics

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Abstract In this paper, we propose a measure of weighted cumulative residual inaccuracy between survival function of the first-order statistic and parent survival function $\bar{F}$. We also consider weighted cumulative inaccuracy measure between distribution of the last- order statistic and parent distribution F. For these concepts, we obtain some reliability properties and characterization results such as relationships with other functions, bounds, stochastic ordering and effect of linear transformation. Dynamic versions of these weighted measures are considered.

Keywords Cumulative inaccuracy, Order statistics, Empirical approach.

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1. Introduction

Let X denote the lifetime of a device or a system with probability density function (pdf) f and cumulative distribution function (cdf) F, respectively. Then, the differential entropy known as Shannon entropy, is defined by Shannon [18] as follows:

$$H(X) = -\int_0^{+\infty} f(x) \log f(x) dx, \tag{1}$$

where, by convention, $0 \log 0 = 0$. Di Crescenzo and Longobardi [6] introduced the concept of weighted differential entropy which is given by

$$H^w(X) = -\int_0^{+\infty} x f(x) \log f(x) dx, \tag{2}$$

Recently, new measures of information are proposed in literatures: replacing the pdf by the survival function $\bar{F} = 1 - F$ in Shannon entropy, the cumulative residual entropy (CRE) is defined by Rao et al. [16] as

$$\mathcal{E}(X) = \int_0^{+\infty} \bar{F}(x) \Lambda(x) dx,$$

where $\Lambda(x) = -\log \bar{F}(x)$. Properties of the CRE can be found in [13], [15], and [22]. A new information measure similar to CRE has been proposed by Di Crescenzo and Longobardi [7] as follows:

$$\mathcal{CE}(X) = \int_0^{+\infty} F(x) \tilde{\Lambda}(x) dx, \tag{3}$$

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where $\tilde{\Lambda}(x) = -\log F(x)$. Analogous to (2), Misagh et al. [12] proposed weighted cumulative residual entropy (WCRE) as

$$\mathcal{E}^w(X) = \int_0^{+\infty} x\bar{F}(x)\Lambda(x)dx. \tag{4}$$

Similarly, Misagh et al. [12] proposed weighted cumulative entropy (WCE) as

$$\mathcal{CE}^w(X) = \int_0^{+\infty} xF(x)\tilde{\Lambda}(x)dx. \tag{5}$$

Now, suppose that X and Y are two non-negative random variables with reliability functions $\bar{F}(x)$, $\bar{G}(x)$, respectively. If $\bar{F}(x)$ is the actual survival function corresponding to the observations and $\bar{G}(x)$ is the survival function assigned by the experimenter, then Kumar and Taneja [11] defined the cumulative residual inaccuracy (CRI) based on $\bar{F}(x)$ and $\bar{G}(x)$ as follows:

$$I(\bar{F}, \bar{G}) = -\int_0^{+\infty} \bar{F}(x)\log\bar{G}(x)dx. \tag{6}$$

In analogy with (6), a measure of cumulative past inaccuracy (CPI) associated with F and G is given by

$$\tilde{I}(F, G) = -\int_0^{+\infty} F(x)\log G(x)dx. \tag{7}$$

Order statistics play an important role in problems such as industrial stress testing, meteorological analysis, hydrology, economics and other related fields. Different order statistics can be used in different applications; for example, the maximum is of interest in the study of floods and other meteorological phenomena while the minimum is often used in reliability and survival analysis, etc. For more details about order statistics and their applications, one may refer to [1]. Let $X_1, X_2, ..., X_n$ be a random sample of size n from an absolutely continuous cumulative distribution function $F(x)$. If $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ represent the order statistics of the sample $X_1, X_2, ..., X_n$. Then the empirical measure of $F(x)$ is defined as

$$\hat{F}_n(x) = \begin{cases} 0, & x < X_{(1)}, \\ \frac{k}{n}, & X_{(k)} \leq x \leq X_{(k+1)}, \quad k = 1, 2, ..., n-1 \\ 1, & x > X_{(k+1)}. \end{cases}$$

Recently Thapliyal and Taneja [21] have introduced the measure of residual inaccuracy of order statistics and proved a characterization result for it. Tahmasebi and Daneshi [19] and Tahmasebi et al. [20] have obtained some results of inaccuracy measures in record values. Eskandarzadeh et al. [5] have discussed the cumulative measure of inaccuracy in k-lower record values and studied characterization results of dynamic cumulative inaccuracy. Daneshi et al. [4] have proposed a weighted cumulative past (residual) inaccuracy of record values and studied its characterization results. The paper is organized as follows: In Section 2, we consider a measure of weighted cumulative residual inaccuracy (WCRI) between $\bar{F}_{X_{(1:n)}}$ and $\bar{F}$ and study its properties. In Section 3, we also propose the weighted cumulative past inaccuracy (WPCI) between $F_{X_{(n:n)}}$ and F and obtain an estimator of cumulative inaccuracy using empirical approach .

2. WCRI For Minimum of Order Statistics

In this section, we propose the WCRI between $\bar{F}_{X_{(1:n)}}$ and $\bar{F}$ as follows:

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = -\int_0^{+\infty} x\bar{F}_{X_{(1:n)}}(x)\Lambda(x)dx = \frac{1}{n}\mathcal{E}^w(X_{(1:n)}), \tag{8}$$

where

$$\mathcal{E}^w(X_{(1:n)}) = n\int_0^{+\infty} x[\bar{F}(x)]^n\Lambda(x)dx.$$

Hereafter we present some properties of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$.

Proposition 1
Let X be an absolutely continuous non-negative random variable with $I(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for $n \geq 1$. Then, we have

$$\begin{aligned} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) &= \int_0^{+\infty} x[\bar{F}(x)]^n \left(\int_0^x \frac{f(z)}{\bar{F}(z)}dz\right)dx \\ &= \int_0^{+\infty}\int_z^{+\infty} \lambda(z)x[\bar{F}(x)]^n dxdz, \end{aligned} \tag{9}$$

where $\lambda(.)$ is the failure rate function.

Proposition 2
Let X be an absolutely continuous non-negative random variable with $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for $n \geq 1$. Then, we have

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = E\left(M^w_{(1:n)}(Z)(\bar{F}(Z))^{n-1}\right), \tag{10}$$

where

$$M^w_{(1:n)}(z) = \frac{1}{(\bar{F}(z))^n}\int_z^{+\infty} x(\bar{F}(x))^n dx$$

is the weighted mean residual lifetime (wmrl) of $X_{(1:n)}$.

Proof
From (9), we have

$$\begin{aligned} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) &= \int_0^{+\infty}\int_z^{+\infty} \lambda(z)x[\bar{F}(x)]^n dxdz \\ &= \int_0^{+\infty} \frac{f(z)}{\bar{F}(z)}dz\left[\int_z^{+\infty} x[\bar{F}(x)]^n dx\right] \\ &= \int_0^{+\infty} \frac{f(z)}{\bar{F}(z)}[\bar{F}(z)]^n M^w_{(1:n)}(z)dz = \int_0^{+\infty} f(z)[\bar{F}(z)]^{n-1} M^w_{(1:n)}(z)dz. \end{aligned} \tag{11}$$

So, the proof is complete. □

Proposition 3
Let $a, b > 0$. For $n = 1, 2, ...$ it holds that

$$I^w(\bar{F}_{aX_{(1:n)}+b}, \bar{F}_{aX+b}) = aI^w(\bar{F}_{X_{(1:n)}}, \bar{F}).$$

The next propositions give some lower and upper bounds for $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$.

Proposition 4
For a non-negative random variable X and $n \geq 1$, it holds that

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \geq M^w_{(1:n)}(t)|\log \bar{F}(t)|[\bar{F}(t)]^n, \tag{12}$$

where $M^w_{(1:n)}(t)$ is the wmrl of $X_{(1:n)}$.

Proof
The proof follows from Baratpour [2].

□

Proposition 5
Let X be an absolutely continuous non-negative random variable with $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for $n \geq 1$. Then, we have

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \geq \int_0^{+\infty} x(\bar{F}(x))^n F(x)dx. \tag{13}$$

Proof
Recalling that $-\log \bar{F}(x) \geq F(x)$, the proof then finally follows. □

Proposition 6
Let X be an absolutely continuous non-negative random variable with $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for $n \geq 1$. Then, we have

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \leq \mathcal{E}^w(X). \tag{14}$$

Proof
Since $\bar{F}(x) \geq [\bar{F}(x)]^n$, $x \geq 0$, when $n \geq 1$, the proof then finally follows. □

Proposition 7
If X is IFRA (DFRA), then

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \leq (\geq) \mathbb{E}\left(X^2(\bar{F}(X))^{n-1}\right). \tag{15}$$

Proof
Since X is IFRA (DFRA), $\frac{\Lambda(x)}{x}$ is increasing (decreasing) with respect to $x > 0$, which implies that

$$\bar{F}(x)\Lambda(x) \leq (\geq) x f(x), \quad x > 0. \tag{16}$$

By multiplying $x[\bar{F}(x)]^{n-1} \geq 0$ in (16) and then integrating, the result follows. □

Proposition 8
Let X be an absolutely continuous non-negative random variable with $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for $n \geq 1$. Then, we have

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = \mathbb{E}\left[h^w(X)\right], \tag{17}$$

where

$$h^w(x) = \int_0^x z\left[-\log \bar{F}(z)\right][\bar{F}(z)]^{n-1}dz, \quad x \geq 0.$$

Proof
From (8) and using Fubini's theorem, we obtain

$$\begin{aligned}
I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) &= \int_0^\infty z[-\log \bar{F}(z)][\bar{F}(z)]^{n-1}\bar{F}(z)dz \\
&= \int_0^\infty \left[\int_z^\infty f(x)dx\right] z[\bar{F}(z)]^{n-1}[-\log \bar{F}(z)]dz \\
&= \int_0^\infty f(x)\left[\int_0^x z[\bar{F}(z)]^{n-1}[-\log \bar{F}(z)]dz\right]dx = \mathbb{E}\left[h^w(X)\right].
\end{aligned}$$

□

Proposition 9
Let X and Y be two non-negative random variables with reliability functions $\bar{F}(x), \bar{G}(x)$, respectively. If $X \leq^{icx} Y$, then

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \leq I^w(\bar{G}_{Y_{(1:n)}}, \bar{G}).$$

Proof
Since $h^w(.)$ is an increasing convex function for $n \geq 1$, it follows by Shaked and Shanthikumar [17] that $X \leq^{icx} Y$ implies $h^w(X) \leq^{icx} h^w(Y)$. By recalling the definition of increasing convex order and Proposition 8 proof is complete. □

Proposition 10
Let X and Y be two non-negative random variables with survival function $\bar{F}(x)$ and $\bar{G}(x)$, respectively. If $X \leq^{hr} Y$, then for $n = 1, 2, ...$, it holds that

$$\frac{I^w(\bar{F}_{X_{(1:n)}}, \bar{F})}{E(X)} \leq \frac{I^w(\bar{G}_{Y_{(1:n)}}, \bar{G})}{E(Y)}.$$

Proof
By noting that the function $h^w(x) = \int_0^x z[\bar{F}(z)]^{n-1}[-\log \bar{F}(z)]dz$ is an increasing convex function, under the assumption $X \leq^{hr} Y$, it follows by Shaked and Shanthikumar [17],

$$\frac{E\left[h^w(X)\right]}{E(X)} \leq \frac{E\left[h^w(Y)\right]}{E(Y)}.$$

Hence, the proof is completed by recalling (17). □

Proposition 11
(i) Let X be a continuous random variable with survival function $\bar{F}(.)$ that takes values in $[0, b]$, with finite b. Then,

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \leq bI(\bar{F}_{X_{(1:n)}}, \bar{F}).$$

(ii) Let X be a non-negative continuous random variable with survival function $\bar{F}(.)$ that takes values in $[a, \infty)$, with finite $a > 0$. Then,

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \geq aI(\bar{F}_{X_{(1:n)}}, \bar{F}).$$

Assume that X_θ^* denotes a non-negative absolutely continuous random variable with the survival function $\bar{H}_\theta(x) = [\bar{F}(x)]^\theta, x \geq 0$. This model is known as a proportional hazards rate model. We now obtain the weighted cumulative residual measure of inaccuracy between $\bar{H}_{X_{(1:n)}}$ and $\bar{H}$ as follows:

$$\begin{aligned} I^w(\bar{H}_{X_{(1:n)}}, \bar{H}) &= -\int_0^{+\infty} x\bar{H}_{X_{(1:n)}}(x) \log\left(\bar{H}(x)\right) dx \\ &= -\theta \int_0^{+\infty} x(\bar{F}(x))^{n\theta} \log \bar{F}(x) dx. \end{aligned} \tag{18}$$

Proposition 12
If $\theta \geq (\leq) 1$, then for any $n \geq 1$, we have

$$I^w(\bar{H}_{X_{(1:n)}}, \bar{H}) \leq (\geq) \theta I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \leq (\geq) \theta \mathcal{E}^w(X).$$

Proof
Suppose that $\theta \geq (\leq) 1$, then it is clear $[\bar{F}(x)]^\theta \leq (\geq) \bar{F}(x)$, and hence (18) yields

$$I^w(\bar{H}_{X_{(1:n)}}, \bar{H}) \leq (\geq) \theta \mathcal{E}^w(X).$$

□

Theorem 13
$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = 0$, if and only if, X is degenerate.

Proof
Suppose X is degenerate at point a, obviously by definition of degenerate function and definition of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$, we have $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = 0$.
Now, suppose that $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = 0$, i.e.

$$-\int_0^{\infty} x[\bar{F}(x)]^n \log \bar{F}(x) dx = 0. \tag{19}$$

Then, by noting that integrand function of (19) is non-negative, we conclude that $-x[\bar{F}(x)]^n \log \bar{F}(x) = 0$, for almost all $x \in \mathbb{R}^+$. Thus, $\bar{F}(x) = 0$ *or* 1, for almost all $x \in \mathbb{R}^+$. □

Recently, Cali et al. [3] introduced the generalized CPI of order m defined as

$$I_m(F, G) = \frac{1}{m!} \int_0^{+\infty} F(x)[-\log G(x)]^m dx. \tag{20}$$

In analogy with the measure defined in (20), we now introduce the weighted generalized CRI (WGCRI) of order m defined as

$$I_m^w(\bar{F}, \bar{G}) = \frac{1}{m!} \int_0^{+\infty} x\bar{F}(x)[-\log \bar{G}(x)]^m dx. \tag{21}$$

Remark 1
Let X be a non-negative absolutely continuous random variable with cdf F. Then, the WGCRI of order m between $\bar{F}_{X_{(1:n)}}$ and F is

$$\begin{aligned} I_m^w(\bar{F}_{X_{(1:n)}}, \bar{F}) &= \frac{1}{m!} \int_0^{\infty} x[\bar{F}(x)]^n [-\log \bar{F}(x)]^m dx \\ &= \frac{1}{n^m} \mathcal{E}_m^w(X_{(1:n)}), \end{aligned} \tag{22}$$

where

$$\mathcal{E}_m^w(X) = \int_0^{\infty} x \frac{\left[-\log \bar{F}(x)\right]^m}{m!} \bar{F}(x) dx,$$

is a weighted generalized cumulative residual entropy (WGCRE) which introduced by Kayal [9].

Remark 2
In analogy with (8), a measure of WCRI associated with $\bar{F}$ and $\bar{F}_{X_{(1:n)}}$ is given by

$$I^w(\bar{F}, \bar{F}_{X_{(1:n)}}) = -\int_0^{+\infty} x\bar{F}(x) \log\left(\bar{F}_{X_{(1:n)}}(x)\right) dx = n\mathcal{E}^w(X). \tag{23}$$

In the remainder of this section, we study dynamic version of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$. Let X be the lifetime of a system under condition that the system has survived up to age t. Analogously, we can also consider the dynamic version of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ as

$$\begin{aligned} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) &= -\int_t^{+\infty} x \frac{\bar{F}_{X_{(1:n)}}(x)}{\bar{F}_{X_{(1:n)}}(t)} \log\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) dx \\ &= \log \bar{F}(t) M_{(1:n)}^w(t) - \int_t^{+\infty} x \frac{\bar{F}_{X_{(1:n)}}(x)}{\bar{F}_{X_{(1:n)}}(t)} \log\left(\bar{F}(x)\right) dx \end{aligned}$$

$$= \log \bar{F}(t) M^{w}_{(1:n)}(t) - \frac{1}{(\bar{F}(t))^n} \int_t^{+\infty} x(\bar{F}(x))^n \log \bar{F}(x) dx. \tag{24}$$

Note that $\lim_{t\to 0} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) = I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$. Since $\log \bar{F}(t) \leq 0$ for $t \geq 0$, we have

$$\begin{aligned} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) &\leq -\frac{1}{(\bar{F}(t))^n} \int_t^{+\infty} x(\bar{F}(x))^n \log \bar{F}(x) dx \\ &\leq -\frac{1}{(\bar{F}(t))^n} \int_0^{+\infty} x(\bar{F}(x))^n \log \bar{F}(x) dx = \frac{I^w(\bar{F}_{X_{(1:n)}}, \bar{F})}{(\bar{F}(t))^n}. \end{aligned}$$

Theorem 14
Let X be a non-negative continuous random variable with distribution function $F(.)$. Let the weighted dynamic cumulative inaccuracy of the 1th order statistics denoted by $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) < \infty$, $t \geq 0$. Then $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t)$ characterizes the distribution function.

Proof
From (24) we have

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) = \log \bar{F}(t) M^{w}_{(1:n)}(t) - \frac{1}{(\bar{F}(t))^n} \int_t^{+\infty} x(\bar{F}(x))^n \log \bar{F}(x) dx. \tag{25}$$

Differentiating both side of (25) with respect to t we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}[I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t)] &= -\lambda_F(t) M^{w}_{(1:n)}(t) + n\lambda_F(t) I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) \\ &= \lambda_F(t) \left[n I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) - M^{w}_{(1:n)}(t) \right]. \end{aligned}$$

Taking derivative with respect to t again we get

$$\acute{\lambda}_F(t) = \frac{(\lambda_F(t))^2 \left(n\lambda_F(t) M^{w}_{(1:n)}(t) + n\frac{\partial}{\partial t} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) - t \right)}{\frac{\partial}{\partial t} I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t)}. \tag{26}$$

Suppose that there are two functions F and F^* such that

$$I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t) = I^w(\bar{F}^*_{X_{(1:n)}}, \bar{F}^*; t) = z(t).$$

Then for all t, from (26) we get

$$\acute{\lambda}_F(t) = \varphi(t, \lambda_F(t)), \quad \acute{\lambda}_{F^*}(t) = \varphi(t, \lambda_{F^*}(t)),$$

where

$$\varphi(t, y) = \frac{y^2 \left[n y s(t) + n\acute{z}(t) - t \right]}{\acute{z}(t)},$$

and $s(t) = M^{w}_{(1:n)}(t)$. By using Theorem 3.2 and Lemma 3.3 of Gupta and Kirmani [8], we have, $\lambda_F(t) = \lambda_{F^*}(t)$, for all t. Since the hazard rate function characterizes the distribution function uniquely, we complete the proof. □

Proposition 15
If $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ represent the order statistics of the sample $X_1, X_2, ..., X_n$. Then, the empirical measure of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ is obtained as

$$\hat{I}^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = -\int_0^{+\infty} x[\hat{\bar{F}}_n(x)]^n \log \hat{\bar{F}}_n(x) dx$$

$$= -\sum_{k=1}^{n-1} \int_{X_{(k)}}^{X_{(k+1)}} x \left(1-\frac{k}{n}\right)^n \log\left(1-\frac{k}{n}\right) dx$$
$$= -\sum_{k=1}^{n-1} U_k \left(1-\frac{k}{n}\right)^n \log\left(1-\frac{k}{n}\right), \tag{27}$$

where $U_k = \frac{X_{(k+1)}^2 - X_{(k)}^2}{2}, \quad k = 1, 2, ..., n-1.$

Theorem 16

Let X be a absolutely continue non-negative random variable whit $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) < \infty$, for all $n \geq 1$. Then we have

$$\hat{I}^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \longrightarrow I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) \quad a.s.$$

Proof

From (27) we have

$$\begin{aligned}
\hat{I}^w(\bar{F}_{X_{(1:n)}}, \bar{F}) &= \int_0^\infty x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx \\
&= \int_0^1 x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx + \int_1^\infty x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx \\
&=: W_1 + W_2,
\end{aligned} \tag{28}$$

where

$$W_1 = \int_0^1 x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx,$$
$$W_2 = \int_1^\infty x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx.$$

Using dominated convergence theorem (DCT) and Glivenko-Cantelli, we have

$$\int_0^1 x(-\log \hat{\bar{F}}_n(x))(\hat{\bar{F}}_n(x))^n dx \longrightarrow \int_0^1 x(-\log \bar{F}(x))(\bar{F}(x))^n dx \quad as\ m \to \infty. \tag{29}$$

It follows that

$$x^p \hat{\bar{F}}_n(x) \leq \frac{1}{n}\sum_{i=1}^n X_i^p.$$

Morever, by using SLLN, $\frac{1}{n}\sum_{i=1}^n X_i^p \longrightarrow \mathbb{E}(X^p)$ and $\sup_n(\frac{1}{n}\sum_{i=1}^n X_i^p) < \infty$, then $\hat{\bar{F}}_n(x) \leq x^{-p}\left(\sup_n(\frac{1}{n}\sum_{i=1}^n X_i^p)\right) = Cx^{-p}$. Now applying the DCT we have

$$\lim_{n\to\infty} W_2 = \int_1^\infty x(-\log \bar{F}(x))(\bar{F}(x))^n dx. \tag{30}$$

Finally by using (28) the result follows. □

3. WCPI For Maximum of Order Statistics

We consider the WCPI between $F_{X_{(n:n)}}$ and F as follows:

$$\tilde{I}^w(F_{X_{(n:n)}}, F) = -\int_0^{+\infty} xF_{X_{(n:n)}}(x)\log(F(x))\, dx = \frac{1}{n}\mathcal{CE}^w(X_{(n:n)}), \tag{31}$$

where

$$\mathcal{CE}^w(X_{(n:n)}) = n\int_0^{+\infty} x[F(x)]^n\tilde{\Lambda}(x)dx.$$

Hereafter we consider some properties of $\tilde{I}^w(F_{X_{(n:n)}}, F)$.

Proposition 17
Let X be an absolutely continuous non-negative random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\begin{aligned}\tilde{I}^w(F_{X_{(n:n)}}, F) &= \int_0^{+\infty} x[F(x)]^n\left(\int_x^{\infty}\frac{f(z)}{F(z)}dz\right)xdx \\ &= \int_0^{+\infty}\int_0^{z}\tilde{\lambda}(z)x[F(x)]^n dxdz,\end{aligned} \tag{32}$$

where $\tilde{\lambda}(.)$ is the reversed failure rate function.

Proposition 18
Let X be an absolutely continuous non-negative random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F) = E\left(\tilde{M}^w_{(n:n)}(Z)(F(Z))^{n-1}\right), \tag{33}$$

where

$$\tilde{M}^w_{(n:n)}(z) = \frac{1}{(F(z))^n}\int_0^z x(F(x))^n dx$$

is the weighted mean inactivity time (WMIT) of $X_{(n:n)}$.

Proof
By (32), we obtain

$$\begin{aligned}\tilde{I}^w(F_{X_{(n:n)}}, F) &= \int_0^{+\infty}\int_0^{z}\tilde{\lambda}(z)x[F(x)]^n dxdz \\ &= \int_0^{+\infty}\frac{f(z)}{F(z)}dz\left[\int_0^z x[F(x)]^n dx\right] \\ &= \int_0^{+\infty}\frac{f(z)}{F(z)}[F(z)]^n\tilde{M}^w_{(n:n)}(z)dz = \int_0^{+\infty} f(z)[F(z)]^{n-1}\tilde{M}^w_{(n:n)}(z)dz.\end{aligned} \tag{34}$$

Thus, the proof is complete. □

Proposition 19
Let $a, b > 0$. For $n = 1, 2, ...$ it holds that

$$\tilde{I}^w(F_{aX_{(n:n)}+b}, F_{aX+b}) = a\tilde{I}^w(F_{X_{(n:n)}}, F).$$

The next propositions give some lower and upper bounds for $\tilde{I}^w(F_{X_{(n:n)}}, F)$.

Proposition 20
For a non-negative random variable X and $n \geq 1$, it holds that

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \geq \tilde{M}^w_{(n:n)}(t)|\log F(t)|[F(t)]^n, \tag{35}$$

where $\tilde{M}_{(n:n)}(t)$ is the WMIT of $X_{(n:n)}$.

Proof
The proof follows from Baratpour [2].

□

Proposition 21
Let X be an absolutely continuous non-negative random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \geq \int_0^{+\infty} x(F(x))^n \bar{F}(x)dx. \tag{36}$$

Proof
Recalling that $-\log F(x) \geq \bar{F}(x)$, the proof then finally follows. □

Proposition 22
Let X be an absolutely continuous non-negative random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \leq \mathcal{CE}^w(X). \tag{37}$$

Proof
Since $F(x) \geq [F(x)]^n$, $x \geq 0$, when $n \geq 1$, the proof then finally follows. □

Proposition 23
If X is DRFRA, then

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \leq \mathbb{E}\left(X^2(F(X))^{n-1}\right). \tag{38}$$

Proof
Since X is DRFRA, $\frac{\tilde{\Lambda}(x)}{x}$ is decreasing with respect to $x > 0$, which implies that

$$F(x)\tilde{\Lambda}(x) \leq xf(x), \quad x > 0. \tag{39}$$

By multiplying $x[F(x)]^{n-1} \geq 0$ in (39) and then integrating, the result follows. □

Proposition 24
Let X be an absolutely continuous non-negative random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F) = \mathbb{E}\left[\tilde{h}^w(X)\right], \tag{40}$$

where

$$\tilde{h}^w(x) = \int_x^{\infty} z\left[-\log F(z)\right]\left[F(z)\right]^{n-1}dz, \quad x \geq 0.$$

Proof
From (31) and using Fubini's theorem, we obtain

$$\begin{aligned}
\tilde{I}^w(F_{X_{(n:n)}}, F) &= \int_0^{\infty} z[-\log F(z)][F(z)]^{n-1}F(z)dz \\
&= \int_0^{\infty}\left[\int_0^z f(x)dx\right] z[F(z)]^{n-1}[-\log F(z)]dz \\
&= \int_0^{\infty} f(x)\left[\int_x^{\infty} z[F(z)]^{n-1}[-\log F(z)]dz\right]dx = \mathbb{E}\left[\tilde{h}^w(X)\right].
\end{aligned}$$

□

Proposition 25
Let X and Y be two non-negative random variables with reliability functions $\bar{F}(x)$, $\bar{G}(x)$, respectively. If $X \leq^{icx} Y$, then

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \leq \tilde{I}^w(G_{Y_{(n:n)}}, G).$$

Proof
Since $\tilde{h}^w(.)$ is an increasing convex function for $n \geq 1$, it follows by Shaked and Shanthikumar [17] that $X \leq^{icx} Y$ implies $\tilde{h}^w(X) \leq^{icx} \tilde{h}^w(Y)$. By recalling the definition of increasing convex order and Proposition 24 proof is complete. □

Proposition 26
Let X and Y be two non-negative random variables with survival function $\bar{F}(x)$ and $\bar{G}(x)$, respectively. If $X \leq^{hr} Y$, then for $n = 1, 2, ...$, it holds that

$$\frac{\tilde{I}^w(F_{X_{(n:n)}}, F)}{E(X)} \leq \frac{\tilde{I}^w(G_{Y_{(n:n)}}, G)}{E(Y)}.$$

Proof
By noting that the function $\tilde{h}^w(x) = \int_x^\infty z[F(z)]^{n-1}[-\log F(z)]dz$ is an increasing convex function, under the assumption $X \leq^{hr} Y$, it follows by Shaked and Shanthikumar [17],

$$\frac{E\left[\tilde{h}^w(X)\right]}{E(X)} \leq \frac{E\left[\tilde{h}^w(Y)\right]}{E(Y)}.$$

Hence, the proof is completed by recalling (40). □

Proposition 27
(i) Let X be a continuous random variable with survival function $\bar{F}(.)$ that takes values in $[0, b]$, with finite b. Then,

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \leq b\tilde{I}(F_{X_{(n:n)}}, F).$$

(ii) Let X be a non-negative continuous random variable with survival function $\bar{F}(.)$ that takes values in $[a, \infty)$, with finite $a > 0$. Then,

$$\tilde{I}^w(F_{X_{(n:n)}}, F) \geq a\tilde{I}(F_{X_{(n:n)}}, F).$$

Assume that X^*_θ denotes a non-negative absolutely continuous random variable with the distribution function $H_\theta(x) = [F(x)]^\theta, x \geq 0$. This model is known as a proportional hazards rate model. We now obtain the weighted cumulative past measure of inaccuracy between $H_{X_{(n:n)}}$ and H as follows:

$$\begin{aligned} \tilde{I}^w(H_{X_{(n:n)}}, H) &= -\int_0^{+\infty} x H_{X_{(n:n)}}(x) \log\left(H(x)\right) dx \\ &= -\theta \int_0^{+\infty} x(F(x))^{n\theta} \log F(x) dx. \end{aligned} \tag{41}$$

Proposition 28
If $\theta \geq (\leq)1$, then for any $n \geq 1$, we have

$$\tilde{I}^w(H_{X_{(n:n)}}, H) \leq (\geq)\theta\tilde{I}^w(F_{X_{(n:n)}}, F) \leq (\geq)\theta\mathcal{CE}^w(X).$$

Proof
Suppose that $\theta \geq (\leq)1$, then it is clear $[F(x)]^\theta \leq (\geq)F(x)$, and hence (41) yields

$$\tilde{I}^w(H_{X_{(n:n)}}, H) \leq (\geq)\theta\mathcal{CE}^w(X).$$

□

Theorem 29
$\tilde{I}^w(F_{X_{(n:n)}}, F) = 0$, if and only if, X is degenerate.

Proof
Suppose X is degenerate at point a, obviously by definition of degenerate function and definition of $\tilde{I}^w(F_{X_{(n:n)}}, F)$, we have $\tilde{I}^w(F_{X_{(n:n)}}, F) = 0$.
Now, suppose that $\tilde{I}^w(F_{X_{(n:n)}}, F) = 0$, i.e.

$$-\int_0^\infty x[F(x)]^n \log F(x) dx = 0. \tag{42}$$

Then, by noting that integrand function of (42) is non-negative, we conclude that $-x[F(x)]^n \log F(x) = 0$, for almost all $x \in \mathbb{R}^+$. Thus, $F(x) = 0$ *or* 1, for almost all $x \in \mathbb{R}^+$. □

Proposition 30
Let X be an absolutely continuous random variable with $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for $n \geq 1$. Then, we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F) = \int_0^1 F_X^{-1}(u) \frac{\psi_n(u)}{f_X\left(F_X^{-1}(u)\right)} du,$$

where $\psi_n(u) = -u^n \log u$, for $0 < u < 1$. Note that $\psi_n(0) = \psi_n(1) = 0$.

Recently, Cali et al. [3] introduced the generalized CPI of order m defined as

$$I_m(F, G) = \frac{1}{m!} \int_0^{+\infty} F(x)[-\log G(x)]^m dx. \tag{43}$$

In analogy with the measure defined in (43), we now introduce the weighted generalized CPI (WGCPI) of order m defined as

$$I_m^w(F, G) = \frac{1}{m!} \int_0^{+\infty} xF(x)[-\log G(x)]^m dx. \tag{44}$$

Remark 3
Let X be a non-negative absolutely continuous random variable with cdf F. Then, the WGCPI of order m between $F_{X_{(n:n)}}$ and F is

$$\begin{aligned} \tilde{I}_m^w(F_{X_{(n:n)}}, F) &= \frac{1}{m!} \int_0^\infty x[F(x)]^n [-\log F(x)]^m dx \\ &= \frac{1}{n^m} \mathcal{CE}_m^w(X_{(n:n)}), \end{aligned} \tag{45}$$

where

$$\mathcal{CE}_m^w(X) = \int_0^\infty x \frac{[-\log F(x)]^m}{m!} F(x) dx,$$

is a weighted generalized cumulative entropy (WGCE) which introduced by Kayal and Moharana [10].

Remark 4
In analogy with (31), a measure of WCPI associated with F and $F_{X_{(n:n)}}$ is given by

$$\tilde{I}^w(F, F_{X_{(n:n)}}) = -\int_0^{+\infty} xF(x) \log\left(F_{X_{(n:n)}}(x)\right) dx = n\mathcal{CE}^w(X). \tag{46}$$

In the remainder of this section, we study dynamic version of $\tilde{I}^w(F_{X_{(n:n)}}, F)$. If a system that begins to work at time 0 is observed only at deterministic inspection times, and is found to be down at time t, then we consider a dynamic version of $\tilde{I}^w(F_{X_{(n:n)}}, F)$ as

$$\begin{aligned}
\tilde{I}^w(F_{X_{(n:n)}}, F; t) &= -\int_0^t x \frac{F_{X_{(n:n)}}(x)}{F_{X_{(n:n)}}(t)} \log\left(\frac{F(x)}{F(t)}\right) dx \\
&= \log F(t) \tilde{M}^w_{(n:n)}(t) - \int_0^t x \frac{F_{X_{(n:n)}}(x)}{F_{X_{(n:n)}}(t)} \log\left(F(x)\right) dx \\
&= \log F(t) \tilde{M}^w_{(n:n)}(t) - \frac{1}{(F(t))^n} \int_0^t x (F(x))^n \log F(x) dx. \qquad (47)
\end{aligned}$$

Note that $\lim_{t\to 0} \tilde{I}^w(F_{X_{(n:n)}}, F; t) = \tilde{I}^w(F_{X_{(n:n)}}, F)$. Since $\log F(t) \leq 0$ for $t \geq 0$, we have

$$\begin{aligned}
\tilde{I}^w(F_{X_{(n:n)}}, F; t) &\leq -\frac{1}{(F(t))^n} \int_0^t x (F(x))^n \log F(x) dx \\
&\leq -\frac{1}{(F(t))^n} \int_0^{+\infty} x (F(x))^n \log F(x) dx = \frac{\tilde{I}^w(F_{X_{(n:n)}}, F)}{(F(t))^n}.
\end{aligned}$$

Theorem 31
Let X be a non-negative continuous random variable with distribution function $F(.)$. Let the weighted dynamic cumulative inaccuracy of the nth order statistics denoted by $\tilde{I}^w(F_{X_{(n:n)}}, F; t) < \infty$, $t \geq 0$. Then $\tilde{I}^w(F_{X_{(n:n)}}, F; t)$ characterizes the distribution function.

Proof
From (47) we have

$$\tilde{I}^w(F_{X_{(n:n)}}, F; t) = \log F(t) \tilde{M}^w_{(n:n)}(t) - \frac{1}{(F(t))^n} \int_0^t x (F(x))^n \log F(x). \qquad (48)$$

Differentiating both side of (48) with respect to t we obtain:

$$\begin{aligned}
\frac{\partial}{\partial t}[\tilde{I}^w(F_{X_{(n:n)}}, F; t)] &= -\tilde{\lambda}_F(t) \tilde{M}^w_{(n:n)}(t) - n \tilde{\lambda}_F(t) \tilde{I}^w(F_{X_{(n:n)}}, F; t) \\
&= -\tilde{\lambda}_F(t) \left[\tilde{M}^w_{(n:n)}(t) - n \tilde{I}^w(F_{X_{(n:n)}}, F; t)\right]
\end{aligned}$$

Taking derivative with respect to t again we get

$$\acute{\tilde{\lambda}}_F(t) = \frac{\left(\tilde{\lambda}_F(t)\right)^2 \left(n \tilde{\lambda}_F(t) \tilde{M}^w_{(n:n)}(t) + n \frac{\partial}{\partial t} \tilde{I}^w(F_{X_{(n:n)}}, F; t) - t\right)}{\frac{\partial}{\partial t} \tilde{I}^w(F_{X_{(n:n)}}, F; t)}. \qquad (49)$$

Suppose that there are two functions F and F^* such that

$$\tilde{I}^w(F_{X_{(n:n)}}, F; t) = \tilde{I}^w(F^*_{X_{(n:n)}}, F^*; t) = z(t).$$

Then for all t, from (49) we get

$$\acute{\tilde{\lambda}}_F(t) = \varphi(t, \lambda_F(t)), \quad \acute{\tilde{\lambda}}_{F^*}(t) = \varphi(t, \lambda_{F^*}(t)),$$

where

$$\varphi(t,y) = \frac{y^2\left[nys(t) + n\acute{z}(t) - t\right]}{\acute{z}(t)},$$

and $\tilde{s}(t) = \tilde{M}^w_{(n:n)}(t)$. By using Theorem 3.2 and Lemma 3.3 of Gupta and Kirmani [8], we have, $\lambda_F(t) = \lambda_{F^*}(t)$, for all t. Since the hazard rate function characterizes the distribution function uniquely, we complete the proof. □

Proposition 32
If $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denote the order statistics of the sample $X_1, X_2, ..., X_n$. Then, the empirical measure of $\tilde{I}^w(F_{X_{(n:n)}}, F)$ is given by

$$\begin{aligned}
\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F) &= -\int_0^{+\infty} x[\hat{F}_n(x)]^n \log \hat{F}_n(x)dx \\
&= -\sum_{k=1}^{n-1} \int_{X_{(k)}}^{X_{(k+1)}} x\left(\frac{k}{n}\right)^n \log\left(\frac{k}{n}\right) dx \\
&= \frac{-1}{n^n}\sum_{k=1}^{n-1} k^n U_k \log\left(\frac{k}{n}\right),
\end{aligned} \tag{50}$$

where $U_k = \frac{X^2_{(k+1)} - X^2_{(k)}}{2}, \quad k = 1, 2, ..., n-1.$

Example 1
Consider the random sample $X_1, X_2, ..., X_n$ from a Weibull distribution with density function

$$f(x) = 2\lambda \exp(-\lambda x^2).$$

Then $Y_k = X_k^2$ has an exponential distribution with mean $\frac{1}{\lambda}$. In this case, the sample spacings $2U_k = X^2_{(k+1)} - X^2_{(k)}$ are independent and exponentially distributed with mean $\frac{1}{\lambda(n-k)}$ (for more details see Pyke [14]). Now from (50) we obtain

$$\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = \frac{-1}{n^n}\sum_{k=1}^{n-1} \frac{k^n}{2\lambda(n-k)} \log\frac{k}{n}, \tag{51}$$

and

$$Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = \frac{1}{n^{2n}}\sum_{k=1}^{n-1} \frac{k^{2n}}{4\lambda^2(n-k)^2}\left(\log\frac{k}{n}\right)^2. \tag{52}$$

We have computed the values of $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ and $Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ for sample sizes $n = 10, 15, 20$ and $\lambda = 0.5, 1, 2$ in Table 1. We can easily see that $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ is decreasing in n. Also, we consider that $\lim_{n\to\infty} Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = 0$.

Example 2
Let $X_1, X_2, ..., X_n$ be a random sample from a population with pdf $f(x) = 2x,\ 0 < x < 1$. Then the sample spacings $2U_k$ are independent of beta distribution with parameters 1 and n (for more details see Pyke [14]). Now from (50) we obtain

$$\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = \frac{-1}{n^n}\sum_{k=1}^{n-1} \frac{k^n}{2(n+1)} \log\frac{k}{n}, \tag{53}$$

and

$$Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = \frac{1}{n^{2n-1}}\sum_{k=1}^{n-1} \frac{k^n}{4(n+1)^2(n+2)}\left(\log\frac{k}{n}\right)^2. \tag{54}$$

We have computed the values of $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ and $Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ for sample sizes $n = 10, 15, 20$ in Table 2. We can easily see that $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ is decreasing in n. Also, we consider that $\lim_{n\to\infty} Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)] = 0$.

Table 1. Numerical values of $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ and $Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ for Weibull distribution.

	$\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$			$Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$		
n	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
10	0.0530	0.0265	0.0132	0.00150	0.00037	0.00009
15	0.0365	0.0182	0.0091	0.00068	0.00017	0.00004
20	0.0278	0.0139	0.0069	0.00038	0.00009	0.00002

Table 2. Numerical values of $\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ and $Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$ for beta distribution.

$\mathbb{E}[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$			$Var[\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F)]$		
n=10	n=15	n=20	n=10	n=15	n=20
0.00339	0.00166	0.00098	2.57e-15	1.43e-23	2.11e-32

Theorem 33
Let X be an absolutely continuous non-negative random variable whit $\tilde{I}^w(F_{X_{(n:n)}}, F) < \infty$, for all $n \geq 1$. Then we have

$$\hat{\tilde{I}}^w(F_{X_{(n:n)}}, F) \longrightarrow \tilde{I}^w(F_{X_{(n:n)}}, F) \quad a.s.$$

Proof
From (50) we have

$$\begin{aligned} \hat{\tilde{I}}^w(F_{X_{(n:n)}}, F) &= \int_0^\infty x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx \\ &= \int_0^1 x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx + \int_1^\infty x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx \\ &=: R_1 + R_2, \end{aligned} \tag{55}$$

where

$$R_1 = \int_0^1 x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx,$$
$$R_2 = \int_1^\infty x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx.$$

Using dominated convergence theorem (DCT) and Glivenko-Cantelli, we have

$$\int_0^1 x(-\log \hat{F}_n(x))(\hat{F}_n(x))^n dx \longrightarrow \int_0^1 x(-\log F(x))(F(x))^n dx \quad as\ m \to \infty. \tag{56}$$

It follows that

$$x^p \hat{F}_n(x) \leq \frac{1}{n}\sum_{i=1}^n X_i^p.$$

Morever, by using SLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^p \longrightarrow \mathbb{E}(X^p)$ and $\sup_n(\frac{1}{n}\sum_{i=1}^{n} X_i^p) < \infty$, then $\hat{F}_n(x) \leq x^{-p}\left(\sup_n(\frac{1}{n}\sum_{i=1}^{n} X_i^p)\right) = Cx^{-p}$. Now applying the DCT we have

$$\lim_{n\to\infty} R_2 = \int_1^{\infty} x(-\log F(x))(F(x))^n dx. \tag{57}$$

Finally by using (55) the result follows. □

In the following example, we calculate $\tilde{I}^w(F_{X_{(n:n)}}, F)$ and $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ for some specific lifetime distributions which are widely used in reliability theory and life testing.

Example 3
(a) If X is uniformly distributed in $[0,\theta]$, then it is easy to see that $\tilde{I}^w(F_{X_{(n:n)}}, F) = \frac{\theta^2}{(n+2)^2}$, and $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = \frac{(2n+3)\theta^2}{(n^2+3n+2)^2}$, for all integers $n \geq 1$. Note that $\tilde{I}^w(F_{X_{(n:n)}}, F)$ and $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ are decreasing functions of n.
(b) If X has a Weibull distribution with survival function $\bar{F}(x) = e^{-\lambda^q x^q},\ x > 0,\ \lambda, q > 0$, then for all integers $n \geq 1$, we obtain $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = \frac{2}{(\lambda q)^2 n^{\frac{q+2}{q}}}\Gamma(\frac{2}{q})$.
(c) If X has a Pareto distribution with pdf $f(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}},\ x \geq \beta,\ \beta > 0,\ \alpha > 0$, then $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = \frac{\alpha\beta^2}{(n\alpha-2)^2}$, for all integers $n > \frac{2}{\alpha}$. Note that $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ is a decreasing function of n for all $\alpha > \frac{2}{n}$.
(d) Let X be an exponential distribution with mean $\frac{1}{\lambda}$, then $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}) = \frac{2}{n^3\lambda^2}$. Note that $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ is a decreasing function of n.
(e) Let X be a non-negative random variable which has an inverse Weibull distribution with the cdf $F(x) = \exp(-(\frac{\alpha}{x})^\beta),\ x > 0$, then for all integers $n \geq 1$, we obtain $\tilde{I}^w(F_{X_{(n:n)}}, F) = \frac{\alpha^2 n^{\frac{2-\beta}{\beta}}}{\beta}\Gamma(\frac{\beta-2}{\beta})$.

4. Conclusion

In this paper, we discussed on concept of a weighted cumulative residual inaccuracy measure between $\bar{F}_{X(1:n)}$ and $\bar{F}$ and studied some properties of its. We proposed a dynamic version of WCRI and studied characterization results of it. It is also proved that $I^w(\bar{F}_{X_{(1:n)}}, \bar{F}; t)$ can uniquely determine the parent distribution F. Moreover, we studied some new basic properties of $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ such as the effect of linear transformation, relationships with other reliability functions, bounds and stochastic order properties. We estimated the WCRI by means of the empirical cumulative inaccuracy for minimum of order statistics and proved that $\hat{I}^w(\bar{F}_{X_{(1:n)}}, \bar{F})$ converge to $I^w(\bar{F}_{X_{(1:n)}}, \bar{F})$. Finally, similarly, we proposed the WCPI measure between $F_{X(n:n)}$ and F. We also studied some properties of $\tilde{I}^w(F_{X_{(n:n)}}, F)$ such as the connections with other reliability functions, several useful bounds and stochastic orderings. These concepts can be applied in measuring the weighted inaccuracy contained in the associated residual (past) lifetime.

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