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A Note on Multiset Dimension and Local Multiset Dimension of Graphs

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Abstract All graphs in this paper are nontrivial and connected simple graphs. For a set $W = \{s_1, s_2, \ldots, s_k\}$ of vertices of G, the multiset representation of a vertex v of G with respect to W is $r(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$ where $d(v, s_i)$ is the distance between of v and s_i. If the representation $r(v|W) \neq r(u|W)$ for every pair of vertices u, v of a graph G, the W is called the resolving set of G, and the cardinality of a minimum resolving set is called the multiset dimension, denoted by $md(G)$. A set W is a local resolving set of G if $r(v|W) \neq r(u|W)$ for every pair of adjacent vertices u, v of a graph G. The cardinality of a minimum local resolving set W is called local multiset dimension, denoted by $\mu_l(G)$. In our paper, we discuss the relationship between the multiset dimension and local multiset dimension of graphs and establish bounds of local multiset dimension for some families of graph.

Keywords Resolving set, Multiset Dimension, Local Resolving set, Local Multiset Dimension, Cartesian product, Hypercube Graph, Almost Hypercube Graph, Trees

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1. Introduction

All graphs discussed in this paper are finite, simple and connected graph. The cartesian product graph of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where vertex (x, u) is adjacent to vertex (y, v) whenever $xy \in E(G_1)$ and $u - v$, or $x - y$ and $uv \in V(G_2)$. In the rest of the paper, we use the terminology defined in [1, 2, 3]. The application of metric dimension in networks is one of the describe navigation robots. The each place is called vertices and edges denote the connections between vertices. The minimum number of the robots required to locate each and the vertex of a some network is called as metric dimension, for more detail this application in [4].

The concept of metric dimension was independently introduced by Slater [5] and Harary and Melter [6]. In his paper, Slater called this concept the *locating set*. Let u, v be two vertices in G, the distance $d(u, v)$ is the length of a shortest path between two vertices u and v in graph G. An ordered set $W = \{w_1, w_2, ..., w_k\}$ subset of vertex set $V(G)$. The $representation$ $r(v|W)$ of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is called the $resolving\ set$ of G if every vertices of G have distinct representation with respect to W. Let u and v be any two vertices in G if $r(u|W) = r(v|W)$ implies that $u = v$. Hence if W is a resolving set of cardinality k for a graph G, then the representation set $r(v|W), v \in V(G)$ consists of $|V(G)|$ distinct k-vector. The minimum cardinality of resolving set of a graph G is called $metric\ dimension$ of G, denoted by $dim(G)$.

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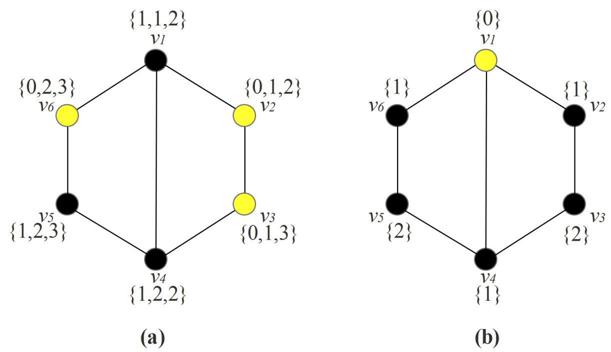

Figure 1. (a) A graph with multiset dimension 3; (b) A graph with local multiset dimension 1

Simanjuntak *et al* [7] introduced the definition of $multiset\ dimension$ of G. Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \ldots, s_k\}$ is a subset (note, not an ordered set as in metric dimension) of the vertex set $V(G)$, the $representation\ multiset$ of a vertex v of G with respect to W is the multiset $r(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$ where $d(v, s_i)$ is the distances between of v and the vertices in W together with their multiplicities. A resolving set having minimum cardinality is called a $multiset\ basis$. If G has a multiset basis, then its cardinality is called the *multiset dimension* of G, denoted by $md(G)$. There are some related research about this topic in [9, 10, 11].

Alfarisi, *et. al* [8] defined a new notion based on the multiset dimension of G, namely a $local\ multiset\ dimension$. The definition of local multiset dimension is below:

Definition 1.1
Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of the vertex set $V(G)$, the representation multiset of a vertex v of G with respect to W is $r(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$, where $d(v, s_i)$ is a multiset of distances between of v and the vertices in W together with their multiplicities. The resolving set W is a $local\ resolving\ set$ of G if $r(v|W) \neq r(u|W)$ for every pair of adjacent vertices u, v of a graph G. The minimum local resolving set W is called $local\ multiset\ dimension$, denoted by $\mu_l(G)$.

We illustrate this concept in Figure 1. In this case, the resolving set is $W = \{v_2, v_3, v_6\}$, shown in Figure 1 (a). The multiset dimension is $md(G) = 3$. The representations of $v \in V(G)$ with respect to W are all distinct . For the local multiset dimension, we only need to make sure the adjacent vertices having distinct representations. Thus we could have the local resolving set $W = \{v_1\}$, shown in Figure 1 (b). Thus, the local multiset dimension is $\mu_l(G) = 1$.

$$\begin{array}{lll} r(v_1|\Pi) = \{0\}, & r(v_2|\Pi) = \{1\}, & r(v_3|\Pi) = \{2\} \\ r(v_4|\Pi) = \{1\}, & r(v_5|\Pi) = \{2\}, & r(v_6|\Pi) = \{1\} \end{array}$$

2. Multiset Dimension

Different to the metric dimension, given a multiset basis, it is impossible to construct the original graph from the representation of the vertices. Fig 3 gives an example of two non-isomorphic graphs with the same multiset basis and representations for vertices.

Lemma 2.1
The multiset dimension is not monotonic to the number of vertices and the number of edges of a graph.

Let G be a connected graphs. The number of vertices, edges and the multiset dimension do not show a monotonic relationship. Assume the graph G with n vertices has $md(G) = k$, if we put m vertices in graph G, then we get new graphs G' with $n + m$ vertices such that we have some condition as follows:

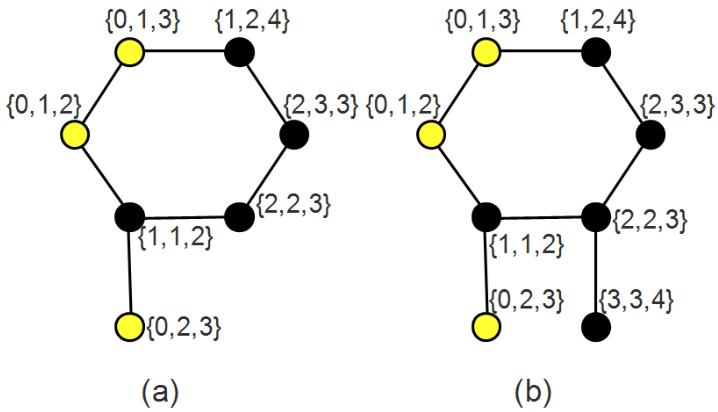

Figure 2. (a) $md(G) = 3$; (b) $md(G \cup \{v\}) = 3$

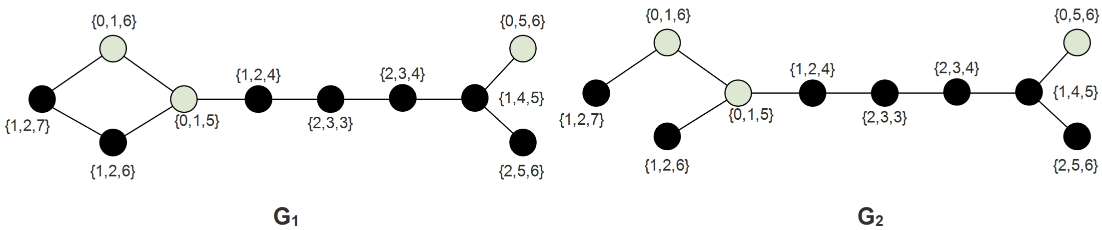

Figure 3. For $G_1 \neq G_2$

i. if $G \cong G'$, then $md(G) = md(G') = k$. (*not monotonic*)
ii. if $G \ncong G'$, then we have $md(G) \geq md(G')$ or $md(G) < md(G')$.

We use a counter example for showing the Lemma 2.1. Assume the number of vertices and edges of G increase, the multiset dimension of G increase or decrease (monotonically). We choose a unicyclic graph G as example. From Figure 2 (b) we increase the vertices and edges in graph G which have the multiset dimension of $G \cup \{v\}$ namely $md(G \cup \{v\}) = 3$ where v is a vertex not in the graph G. Furthermore, we can say that if we increase the number of vertices and edges in graphs G, then the number of resolving set does not increase or the multiset dimension is constant. it is a counter example of Lemma 2.1.

In Simanjuntak, *et. al.* [7], some bounds are given for the multiset dimension of graphs. For example,

Theorem 2.1
[7] Let G be a graph other than a path. Then $md(G) \geq 3$

If we look at the resolving set, since the vertex has distance 0 to itself, then it is easy to get a better bound than Theorem 2.2. For positive integers n and d, we define $f(n, d)$ to be the least positive integer k for which $\frac{(k+d1)!}{k!(d1)!} + k \geq n$.

Theorem 2.2
If G is a graph of order $n \geq 3$ and diameter d, then $md(G) \geq f(n, d)$

Proof. Let W be a multiset basis of G having k vertices. If x is a vertex in W, then $r(x|W) = \{0, 1^{m_1}, 2^{m_2}, \ldots, d^{m_d}\}$, where $m_1 + m_2 + \cdots + m_d = k - 1$ and $0 m_i k$ for each $i = 1, 2, \ldots, k$. Then there are $C(k - 1 + d - 2, d - 2)$ different possibilities for representation of x. Since we have k vertices in W, then $C(k - 1 + d - 2, d - 2)$ must be at least k.

Furthermore, look at the degree of the graph, we could have the following bounds.

Theorem 2.3
Let G be a connected graphs and let d be the maximum degree of G and $md(G) = k$, we have for $k \geq 3$ and $d < 3k$.

Proof. As $md(G) = k$, so let W be a multiset basis of G having k vertices. If x is a vertex not in W, then $r(x|W) = \{0, 1^{m_1}, 2^{m_2}, \ldots, d^{m_d}\}$, where $m_1 + m_2 + \cdots + m_d = k$ and $0 \leq m_i \leq k$ for each $i = 1, 2, \ldots, k$. As W is a minimum resolving set, removing a vertex from W, there will be two vertices in the graph G which have the same representation. Let's assume that there are two vertices v_x and v_y which are in G. Formally, we have $r(v_x|W) \cap r(v_y|W) = \{1^{m_1}, 2^{m_2}, \ldots, d^{m_d}\}$, where $m_1 + m_2 + \cdots + m_d = k - 1$. Considering the neighbour of these two vertices, if v_x has distance t to a vertex $w \in W$, then the neighbour of v_x would have distance $t - 1$ or t or $t + 1$ to w. So, if v_x has distance 1 to m_1 vertices in W, then among v_x's neighbour, there are at most m_1 vertices having distance 0 to vertices in W, i.e. in W. if v_x has distance 2 to m_2 vertices in W, then among v_x's neighbour, there are at most m_2 vertices having distance 1 to vertices in W, furthermore, v_x's neighbour having distance less than 3 to at least m_2 vertices in W. More general, if v_x having distance x to m_x vertices in W, then among the neighbours of v_x, there are less than m_x vertices of distance $x - 1$ to W and there are no more than m_x vertices having distance $x + 1$ to vertices in W.

Thus, the number of different representations for the neighbour of v_x is $m_1(m_1 + m_2)(m_2 + m + 3)..(m_{d-1} + md)d$ and the representations are shared by neighbours of v_x and v_y. This will give us a bound for the degree, diameter and multiset dimension. □

We shall define a new graph which is based on the well-known Hypercubes. The Hypercube is defined the graph formed from the vertices and edges of an n-dimensional hypercube, we shall remove some edges from the hypercube, denoted by AHQ_n, is called almost hypercube graphs. Almost hypercube graph satisfies $AHQ_n = (HQ_{n-1} \times P_2) - \{e\}$ for $n \geq 3$, where e is correspondence edge of subgraph $(HQ_{n-1})_1$ and $(HQ_{n-1})_2$. We know that $HQ_{n-1} \times P_2$ has two isomorphic graphs $(HQ_{n-1})_1$ and $(HQ_{n-1})_2$ with $\{e\}$ is the correspondence edge set.

Theorem 2.4
$md(AHQ_n) \geq 2^{n-1} - 1$, for $n \geq 3$

Proof: The cardinality of vertex set of almost hypercube graphs, denoted by AHQ_n, is 2^n for $n \geq N \cup \{0\}$. We can prove $md(AHQ_n) \geq 2^{n-1} - 1$.
Case 1: For $n = 3$, we know that $md(AHQ_3) \geq 3$, $md(AHQ_3) > 3$. If the resolving set of AHQ_3 is 2, then we have some condition for position resolving set in $AHQ_3 = (HQ_1 \times P_2) - \{e\}$ in the following.

i. If three vertices in HQ_2', then there is at least two vertices which have same representation.
ii. If two vertices in HQ_2' and one vertex in HQ_2", then always two vertices with respect to $v \in W(HQ_2")$ which have same representation.

Based on cases above, we know that $md(AHQ_3) \geq 3$
Case 2: For $n = k$, we know that $md(AHQ_k) \geq 2^{k-1} - 1$, $md(AHQ_k) > 2^{k-1} - 1$. If the resolving set of AHQ_k is $2^{k-1} - 2$, then we have some condition for position resolving set in $AHQ_k = (HQ_{k-1} \times P_2) - \{e\}$. We have some condition for proof this cases which divided into some cases as follows.

i. Shortest path between two vertices in the same components is within the component
ii. If $2^{k-1} - 2$ vertices in HQ_{n-1}', then there is at least two vertices which have same representation. Let $v_1, v_2 \in V(AHQ_n)$ be a correspondence vertices of $v_1 \in V(HQ_{n-1}')$ and $v_2 \in V(HQ_{n-1}")$ such that $d(v_1, r) \neq d(v_2, r) = d(v_1, r) + 1$ for $r \in W$. We assume that the representation of v_1 and v_2 respect to resolving set W namely $r(v_1|W) = \{d_1, d_2, d_3, \ldots, d_{x-3}, d(v_1, s) + 1, d(v_1, s)\}$ and $r(v_2|W) = \{d_1, d_2, d_3, \ldots, d_{x-3}, d(v_1, s), d(v_2, s)\}$ such that $r(v_1|W) = r(v_2|W)$, it is contradiction.
iii. If 2^{k-2} vertices in HQ_{n-1}' and $2^{k-2} - 2$ vertex in HQ_{n-1}", then always two vertices with respect to $v \in W(HQ_{n-1}")$ which have same representation. Let $v_1, v_2 \in V(HQ_{n-1}')$ be a vertex in first position HQ_{n-1}' which have same representation respect to resolving set in HQ_{n-1}' namely $r(v_1|W') = r(v_2|W') = \{d_1, d_2, d_3, \ldots, d_{\alpha-l}\}$. The distance of v_1 and v_2 respect to resolving set in second position HQ_{n-1}" is symmetric distance such that $d(v_1, r) = d(v_2, r)$. Thus, we have the representation $v_1, v_2 \in AHQ_n$ namely $r(v_1|W) = r(v_2|W) = \{d_1, d_2, d_3, \ldots, d_{\alpha-l}, d_{\alpha-l+1}, \ldots, d_\alpha\}$, it is contradiction.

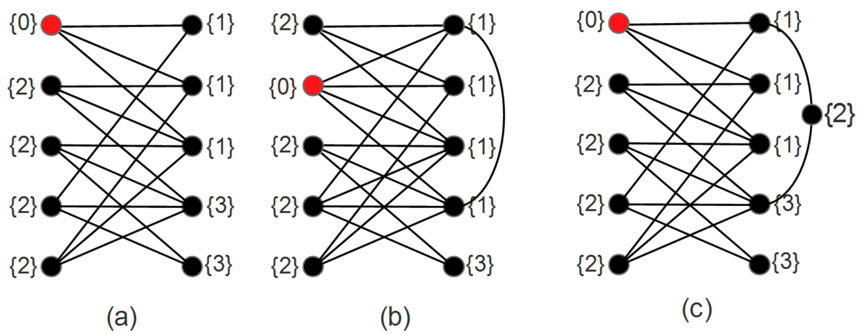

Figure 4. (a) $\mu_l(B_{5,5}) = 1$; (b) $\mu_l(B_{5,5} \cup \{e\}) = 1$; (c) $\mu_l(B_{5,5} \cup v) = 1$

Based on both cases, we obtain the bounds of multiset dimension of almost hypercube graph namely $md(AHQ_n) \geq 2^{n-1} - 1$. □

3. Local Multiset Dimension

In this section, we give some results about local multiset dimension of graphs. Firstly, we show that the local multiset dimension for AHQ is 1, which is different to the results we got for multiset dimension.

Corollary 3.1
The difference between Multiset dimension and local multiset dimension can be arbitrarily large.

Observation 3.1
If G is complete graph, then the graph G does not have a local resolving set.

Lemma 3.1
The local multiset dimension is not monotonic to the number of vertices and the number of edges of a graph.

Let G be a connected graphs. The number of vertices, edges and the local multiset dimension do not show a monotonic relationship. Assume the graph G with n vertices has $\mu_l(G) = k$, if we put m vertices in graph G, then we get new graphs G' with $n + m$ vertices such that we have some condition as follows:

i. if $G \cong G'$, then $\mu_l(G) = \mu_l(G') = k$. (*not monotonic*)
ii. if $G \ncong G'$, then we have $\mu_l(G) \geq \mu_l(G')$ or $\mu_l(G) < \mu_l(G')$.

We use a counter example for showing the Lemma 3.1. Assume the number of vertices and edges of G increase, the local multiset dimension of G increase or decrease (monotonically). We choose a bipartite graph $B_{5,5}$ as example. From Figure 4 (a) the local multiset dimension of $B_{5,5}$ is $\mu_l(B_{5,5}) = 1$, we add some edges in $B_{5,5}$ in Figure 4 (b) such that we have the local multiset dimension of $B_{5,5} \cup \{e\}$ namely $\mu_l(B_{5,5} \cup \{e\}) = 1$. Figure 4 (c) we increase the vertices and edges in graph $B_{5,5}$ which have the local multiset dimension of $B_{5,5} \cup \{v\}$ namely $\mu_l(B_{5,5} \cup \{v\}) = 1$ where v is a vertex not in the graph $B_{5,5}$. Furthermore, we can say that if we increase the number of vertices and edges in graphs $B_{5,5}$, then the number of resolving set does not increase or the local multiset dimension is constant. it is a counter example of Lemma 3.1.

The following, we show the new bound of local multiset dimension of cartesian product of graphs. Let G_1 and G_2 be two connected graphs.

Lemma 3.2
Given that two connected graphs G_1 and G_2, $\mu_l(G_1 \times G_2) \geq min\{\mu_l(G_1), \mu_l(G_2)\}$

Proof. A graph G_1 has n_1 vertices and G_2 has n_2 vertices. The cartesian product graph of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where vertex (x, u) is adjacent to vertex (y, v) whenever

$xy \in E(G_1)$ and $u - v$, or $x - y$ and $uv \in V(G_2)$. For a fixed x of G_1, the vertices $\{(x,u)|u \in V(G_2)\}$ induces a subgraph of $G_1 \times G_2$ isomorphic to G_2 and we call it as G_2-layer. Such that, we have G_1-layers or G_2-layers. Assume that we have local multiset dimension of G_1 and G_2, respectively are $\mu_l(G_1) = k_1$ and $\mu_l(G_2) = k_2$.

Case 1: If we choose $k_1 \leq k_2$, such that we have $|W(G_1)| = k_1$. The position of resolving set in first layer of subgraph G_2, denoted by $(G_2)_1$, so we have distinct vertex representation in $(G_2)_1$. For the vertex in $(G_2)_2$ adjacent to the vertex in $(G_2)_1$, we have for$d_{(G_1 \times G_2)}((x_j,u),(x_1,u^*)) = d_{(G_1 \times G_2)}((x_j,u),(x_1,u)) + d_{((G_2)_1)}((x_1,u),(x_1,u^*))$ or $d_{G_1 \times G_2}((x_j,u),(x_1,u*)) > d_{(G_2)_1}((x_1,u),(x_1,u^*))$. Such that for two adjacent vertices of vertex in $(G_2)_1$ and $(G_2)_j$ have distinct representation. For two adjacent vertex $(x_1,u_i),(x_1,u_l) \in V((G_2)_1)$ where $d_{(G_2)_1}((x_1,u_i),(x_1,u^*)) \neq d_{(G_2)_1}((x_1,u_i),(x_1,u^*))$. If we have two adjacent vertices in $(G_2)_j$-layers for $(x_j,u_i),(x_j,u_l) \in V((G_2)_j)$ such that we have $d_{G_1 \times G_2}((x_j,u_i),(x_1,u^*)) = d_{G_1 \times G_2}((x_j,u_i),(x_1,u_i)) + d_{(G_2)_1}((x_1,u_i),(x_1,u^*))$ and $d_{G_1 \times G_2}((x_j,u_l),(x_1,u^*)) = d_{G_1 \times G_2}((x_j,u_l),(x_1,u_l)) + d_{(G_2)_1}((x_1,u_l),(x_1,u^*))$ and we know that $d_{G_1 \times G_2}((x_j,u_i),(x_1,u^*)) \neq d_{G_1 \times G_2}((x_j,u_l),(x_1,u^*))$ for $(x_1,u^*) \in W \subset V((G_2)_1)$.

Case 2: If we choose $k_2 < k_1$, such that we have $|W(G_2)| = k_2$. The position of resolving set in first layer of subgraph G_1, denoted by $(G_1)_1$, so we have distinct vertex representation in $(G_1)_1$. From **Case 1**, we have same characterization of the vertex representation.

Based on both cases, we can claim that $|W(G_1 \times G_2)| = k_1$ for $k_1 \leq k_2$ and $|W(G_1 \times G_2)| = k_2$ for $k_2 < k_1$. Thus, we get $|W(G_1 \times G_2)| = min\{k_1,k_2\}$. Such that, $\mu_l(G_1 \times G_2) \geq min\{\mu_l(G_1),\mu_l(G_2)\}$. □

The cartesian product of graph G and tree graph T with characterization for $\mu_l(G) = 1$ and we get the results as follows.

Theorem 3.1
Given that a connected graph G and a path P_n, $\mu_l(G \times P_n) = \mu_l(G)$

Proof: The graph $G \times P_n$ has n copies subgraph G_i, $1 \leq i \leq n$. Let W be a local resolving set of $G = G_i$ so that every vertices $u,v \in V(G)$ for u adjacent to v has different representation. If we assume that W is a set of $G \times P_n$, then we prove that W is local resolving set of $G \times P_n$,

i. We know that for every vertices $u \in W$ belong to in subgraph G_1 or first copy (first layer).
ii. Every two adjacent vertices $u,v \in V(G_1) - W$, has different representation. Since, a set W is the local resolving set of $G = G_1$.
iii. For every two adjacent vertices $u \in V(G_1) - W$ and $v \in V(G_j)$ such that for $w \in W$, $d(u,w) = d'$ and $d(v,w) = d(v,u) + d(u,w) = d(u,v) + d' > d' = d(u,w)$ which $d(u,w) \neq d(v,w)$. Thus, $r(u|W) \neq r(v|W)$.
iv. For G_i and G_j, $1 \leq i < j \leq n$. Choose two adjacent vertices $u \in V(G_i)$ and $v \in V(G_j)$ such that for $w \in W$, $d(u,w) = d(u,x) + d(x,w)$, $\forall x \in V(G_1)$. We know that $i < j$ such that $d(v,w) = d(v,u) + d(u,x) + d(x,w) > d(u,x) + d(x,w) = d(u,w)$ so $d(u,w) \neq d(v,w)$. Thus, $r(u|W) \neq r(v|W)$.
v. For every two adjacent vertices $u,v \in V(G_i)$ which have $d(u,w) = d(u,x) + d(x,w)$ and $d(v,w) = d(v,y) + d(y,w)$ where $x,y,w \in V(G_1)$ such that $d(x,w) \neq d(y,w)$ and $d(u,x) = d(v,y)$. We have $d(u,w) \neq d(v,w)$. Thus, $r(u|W) \neq r(v|W)$.

Based on five cases $(i) - (v)$, W is a local resolving set of $G \times P_n$. Thus, we have upper bound of local multiset dimension of $G \times P_n$ is $\mu_l(G \times P_n) \leq \mu_l(G)$.

Furthermore, we show that the lower bound of local multiset dimension of $G \times P_n$ is $\mu_l(G \times P_n) \geq \mu_l(G)$. Assume that $|W_{G \times P_n}| < |W_G|$, by taking $|W_{G \times P_n}| = |W_G| - 1$.

i. For every vertices $v \in W_{G \times P_n}$ belong to in subgraph G_1 such that there exists at least two adjacent vertices has same representation.
ii. Let $u,v \in V(G_1)$ where u adjacent to v, $d(u,w) = d(v,w)$. Thus, $r(u|W) = r(v|W)$. It is a contradiction.
iii. If some vertices of resolving set not all in subgraph G_1, then there is at least one vertex of W in G_i, $1 \leq i \leq n$.

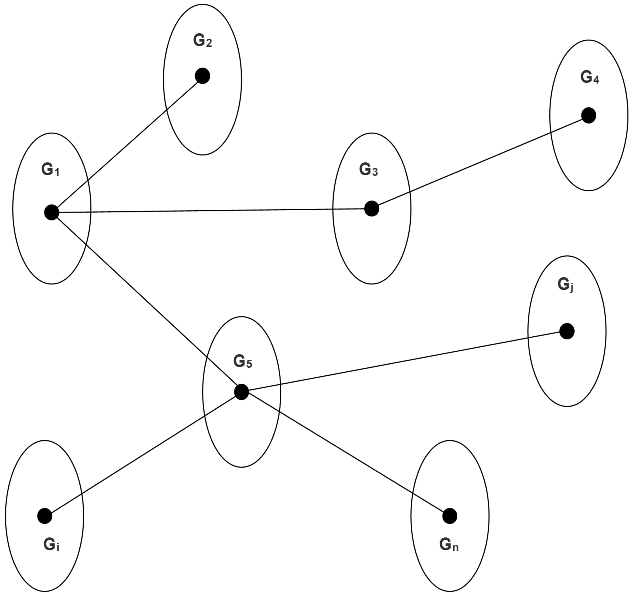

Figure 5. Illustration of $G \times T$

iv. For any two adjacent vertices $u, v \in G_i$, we have $d(u, w_i) = d(v, w_i)$ such that $d(u, w) = d(u, x) + d(x, w)$ and $d(v, w) = d(v, y) + d(y, w)$, we have $d(u, w) = d(v, w)$. Thus, $r(u|W) = r(v|W)$. It is a contradiction.

Based cases above, we have the local resolving set of $G \times P_n$ at least $|W_G|$ or $|W_{G\times P_n}| \geq |W_G|$. Hence, we have lower bound of local multiset dimension of $G \times P_n$ is $\mu_l(G \times P_n) \geq \mu_l(G)$. Thus, the local multiset dimension of $G \times P_n$ is $\mu_l(G \times P_n) = \mu_l(G)$. □

Theorem 3.2
Given that a connected graph G and a tree T, $\mu_l(G \times T) \leq \mu_l(G)$

Proof: The graph $G \times T$ has n copies subgraph G_i, $1 \leq i \leq n$. Let W be a local resolving set of $G = G_i$ so that every vertices $u, v \in V(G)$ for u adjacent to v has different representation. If we assume that W is a set of $G \times T$, then we prove that W is local resolving set of $G \times T$,

i. We know that for every vertices $u \in W$ belong to in subgraph G_1 or first copy (first layer).
ii. Every two adjacent vertices $u, v \in V(G_1) - W$, has different representation. Since, a set W is the local resolving set of $G = G_1$.
iii. For G_i and G_j, $1 \leq i < j \leq n$. Choose two adjacent vertices $u \in V(G_i)$ and $v \in V(G_j)$ such that for $w \in W$, $d(u, w) = d(u, x) + d(x, w)$, $\forall x \in V(G_1)$. We know that $i < j$ such that $d(v, w) = d(v, u) + d(u, x) + d(x, w) > d(u, x) + d(x, w) = d(u, w)$ so $d(u, w) \neq d(v, w)$. Thus, $r(u|W) \neq r(v|W)$.
iv. For every two adjacent vertices $u, v \in V(G_i)$ which have $d(u, w) = d(u, x) + d(x, w)$ and $d(v, w) = d(v, y) + d(y, w)$ where $x, y, w \in V(G_1)$ such that $d(x, w) \neq d(y, w)$ and $d(u, x) = d(v, y)$. We have $d(u, w) \neq d(v, w)$. Thus, $r(u|W) \neq r(v|W)$.

Based on four cases $(i) - (iv)$, W is a local resolving set of $G \times T$. Thus, we have upper bound of local multiset dimension of $G \times T$ is $\mu_l(G \times T) \leq \mu_l(G)$. □

Corollary 3.2
Given that a connected graph G and a tree T. For $\mu_l(G) = 1$, $\mu_l(G \times T) = 1$

Proof: Alfarisi, *et. al.* [8] determined the local multiset dimension of T is 1. If the local multiset dimension of G is 1, then every two adjacent vertices have distinct representation.

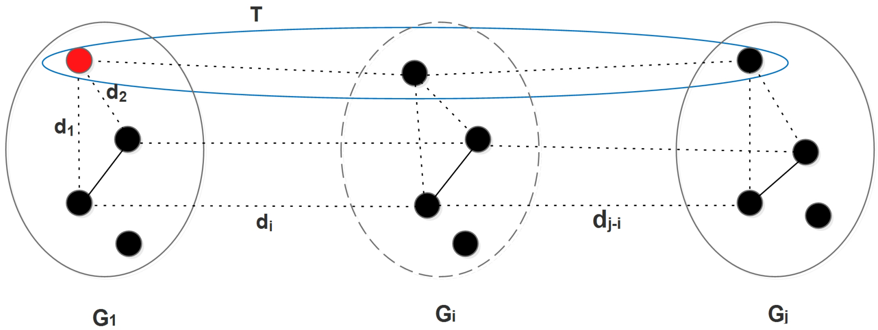

Figure 6. Illustration of $G \times T$ for $d_1 \neq d_2$

From Lemma 3.2 that $\mu_l(G \times T) \geq min\{\mu_l(G), \mu_l(1)\} \geq min\{1, 1\} \geq 1$. Furthermore, we can show $\mu_l(G \times T) \leq 1$ as follows.

i. the graph $G \times T$ has $|V(T)|$-copies of subgraph G, namely $G_j, 1 \leq j \leq |V(T)|$
ii. we choose the local resolving set in first copy G_1, namely $W = \{(x_1, u^*)\}$
iii. we know that $\mu_l(G) = 1$, the distance of two adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V(G_1) - W$ is $d((x_1, u_k), (x_1, u^*)) \neq d((x_1, u_l), (x_1, u^*))$ for $1 \leq k, l \leq V(G)$.
iv. every two adjacent vertices $(x_j, u_k), (x_j, u_l) \in V(G_j)$ has distinct representation, such that $d((x_j, u_k), (x_1, u^*)) \neq d((x_j, u_l), (x_1, u^*))$ where $d((x_j, u_k), (x_1, u^*)) = d((x_j, u_k), (x_1, u_k)) + d((x_1, u_k), (x_1, u^*))$ and $d((x_j, u_l), (x_1, u^*)) = d((x_j, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$.
v. every two adjacent vertices $(x_i, u_l) \in V(G_i)$, $(x_j, u_l) \in V(G_j)$ has distinct representation, such that $d((x_i, u_l), (x_1, u^*)) \neq d((x_j, u_l), (x_1, u^*))$ where $d((x_i, u_l), (x_1, u^*)) = d((x_i, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$ and $d((x_j, u_l), (x_1, u^*)) = d((x_j, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$.

Thus, we obtain $\mu_l(G \times T) \leq 1$. Thus, $\mu_l(G \times T) = 1$, for $\mu_l(G) = 1$ and any tree T. □

Lemma 3.3
For $\mu_l(G) \neq 1$ and any tree T, $\mu_l(G \times T) \geq 2$.

Proof: If local multiset dimension $\mu_l(G) \neq 1$, then we have local resolving set $|W| \geq 2$. From Lemma 3.2, it states that $\mu_l(G \times T) \geq min\{\mu_l(G), 1\} \geq 1$. Assume that $|W| = 1$. There is at least two adjacent vertices which have same representation. Choose the local resolving set in G_1. Every adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V(G_1) - W$ has some distances, namely $d((x_1, u_k), (x_1, u^*) = d((x_1, u_l), (x_1, u^*)$. Thus, the cardinality of the local resolving set of $G \times T$ is $|W| \neq 1$, and the local multiset dimension of $\mu_l(G \times T) \geq 2$. □

Theorem 3.3
For $\mu_l(G) = 1$ and m is even, $\mu_l(G \times C_m) = 1$.

Proof: Alfarisi, *et. al.* [8] determined the local multiset dimension of C_m with m is odd is 1. If the local multiset dimension of G is 1, then every two adjacent vertices has distinct representation.

From Lemma 3.2 that $\mu_l(G \times C_m) \geq min\{\mu_l(G), \mu_l(C_m)\} \geq min\{1, 1\} \geq 1$. Furthermore, we can show $\mu_l(G \times C_m) \leq 1$ as follows.

i. the graph $G \times C_m$ has $|V(C_m)|$-copies of subgraph G, namely $G_j, 1 \leq j \leq |V(C_m)|$
ii. we choose the local resolving set in first copy G_1, namely $W = \{(x_1, u^*)\}$
iii. we know that $\mu_l(G) = 1$, the distance of two adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V(G_1) - W$ is $d((x_1, u_k), (x_1, u^*)) \neq d((x_1, u_l), (x_1, u^*))$ for $1 \leq k, l \leq V(G)$.

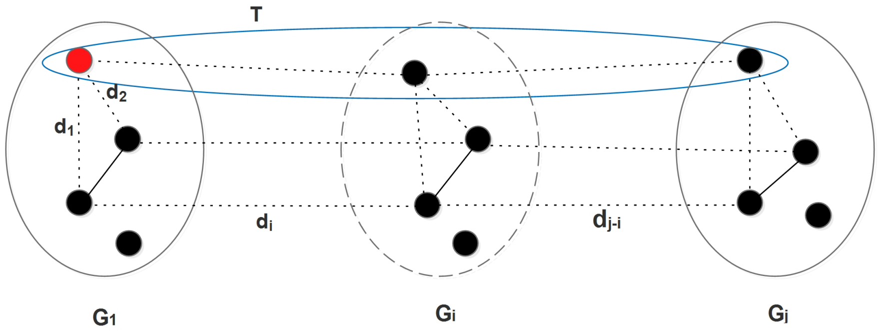

Figure 7. Illustration of $G \times T$ for $d_1 \neq d_2$

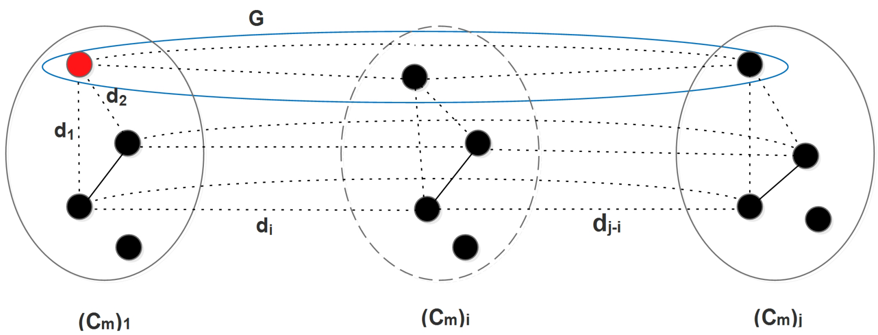

Figure 8. Illustration of $G \times C_m$ for $d_1 = d_2$

iv. every two adjacent vertices $(x_j, u_k), (x_j, u_l) \in V(G_j)$ has distinct representation such that $d((x_j, u_k), (x_1, u^*)) \neq d((x_j, u_l), (x_1, u^*))$ where $d((x_j, u_k), (x_1, u^*)) = d((x_j, u_k), (x_1, u_k)) + d((x_1, u_k), (x_1, u^*))$ and $d((x_j, u_l), (x_1, u^*)) = d((x_j, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$.

v. every two adjacent vertices $(x_i, u_l) \in V(G_i)$, $(x_j, u_l) \in V(G_j)$ has distinct representation such that $d((x_i, u_l), (x_1, u^*)) \neq d((x_j, u_l), (x_1, u^*))$ where $d((x_i, u_l), (x_1, u^*)) = d((x_i, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$ and $d((x_j, u_l), (x_1, u^*)) = d((x_j, u_l), (x_1, u_l)) + d((x_1, u_l), (x_1, u^*))$.

Thus, we obtain that $\mu_l(G \times C_m) \leq 1$. It concludes that $\mu_l(G \times C_m) = 1$, for $\mu_l(G) = 1$ and m is even. □

Lemma 3.4
For ($\mu_l(G) = 1$ and m is odd) or ($\mu_l(G) \neq 1$ and $m \geq 3$), $\mu_l(G \times C_m) \geq 2$.

Proof: Based on Lemma 3.1 that $\mu_l(G_1 \times G_2) = \min\{\mu_l(G_1), \mu_l(G_2)\}$. If one of both graph has local multiset dimension at least one, then $\mu_l(G \times C_m) \geq 1$. We try construct of the sharpest lower bound of $G \times C_m$ for m is odd or ($\mu_l(G) \neq 1$ and m is even) as follows.
Case 1: For $\mu_l(G) = 1$ and m is odd
We know that $\mu_l(C_m) = 3$ for n is odd and $\mu_l(G) = 1$, based Lemma 3.2 that $\mu_l(G \times C_m) \geq min\{1, 3\} \geq 1$. Assume that $|W| = 1$, there is at least two adjacent vertices which have same representation. Choose the local resolving set in $(C_m)_1$, then every adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V((C_m)_1) - W$ has some distance namely $d((x_1, u_k), (x_1, u^*) = d((x_1, u_l), (x_1, u^*)$. Thus, the cardinality of the local resolving set of $G \times C_m$ is $|W| \neq 1$, such that the local multiset dimension of $\mu_l(G \times C_m) \geq 2$. This illustration can be seen in Figure 8

Case 2: For $\mu_l(G) \neq 1$ and m is odd
If local multiset dimension $\mu_l(G) \neq 1$, then we have local resolving set $|W| \geq 2$. From Lemma 3.2 that $\mu_l(G \times$

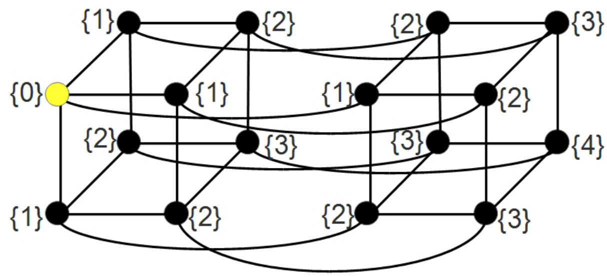

Figure 9. Illustration of HQ_4

$C_m) \geq min\{\mu_l(G), 3\} \geq 2$. Assume that $|W| = 1$ such that there is at least two adjacent vertices which have same representation. Choose the local resolving set in G_1, then every adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V(G_1) - W$ have some distance namely $d((x_1, u_k), (x_1, u^*) = d((x_1, u_l), (x_1, u^*)$. Thus, the cardinality of the local resolving set of $G \times C_m$ is $|W| \neq 1$, such that the local multiset dimension of $\mu_l(G \times C_m) \geq 2$.
Case 3: For $\mu_l(G) \neq 1$ and m is even
If local multiset dimension $\mu_l(G) \neq 1$, then we have local resolving set $|W| \geq 2$. From Lemma 3.2 that $\mu_l(G \times C_m) \geq min\{\mu_l(G), 1\} \geq 1$. Assume that $|W| = 1$ such that there is at least two adjacent vertices which have same representation. Choose the local resolving set in G_1, then every adjacent vertices in $(x_1, u_k), (x_1, u_l) \in V(G_1) - W$ have some distance namely $d((x_1, u_k), (x_1, u^*) = d((x_1, u_l), (x_1, u^*)$. Thus, the cardinality of the local resolving set of $G \times C_m$ is $|W| \neq 1$, such that the local multiset dimension of $\mu_l(G \times C_m) \geq 2$. The local multiset dimension of $G \times C_m$ is $\mu_l(G \times C_m) \geq 2$. □

Next, we study a Hypercube graph, denoted by HQ_n. Hypercube graph is the graph formed from the vertices and edges of an n-dimensional hypercube. It is the n-fold Cartesian product of the two-vertex complete graph, and decomposed into two copies of HQ_{n-1} connected to each other by a perfect matching.

Theorem 3.4
For $n \in N \cup \{0\}$, $\mu_l(HQ_n) = 1$.

Proof: Hypercube graph satisfies $HQ_n = HQ_{n-1} \times P_2$ for $n \geq 0$. For $n = 0$, we have HQ_0 isomorphic to K_1 or trivial graphs. The local multiset dimension of K_1 is $\mu_l(HQ_0) = 1$. To prove this theorem, we can use a mathematical induction or recursive technique below.

i. For $n = 1$, we have $HQ_1 = HQ_0 \times P_2$. Based on Lemma 3.2, it holds $\mu_l(HQ_1) = \mu_l(HQ_0 \times P_2) = 1$ since $\mu_l(HQ_0) = 1$.
ii. For $n = 2$, we have $HQ_2 = HQ_1 \times P_2$. Based on Lemma 3.2, it holds $\mu_l(HQ_2) = \mu_l(HQ_1 \times P_2) = 1$ since $\mu_l(HQ_1) = 1$.

...

iii. Assume that for $n = k$, we have $HQ_k = HQ_{k-1} \times P_2$. Based on Lemma 3.2, it holds $\mu_l(HQ_k) = \mu_l(HQ_{k-1} \times P_2) = 1$ since $\mu_l(HQ_{k-1}) = 1$.
iv. For $n = k + 1$, we prove that $\mu_l(HQ_k \times P_2) = 1$?. From point iii, we have $\mu_l(HQ_k) = 1$. Recalling Lemma 3.2, It implies that $\mu_l(HQ_{k+1}) = \mu_l(HQ_k \times P_2) = 1$ since $\mu_l(HQ_k) = 1$.

Thus, the local multiset dimension of hypercube HQ_n is $\mu_l(HQ_n) = 1$, for $n \in N \cup \{0\}$. Figure 9 is an illustration of local multiset dimension of hypercube graphs for HQ_4. □

Theorem 3.5
$\mu_l(AHQ_n) = 1$, for $n \geq 2$

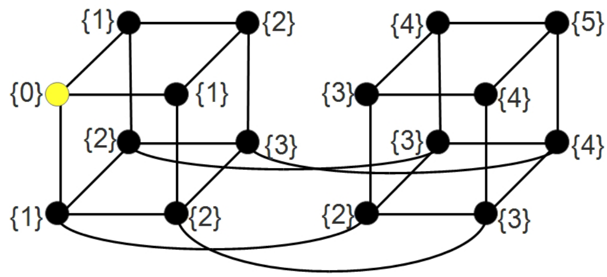

Figure 10. Illustration of AHQ_4

Proof: Almost hypercube graph satisfies $AHQ_n = (HQ_{n-1} \times P_2) - \{e\}$ for $n \geq 3$, where e is correspondence edge of subgraph $(HQ_{n-1})_1$ and $(HQ_{n-1})_2$. We know that $HQ_{n-1} \times P_2$ has two isomorphic graphs $(HQ_{n-1})_1$ and $(HQ_{n-1})_2$ with $\{e\}$ is the correspondence edge set. Based on Theorem 3.4 that $\mu_l(HQ_{n-1}) = 1$ such that for two adjacent vertices $u, v \in V((HQ_{n-1})_1)$ has the distinct representation namely $r(u|W) \neq r(v|W)$ for W in the first copy of HQ_{n-1}.

i. If we choose two adjacent vertices in $u, v \in V((HQ_{n-1})_2)$ with their vertices are correspondence by edge to two adjacent vertices in $x, y \in V((HQ_{n-1})_1)$ where $d(x, w) \neq d(y, w)$ for $w \in W$. We have the distance $d(u, x) = d(v, y) = 1$ and $d(u, w) = d(u, x) + d(x, w) = 1 + d(x, w)$, $d(v, w) = d(v, y) + d(y, w) = 1 + d(y, w)$ then $d(u, w) \neq d(v, w)$. Thus, we have $r(u|W) \neq r(v|W)$.
ii. If we choose two adjacent vertices in $u, v \in V((HQ_{n-1})_2)$ with their vertices are not correspondence by edge to two adjacent vertices in $x, y \in V((HQ_{n-1})_1)$ where $d(x, w) \neq d(y, w)$ for $w \in W$. But We have the distance $d(u, x) > 1, d(v, y) > 1$ and $d(u, w) = d(u, x) + d(x, w) = d + d(x, w)$, $d(v, w) = d(v, y) + d(y, w) = d^* + d(y, w)$ then $d(u, w) \neq d(v, w)$. Thus, we have $r(u|W) \neq r(v|W)$.
iii. If we choose two adjacent vertices in $u, v \in V((HQ_{n-1})_2)$ with one of them are correspondence by edge to adjacent vertices in $x \in V((HQ_{n-1})_1)$ for $w \in W$. But we have the distance $d(u, x) = 1, d(v, x) = d(u, v) + d(v, x) = 1 + 1 = 2$ and $d(u, w) = d(u, x) + d(x, w) = 1 + d(x, w)$, $d(v, w) = d(v, y) + d(y, w) = 2 + d(y, w)$ then $d(u, w) \neq d(v, w)$. Thus, we have $r(u|W) \neq r(v|W)$.

Based on three cases above, the local resolving set of almost hypercube graphs $|W| = 1$. Thus, the local multiset dimension of almost hypercube graphs AHQ_n is $\mu_l(AHQ_n) = 1$, for $n \geq 3$. Figure 10 is an illustration of local multiset dimension of almost hypercube graphs for AHQ_4. □

Kautz graphs $K(d, n)$ for $d \geq 2$ and $n \geq 2$, is defined as follows. The vertex set of $K(d, n)$ is $V(K(d, n)) = \{x_1, x_2, \ldots, x_n | x_i \in \{0, 1, 2, \ldots, d\}, x_i \neq x_{i+1},\ i = 1, 2, \ldots, n-1\}$ and the edge set $E(K(d, n))$ consists of all edges from one vertex $x_1, x_2, \ldots, x_n$ to d order vertices $x_1, x_2, \ldots, x_n, \alpha$, where $\alpha \in \{1, 2, \ldots, d\}$ and $\alpha \neq x_n$. It is clear that $K(dn)$ is d-regular, $|V(d, n)| = d^n + d^{n1}$ and $|E(dn)| = d^{n+1} + d^n$. Moreover, $K(d, n)$ has $\frac{d(d+1)}{2}$ pairs of symmetric edges. The Kautz undirected graph, denoted by $UK(d, n)$, is an undirected graph obtained from $K(d, n)$ by deleting the orientation of all edges and omitting multiple edges. It is clear that $UK(dn)$ has $d^{n+1} + d^n - \frac{d(d+1)}{2}$ edges, the maximum degree is $2d$ for $n \geq 3$ and the minimum degree is $2d - 1$ for $n \geq 2$.

The Kautz undirected graph of cycle, denoted by $UK(C_n)$ is connected graphs with 3-regular isomorphic to $C_n \times P_2$. The vertex set and edge set of C_n respectively, are $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(C_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. The vertex set of $UK(C_n)$ is $V(UK(C_n)) = \{x_{i,(i+1)\text{mod}n}, x_{(i+1)\text{mod}n,i} : 1 \leq i \leq n\}$ and edge set of $UK(C_n)$ is $E(UK(C_n)) = \{x_{i,(i+1)\text{mod}n}x_{(i+1)\text{mod}n,i}, x_{i,(i+1)\text{mod}n}x_{(i+1)\text{mod}n,(i+2)\text{mod}n}, x_{(i+1)\text{mod}n,i}x_{(i+2)\text{mod}n,(i+1)\text{mod}n}; 1 \leq i \leq n\}$.

Corollary 3.3
For $n \geq 4$,

$$\mu_l(UK(C_n)) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

Furthermore, we characterization of relationship between multiset dimension and local multiset dimension of graphs.

Remark 3.1
The relationship of multiset dimension and local multiset dimension of graphs G, $gap(md(G), \mu_l(G) = \infty$.

4. Conclusion

In this paper we have given an result the lower bound of multiset dimension and local multiset dimension of graphs. Hence the following problem aries naturally.

Open Problem 4.1
Determine the local multiset dimension of family graph namely family tree, unicyclic, regular graphs, and others.

Open Problem 4.2
Determine the local multiset dimension of operation graph namely corona product, joint, comb product, and others.

Acknowledgments

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