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Applications of Some Rating Methods to Solve Multicriteria Decision-Making Problems

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Abstract This study proposes a new approach for the solution of multicriteria decision-making problems. The proposed approach is based on using rating/ranking methods. Particularly, in this paper, we investigate the possibility of applying Massey, Colley, Keener, offence-defence, and authority-hub rating methods, which are successfully used in various fields. The proposed approach is useful when no decision-making authority is available or when the relative importance of various criteria has not been previously evaluated. The proposed approach is tested with an example problem to demonstrate its viability and suitability for application.

Keywords Decision making problem, ranking theory, multicriteria optimisation

AMS 2010 subject classifications:90B50

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1. Introduction

This paper proposes a novel approach for solving multi criteria decision-making (MCDM) problems based on using rating/ranking methods. The proposed approach is particularly useful when no decision-making authority is available or when the relative importance of various criteria has not been previously evaluated. The essence of the proposed method is briefly explained. The multi criteria formulation is the typical starting point for theoretical and practical analyses of decision-making problems. Thus, the definition of Pareto optimality and a vast arsenal of different Pareto optimization methods widely used for decision-making purposes. Correspondingly, the literature on Pareto optimization methods for decision-making very extensive and we can indicate here only an insignificant number of theoretical and applied works: see e.g. ([1] [2] [3] [4] [5] [6] [7]).

However, unlike single-objective optimizations, a characteristic feature of Pareto optimality is that the set of Pareto-optimal alternatives (i.e., the set of efficient alternatives) is usually 'large'. In addition, all these Pareto-optimal alternatives must be considered mathematically equal. Hence, the problem of choosing a specific Pareto-optimal alternative for implementation arises because the final decision usually must be unique. Thus, additional factors must be considered to aid a decision-maker in the selection of specific or more favorable alternatives from the set of Pareto-optimal solutions.

In this paper, we show that, for any MCDM problem that can be defined naturally, a special kind of score matrix (and some its modifications) can be used to introduce a rating/rank on the set of alternatives. The best option using this rating/rank alternative can be declared a 'solution' of the MCDM problem under consideration. We investigate the possibility of applying different rating/ranking methods. These are a number of well-known rating methods

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that have already been successfully demonstrated in various fields : sports team rating, web ranking, citation index development, etc. Hence, these rank/rating procedures can be considered well-established for sufficiently difficult rating/ranking problems. Because the proposed score matrix reflects the natural relationship between the alternatives and because the rank/rating procedures are well-tested, the alternative declared above as a ‘ solution ’ can be considered an ‘ objective solution ’ for the considered MCDM problem.

The proposed approach is illustrated using a real-world example MCDM problem of selecting the best material for a sailboat mast. This illustration explains the viability and applicability of the proposed approach to MCDM problems. The solutions of the illustrative example are acceptable, and the proposed method in this study yields a competitive ranking of alternatives for the considered MCDM.

The rest of this paper is structured as follows. In Section 2, preliminaries regarding MCDM and the rating methods are presented, and the proposed methodology is described. Section 3 considers an illustrative example, and Section 4 presents the conclusion.

2. Background

2.1. Preliminaries

2.1.1. General notation For a natural number n, we denote an n -dimensional vector space by $\mathbb{R}^n$ and $\mathbb{N}(n) = \{1, ..., n\}$ .The following notation is used for the special vectors

$$e_k = (0, \ldots, \underset{(k)}{1}, \ldots, 0) \in R^n,\ k = 1, ..., n;$$

$$0_n = \underbrace{(0, \ldots, 0)}_{n} \in \mathbb{R}^n, \quad 1_n = \underbrace{(1, \ldots, 1)}_{n} = \sum\nolimits_{k=1}^{n} e_k \in \mathbb{R}^n$$

and the following sets :

$$\mathbb{R}^n_+ = \{\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n | \xi_k \geq 0\ k = 1, ..., n\}, \quad \Delta_n = \left\{\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n_+ | \sum_{k=1}^{n} \xi_k \ = 1\right\}$$

$$\overset{\circ}{\Delta}_n = \{\xi = (\xi_1, ..., \xi_n) \in \Delta_n | \xi_k > 0,\ k = 1, ..., n\}$$

If not otherwise mentioned, we identify the finite set with the set $\mathbb{N}(n)$,where $n = |A|$ is the capacity of set A.We also identify the matrix $\Pi \in \mathbb{R}^{n\times m}$ with the map $\Pi : \mathbb{N}(n) \times \mathbb{N}(m) \to \mathbb{R}$. For the matrix $\Pi \in \mathbb{R}^{n\times m}$,we denote its transpose by $\Pi^T \in \mathbb{R}^{m\times n}$. For a vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we introduce the matrix $\Lambda(\xi) = [\lambda_{ij}]_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ such that $\lambda_{ii} = \xi_i,\ i = 1, \ldots, n$, and $\lambda_{ij} = 0,\ i, j = 1, \ldots, n, i \neq j$.The Heaviside function is defined as follows:

$$\chi(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad x \in \mathbb{R}.$$

Now, we give a brief overview of some basic notations of ranking theory (see e.g. [8]) which are necessary for further consideration. For a natural number N, the $N \times N$ matrix $S = [S_{ij}],\ 1 \leq i, j \leq N$ is a score matrix if $S_{ij} \geq 0,\ S_{ii} = 0,\ \ 1 \leq i, j \leq N$. Note that we can interpret the elements of $\mathbb{N}(N)$ as athletes, and for each pair of athletes $(i, j),\ \ 1 \leq\ i, j \leq N$ we assume that the joint match $M(i, j)$ includes Ngames. We interpret entry $S_{ij},\ 1 \leq i, j \leq N$ as the number total wins of athlete i in matches $M(i, j)$ and the result of the match $M(i, j)$ is S_{ij} wins of athlete i, S_{ji} wins of athlete j,and $(N - S_{ij} - S_{ji})$ draws. We also introduce the matrix $(G = [G_{ij}(S)],\ 1 \leq i, j \leq N;\ G = S + S^*)$and the function $g_i(S) = \sum_{j=1}^{N} G_{ij}(S),\ 1 \leq i \leq N.$

2.1.2. Pareto optimality The following concepts is drawn from a general treatment of the multicriteria optimization theory. Alternatives are denoted by$A = \{a_1, ..., a_m\}$, criteria are denoted by $c_j : A \to R, j = 1, \ldots, n;\ \ C = \{c_1, ..., c_n\}$, and the MCDM problem is ordered pair $< A, C >$. Obviously, we may assume that the criteria are normalized such that the lower value is preferable for each criterion and the goal of the decision-making procedure is to minimize all criteria simultaneously (see e.g. [1]). Later in the article, we will adhere to this assumption. Furthermore, A is the set of admissible alternatives, and map $\overrightarrow{c} = (c_1, \ldots, c_n) : A \to \mathbb{R}^n$ is the criterion map (correspondingly, $\overrightarrow{c}(A) \subset \mathbb{R}^n$ is the set of admissible values of criteria). An alternative $a_* \in A$ is Pareto optimal (i.e., efficient) if no $a \in A$ exists such that $c_j(a) \leq c_j\,(a_*)$ for all $j \in N\,(n)$ and $c_k(a) < c_k(a_*)$ for some $k \in \mathbb{N}\,(n)$ .The set of all efficient alternatives is denoted as A_e and is called the Pareto set. Correspondingly, $\overrightarrow{c}(A_e)$ is called the efficient front.

2.2. Rating Methods

In this section, we present various ranking methods. The ranking methods presented here were originally proposed in different fields, namely the Massey, Colley, Keener, and offence-defence rating methods originated in the context of team ranking in sports, see ([9] [10] [11]), and the authority-hub method originated in the context of citation index development and for web ranking purposes, see ([12] [13] [14]) .

2.2.1. Massey method To describe the Massey method, we introduce the following notation: m is the number of teams, n_G is the number of the games that have already been played, $r_i^M,\ i = 1, \ldots, m$ is Massey ' s rating of the i team and $r^M = (r_1^M, \ldots, r_m^M)$is Massey ' s rating vector.

The result of the k^{th} game is described as an ordered quadruple $(t^1(k), s^1(k), t^2(k), s^2(k))$ where $t^1(k), t^2(k)$ are teams who played the k^{th} game, $s^1(k), s^2(k)$ are the scores of these teams in the game, and the margin of the k^{th} game (we also consider draws) is assumed to be $y_k = s^1\,(k) - s^2\,(k) \geq\ 0,\ \ y =\ (y_1, \ldots, y_m),\ k = 1, ..., n_G$.

The main assumption of the Massey method is that the rating vector $r^M = (r_1^M, \ldots, r_m^M)$ can be obtained through least squares approximation, as follows:

$$\sum\nolimits_{k=1}^{n_G} \left(y_k - (r^M_{t^1(k)} - r^M_{t^2(k)}) \right)^2 = ||y - Xr^M||^2 \to \underset{r^M}{min},$$

where X is the correspondingly defined matrix. Hence, we can conclude that $\left(X^T X\right) r^M = X^T y$.

To avoid ill definiteness of the matrix, one of the possibilities is that a row (for definiteness — the last one) in the matrix $X^T X$must be replaced with a row of all ones, and the corresponding component of the vector $X^T y$ must be replaced with a zero. The matrix M and the vector v^M that are obtained in such a way are called the Massey matrix and Massey vector, respectively. After this, Massey 's rating vector, r^M, is defined as solution of the linear equation $Mr^M = v^M$.

2.2.2. Colley method The Colley method can be described based on the following notation: m is the number of teams, w_i is the number of wins for team i, l_i is the number of losses for team i, n_i is the total number of games played by team i, and n_{ij} is the number of times teams i, j played, $i, j = 1, \ldots, m$.

From this data, the following objects can be introduced: the Colley matrix, $C = [C_{ij}]_{i,j=1,\ldots,n}$, and the Colley vector, $v^C = (v_1^C, \ldots, v_m^C)$,where

$$C_{ij} = \begin{cases} 2 + n_i, & i = j; \\ -n_{ij} & i \neq j; \end{cases}, \quad v_i^C = 1 + \frac{1}{2}\left(w_i - l_i\right), \quad i, j = 1, \ldots, m.$$

The Colley rating vector, r^C, is obtained as a solution of the equation $Cr^C = v^C$.

2.2.3. Keener method The Keener method can be described based on the following notation: m is the number of teams, and S_{ij} is the score produced by team i against team $i, j = 1, \ldots, m$. Keener's matrix $K = [K_{ij}]_{i,j=1,\ldots,m}$ is defined as follows:

$$K_{ij} = \begin{cases} h\left(\frac{1+S_{ij}}{2+S_{ij}+S_{ji}}\right) & if\ teams\ i\ and\ j\ played\ each\ other; \\ 0, & otherwise; \end{cases}, \quad i, j = 1, ..., m,$$

where

$$h(x) = \frac{1}{2} + \frac{1}{2} sgn\left(x - \frac{1}{2}\right)\sqrt{|2x-1|}.$$

Correspondingly, the rating vector for Keener method, r^K, is obtained as a solution of the eigenvalue problem $Kr^K = \lambda r^K$. If the matrix K is irreducible, the Perron-Frobenius theorem guarantees the existence and uniqueness of the ratings vector r^K, see ([11]) .

2.2.4. Offence-defence method We assume that, given a set A of m objects that should be ranked, a (row) stochastic matrix P is defined for reflecting the set of available data on pairwise comparisons of the considered objects. The offense and defense vectors are defined through the following iteration process: $o^{(k)} = P^T\Lambda(d^{(k-1)})^{-1}1_m,\ d^{(k)} = P\Lambda(o^{(k)})^{-1}1_m$, and $d^{(0)} = 1/m1_m^T$. It is known that, under certain conditions, the convergence is guaranteed, and

$$o = (o_1, \dots, o_m) = \lim_{k\to+\infty} o^{(k)},\quad d = (d_1, \dots, d_m) = \lim_{k\to+\infty} d^{(k)}$$

$$o = P^T\Lambda(d)^{-1}1_m, \quad d = P\Lambda(o)^{-1}1_m,$$

(we can assume that $o, d \in \overset{\circ}{\Delta}_m$.). The aforementioned 'certain condition' is that matrix P has total support. Checking whether a sufficiently large matrix has total support is a difficult task, and it is convenient to realise a small perturbation of matrix P i.e. instead of matrix P we use matrix $P + \varepsilon 1_m 1_m^T$ with sufficiently small $\varepsilon > 0$,. The offense and defence vectors $o,\ d$ can be considered as rating vectors of elements of set A. Moreover, by equalities, $r_i^{od} = o_i/d_i,\ i = 1, \dots, m$ can be defined new rating vector $r^{od} = (r_1^{od}, \dots, r_m^{od})$, called the offence-defence (aggregate) rating vector .

2.2.5. Authority-hub method As in the previous case, we assume that, given set A of m objects, which should be ranked, a (row) stochastic matrix P is defined for reflecting the set of available data on pairwise comparisons of the considered objects. The authority and hub vectors $a,\ h$ can be defined through the following iteration process: $a^{(k)} = P^TPa^{(k-1)},\quad h^{(k)} = PP^Th^{(k-1)}$. If matrices $P^TP,\ PP^T$ are primitive, this iteration process converges to the eigenvectors of these matrices, respectively:

$$a = (a_1, \dots, a_m) = \lim_{k\to+\infty} a^{(k)}, \quad h = (h_1, \dots, h_m) = \lim_{k\to+\infty} h^{(k)};\ \ a = P^TP,\ h = PP^Th,$$

(assuming that $a, h \in \overset{\circ}{\Delta}_m$). If matrices $P^TP,\ PP^T$ are irreducible but not primitive, the above-described iterative process cannot be used for determining vectors $a,\ h$. However, dominant eigenvectors for matrices $P^TP,\ PP^T$exist and are unique up to multiplication by a scalar. The authority and hub vectors $a,\ h$ can be considered as ranking vectors and will be named further as authority rating vector and hub rating vector, respectively.

3. Proposed Methods

In this section, we show that, with each MCDM problem, special matrices are associated, and these matrices can be successfully utilized for ranking alternatives in the MCDM problem. Namely, we will show how these matrices can be used in the ranking methods described above. In the final part of this section, we provide a general formulation of the proposed approach for solving MCDM problems.

We assume further that an MCDM problem $< A, C >,\ A = \{a_1, \dots, a_m\}, C = \{c_1, \dots, c_n\}$is given, and the decision-making goal is to minimize the criteria simultaneously.

3.1. Special Matrices Associated with an MCDM Problem

We propose a special construction of the score matrices of alternatives, S^A, for the MCDM problem $< A, C >$. For clarity, we imagine that the elements of A are athletes who conduct matches with each other and that, for each pair of athletes $a, a^{'} \in A$ their match $M(a, a^{'})$includes m games. Now, for any $a, a^{'} \in A$,we define the following:

$$S^A(a, a^{'}) = \sum\nolimits_{c \in C} s_c^A(a, a^{'}), \;\; s_c^A(a, a^{'}) = \begin{cases} 1, & c(a) < c(a^{'}); \\ 0, & c(a) \geqslant c(a^{'}); \end{cases} \;\; \forall c \in C.$$

Thus, the equality $s_c^A(a, a^{'}) = 1$ means that $c(a) < c(a^{'})$ for criterion $c \in C$ and the alternative (athlete) a receives one point (i.e., athlete $a$wins a game $c \in C$ in match $M(a, a^{'})$). Correspondingly, $S^A(a, a^{'})$indicates the number of total wins of alternative (athlete) a in match $M(a, a^{'})$. An alternative a has defeated an alternative $a^{'}$if $S^A(a, a^{'}) > S^A(a^{'}, a)$. In addition, the result of match $M(a, a^{'})$ is $S^A(a, a^{'})$ wins of the alternative a (losses of alternative $a^{'}$), $S^A(a^{'}, a)$ wins of the alternative $a^{'}$(losses of alternative a), and number of draws $\left(m - \; S^A(a, a^{'}) - \; S^A(a^{'}, a)\right)$. Obviously, $m \geq S^A(a, a^{'}) \geq 0, \;\; S^A(a, a) = 0, \;\;\; \forall a, a^{'} \in A$, and matrix $S^A = \left[S^A(a, a^{'})\right]$is the score matrix.

Now, we can define matrix $\Pi = \left[\Pi(a, a^{'})\right]_{a, a^{'} \in A}$,the adjacency matrix for the MCDM problem $< A, C >$, where

$$\Pi(a, a^{'}) = \begin{cases} S^A(a, a') + \left(m - S^A(a, a^{'}) - S^A(a^{'}, a)\right), & a \neq a^{'}; \\ 0, & a = a; \end{cases} \qquad \forall a, a^{'} \in A.$$

The matrix $\Pi = \left[\Pi(a, a^{'})\right]_{a, a^{'} \in A}$can be transformed into a (row) stochastic matrix : $P(\Pi) = \Lambda(\Pi 1_m + \pi(\Pi))^{-1}\Pi + \pi(\Pi)\xi^T$

Where (usually) $\xi = (1/m)1_m$, and $\pi(\Pi) = (\pi_1(\Pi), \ldots, \pi_m(\Pi))$is a vector defined as follows:

$$\pi_i(\Pi) = \begin{cases} 1, & if \; i^{th} \; row \; \Pi \; is \; 0_m; \\ 0, & otherwise; \end{cases} \; .$$

To justify its name, matrix Π can be interpreted as an adjacency matrix for a directed graph $\Gamma(A, C)$ associated with the MCDM problem $< A, C >$ and, correspondingly, matrix $P(\Pi)$ can be interpreted as a transition probability matrix for the Markov chain determined by the graph $\Gamma(A, C)$.

3.2. Adaptation of the Ranking Methods to the MCDM Problems

Below, we assume that $< A, C >, \;\; A = \{a_1, ..., a_m\}, C = \{c_1, .., c_n\}$,is an MCDM problem under consideration and that $S^A = \left[S_{ij}^A\right]_{i,j=1,\ldots,m}$ is the score matrix, $\Pi = (\pi_{ij})_{i,j=1,\ldots,m}$ is the adjacency matrix, and $P(\Pi)$ is the transition probability matrix for it. By definition, matrix Π is a matrix of points scored (considering a draw, estimated at 0.5 points) by alternatives (teams and athletes) A in the 'tournament' determined by the set of criteria C. Now we propose the following adaptations of the described above ranking methods for the MCDM problem $< A, C >$.

3.2.1. Massey ranking In the Massey method for the MCDM problem $< A, C >$, we have m alternatives/teams, and we can define the number of games played (in the sense of matrix Π i.e., we also consider draws) as $n_G = m(m-1)/2$. We are considered the set $\mathbb{N}(n_G)$ to be the set of games played and construct the mappings $t^1, t^2 : \mathbb{N}(m) \to \mathbb{N}(n_G)$, such that the inequality $y_k = s^1(k) - s^2(k) \geq 0$ holds for the scores $s^1(k) = \pi_{t^1(k)t^2(k)}, \;\; s^2(k) = \pi_{t^2(k)t^1(k)}, \;\; k = 1, \ldots, n_G$.We also construct the matrix X corresponding to the mappings $t^1, t^2$and the Massey matrix M and Massey vector v^M obtained as described in Section 2.3.1. Correspondingly, the Massey rating vector r^M for the MCDM problem $< A, C >$ can be obtained as solution of the equation $Mr^M = v^M$.

3.2.2. Colley ranking In the Colley method, for the MCDM problem $< A, C >$ we have m alternatives/teams, and the number of wins and number of losses for the alternative $a_i \in A$ can be defined as $w_i = \sum_{j=1}^{m} \chi(s_{ij}^A - s_{ji}^A)$, $l_i = \sum_{j=1}^{m} \chi(s_{ji}^A - s_{ij}^A)$, $i = 1, ..., m$, respectively. Correspondingly, the total numbers of no draw games played by alternative $a_i \in A$ is $n_i = w_i + l_i = \sum_{j=1}^{m} n_{ij}$, where $n_{ij} = \chi(s_{ij}^A - s_{ji}^A) + \chi(s_{ji}^A - s_{ij}^A)$ is the number of times team i, j played no draw games, $i, j = 1, \ldots, m$. Hence, for the MCDM problem $< A, C >$, the Colley matrix C and Colley vector v^C can be obtained as described in Section 2.3.2. Correspondingly, the Colley rating vector, r^C, for the MCDM problem $< A, C >$ can be obtained as a solution of the equation $Cr^C = v^C$.

3.2.3. Keener ranking In the Keener method, for the MCDM problem $< A, C >$ we have m lternatives/teams and the score produced by team $i$gainst team j an be defined as $S_{ij} = s_{ij}^A, i, j = 1, \ldots, m$. Hence, for the MCDM problem $< A, C >$, the Keener matrix K can be obtained as described in Section 2.3.3. Correspondingly, the rating vector for the Keener method, r^K, is obtained as a solution of the eigenvalue problem $Kr^K = \lambda r^K$.

3.2.4. Offense, defense, and aggregate offence-defence ranking The offense, defense, and aggregate offence-defence vectors can be defined through the following iteration process (see [2].[3].[4]):

$$d^{(k)} = P(\Pi)\Lambda(o^{(k)})^{-1}1_m, \quad o^{(k)} = P(\Pi)^T\Lambda(d^{(k-1)})^{-1}1_m, \quad d^{(0)} = 1/m1_m^T.$$

As we know, if matrix $P(\Pi)$ is sufficiently good then for $o = \lim_{k\to+\infty} o^{(k)}$ $d = \lim_{k\to+\infty} d^{(k)}$, we have $o = P(\Pi)^T\Lambda(d)^{-1}1_m$ and $d = P(\Pi)\Lambda(o)^{-1}1_m$. Hence, we can define the offense, defense, and aggregate offence-defence rating vectors as follows: $r_i^o = o_i$, $r_i^d = d_i$, $r_i^{od} = o_i/ d_i$, $i = 1, \ldots, m$. where $r^o = (r_1^o, \ldots, r_m^o)$,$r^d = \left(r_1^d, \ldots, r_m^d\right)$, $andr^{od} = (r_1^{od}, \ldots, r_m^{od})$, respectively,

3.2.5. Authority - hub ranking Analogously, authority and hub vectors $a\ and\ h$can be defined through the following iteration process (see 2.3.5):

$$a^{(k)} = P(\Pi)^T P(\Pi)a^{(k-1)}, \quad h^{(k)} = P(\Pi)P(\Pi)^T h^{(k-1)}.$$

If matrix $P(\Pi)$is sufficiently good, Then for $a = \lim_{k\to+\infty} a^{(k)}$,and $h = \lim_{k\to+\infty} h^{(k)}$,we have $a = P(\Pi)^T P(\Pi)a\ and\ \ h = P(\Pi)P(\Pi)^T h$.Now, we can define the authority and hub rating vectors as follows:
$r_i^a = a_i$, $r_i^h = h_i$, $i = 1, \ldots, m$, and $r^a = (r_1^a, \ldots, r_m^a)\ andr^h = (r_1^h, \ldots, r_m^h)$,respectively.

3.2.6. Proposed MCDM procedure Now, we can establish the following procedure:
(i) the matrix $S\ (\ \Pi,\ P(\Pi)\ )$ is calculated and the rating method, ρ, is the choice for the MCDM problem $< A, C >$;
ii) the alternative from the Pareto set, A_e, that is ρ ranked best is declared the solution of the considered MCDM problem.

The following remarks are important in connection with this procedure. Obviously, it would suffice to rank the Pareto set if it is known at the beginning of the proposed procedure. Nevertheless, we prefer the above description because it is more convenient in cases in which the Pareto set is unknown (or partially or approximately known), as is usually the case for complex MCDM problems.

Note also that, instead of the MCDM problem $< A, C >$, the MCDM problem $< C, A >$ can be considered. Applying the described procedure to the MCDM problem $< C, A >$, we can obtain a ranking of the criteria and identify a 'leading criterion'. This possibility may be useful for the development of some hierarchical procedures for solution s to MCDM problems.

In Section 4, we will discuss a particular problem that can be solved using the procedure described above and show that appropriate and competitive results are obtained for it.

4. Example

4.1. The problem statement

This section discusses the example that was considered to demonstrate the practicality of the proposed methods. The example considered here is the problem of selecting material for the mast of a sailboat, see [15] for a detailed description (the decision-making matrix is given in Table 1) . This problem was investigated by several researchers using various methods , see e.g. ([15] [5] [3] [2] [4] [10]), so it can be considered a benchmarking problem.

Table 1. Decision matrix for selecting material for a sailing boat mast

Alternatives		Criteria			
Code	Material	SS 1	SM 2	CR 3	CC 4
01	AISI 1020	35.9	26.9	1	5
02	AISI 1040	51.3	26.9	1	5
03	ASTM A242	42.3	27.2	1	5
04	AISI 4130	194.9	27.2	4	3
05	AISI 316	25.6	25.1	4	3
06	AISI 416	57.1	28.1	4	3
07	AISI 431	71.4	28.1	4	3
08	AA 6061	101.9	25.8	3	4
09	AA 2024	141.9	26.1	3	4
10	AA 2014	148.2	25.8	3	4
11	AA 7075	180.4	25.9	3	4
12	Ti-6AI-4V	208.7	27.6	5	1
13	Epoxy-70%	604.8	28.0	4	2
14	Epoxy-63%	416.2	66.5	4	1
15	Epoxy-62%	637.7	27.5	4	1

Source $. Criteria: SS (specific strength), scale: Numeric; SM (specific modulus), scale: Numeric; CR (corrosion resistance), scale: 1 = poor; 2 = fair; 3 = good; 4 = very good; 5 = excellent; CC(cost category), scale: 1 = very high; 2 = high; 3 = moderate; 4 = low; 5 = very low.

The ranks of the materials obtained in the mentioned articles using the following methods: the multi-objective optimization based on ratio analysis (MOORA), the multiplicative form of MOORA (MULTIMOORA), the reference-point approach (RPA), the fuzzy-logic approach (FLA), the weighted-properties method (WPM), the multicriteria optimization through the concept of a compromise solution (VIKOR), the comprehensive VIKOR (CVIKOR), and the game-theoretic method (GTM).

The auxiliary results are presented in the Appendix: the normalized decision matrix (the upper-lower bound approach was used for normalization, see [1]),Table 5; the ranks of the materials obtained in the articles ([15] [5] [3] [2] [4] [10]) are presented in Table 6; the Massey matrix, Massey vector, and Massey rating vector,Table 7; the Colley matrix, Colley vector, and Colley rating vector,Table 8 ; the Keener matrix and Keener rating vector,Table 9 ; the offense, defense, aggregate offense-defense, authority and hub rating vectors, Table 10 . Note also that the Pareto set for the problem under consideration is $A_e = \{2,\ 3,\ 4,\ 7,\ 9,\ 11,\ 12,\ 13,\ 14,\ 15\}$.

4.2. The obtained results

The obtained results are summarized inTable 2 , where the corresponding rankings for the obtained rating vectors are presented.

Table 3 presents the results of the correlation analysis between the considered rankings. As we can see, the considered methods are significantly correlated among themselves but sometimes may have an ‘opposite orientation’ (thus, we will further consider absolute values of correlation coefficients): the ranks corresponding to $r^M, r^C, r^K, r^o, r^d, r^{aod}, r^a,\ and\ r^h$are strongly correlated (absolute value of correlation coefficients no less than

Table 2. Ranking vectors for the material selection problem

	r^M	r^C	r^K	r^o	r^d	r^{od}	r^a	r^h
01	3	3	2	14	13	14	14	3
02	5	5	4	12	11	12	12	5
03	7	6	5	10	10	10	10	6
04	9	9	9	5	2	5	5	14
05	1	1	1	15	15	15	15	1
06	12	11	10	7	7	7	7	9
07	15	14	12	6	6	6	6	10
08	2	2	3	13	14	13	13	2
09	6	7	7	9	9	9	9	7
10	4	4	6	11	12	11	11	4
11	8	8	8	8	8	8	8	8
12	11	12	13	3	1	3	3	15
13	13	13	14	2	5	1	2	11
14	14	15	15	1	3	2	1	13
15	10	10	11	4	4	4	4	12

For material codes see Table 1 . Gray denote Pareto-optimal (efficient) materials.

0.75) among themselves; at the same time, the ranks corresponding to $r^M, r^C, r^K, and\ r^h$ have one direction, and the ranks corresponding to $r^o, r^d, r^{aod}, and\ r^a$ have opposite directions.

Table 3. Correlation analysis of the proposed ranks

	r^M	r^C	r^K	r^o	r^d	r^{od}	r^a	r^h
$\mathbf{r}^M$	1.00							
$\mathbf{r}^C$	0.99	1.00						
$\mathbf{r}^K$	0.94	0.97	1.00					
r^o	-0.90	-0.94	-0.98	1.00				
$\mathbf{r}^d$	-0.85	-0.88	-0.90	0.95	1.00			
$\mathbf{r}^{od}$	-0.90	-0.93	-0.98	1.00	0.94	1.00		
$\mathbf{r}^a$	-0.90	-0.94	-0.98	1.00	0.95	1.00	1.00	
$\mathbf{r}^h$	0.85	0.88	0.90	-0.95	-1.00	-0.94	-0.95	1.00

Finally, Table 4 presents correlations between the ranks obtained previously in other studies and the ranks proposed in this article. As Table 4 shows, the ranks proposed in this study are quite competitive. Particularly, all ranks introduced in this study (except cases related to weak correlations, i.e., the absolute value of correlation coefficients less than 0.35, with GTM and few cases with MULTIMOORA, RPA, and WPM) are moderately (absolute value of correlation coefficients between 0.35 and 0.75) or strongly correlated (absolute value of correlation coefficients no less than 0.75) with the previously obtained ranks.

5. Conclusions

In this study, we proposed a new approach for solving MCDM problems. In the framework of the proposed approach, we built a special score matrix for a given multicriteria problem, which allows us to use an appropriate ranking method and choose the corresponding best-ranked alternative from the Pareto set as a solution of the MCDM problem. The proposed approach is particularly useful when no decision-making authority is available or when the relative importance of various criteria has not been previously evaluated. To demonstrate the viability and suitability for applications, the proposed approach is illustrated using an example of a material-selection problem.

Table 4. Correlations between proposed and comparable ranks

Method	r^M	r^C	r^K	r^o	r^d	r^{od}	r^a	r^h
M1	-0.51	-0.53	-0.60	0.56	0.39	0.57	0.56	-0.39
M2	-0.28	-0.34	-0.50	0.50	0.33	0.50	0.50	-0.33
M3	-0.29	-0.37	-0.5	0.47	0.29	0.46	0.47	-0.29
M4	-0.53	-0.58	-0.72	0.76	0.69	0.76	0.76	-0.69
M5	-0.24	-0.34	-0.45	0.40	0.34	0.36	0.40	-0.34
M6	-0.52	-0.59	-0.65	0.74	0.70	0.74	0.74	-0.7
M7	-0.72	-0.82	-0.88	0.89	0.81	0.89	0.89	-0.81
M8	0.15	0.16	0.11	-0.12	-0.11	-0.09	-0.12	0.11

Methods: M1- MOORA [2] ; M2-MULTIMOORA [2] ; M3- RPA [2] ; M4- FLA [2] ; M5- Wpm [15] ; M6 -CVIKOR [4] ; M7 - VIKOR [4] ; M8 - GTM [10] .

It is shown that the solutions obtained using the proposed approach are quite competitive. All our numerical experiments were conducted on standard equipment (laptop with 2.59 GHz, 8 GB RAM, and a 64-bit operation system) and required ~1 min for all eight considered rankings in the MATLAB environment without making any effort to optimist the code.

Due to the simplicity and flexibility of the implementation, the proposed approach can be also used in a few interesting directions. For example, the alternative rankings seem important to developing the (post-Pareto) improvements of existing Pareto-optimization algorithms. On the other hand, if we consider the 'transposed' MCDM problem (i.e., the problem for which the criteria of the original problem are alternatives and the alternatives of the original problem are criteria), the proposed approach also allows ranking the criteria and identifying a 'leading criterion'. This possibility may be useful for the development of some hierarchical procedures for the solution to MCDM problems. However, we will limit ourselves here to only mentioning these directions for further investigation.

A. Appendix

Table 5. Normalized decision matrix for the material-selection problem

Material	Criteria			
	1	**2**	**3**	4
01	0.9832	0.9565	1.0000	0.0000
02	0.9580	0.9565	1.0000	0.0000
03	0.9727	0.9493	1.0000	0.0000
04	0.7234	0.9493	0.2500	0.5000
05	1.0000	1.0000	0.2500	0.5000
06	0.9485	0.9275	0.2500	0.5000
07	0.9252	0.9275	0.2500	0.5000
08	0.8753	0.9831	0.5000	0.2500
09	0.8100	0.9758	0.5000	0.2500
10	0.7997	0.9831	0.5000	0.2500
11	0.7471	0.9807	0.5000	0.2500
12	0.7009	0.9396	0.0000	1.0000
13	0.0537	0.9300	0.2500	0.7500
14	0.3619	0.0000	0.2500	1.0000
15	0.0000	0.9420	0.2500	1.0000

For material and criteria codes see Table 1 .

Table 6. Materials ranked by comparable methods

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15
M1	14	15	13	12	4	7	6	11	10	9	5	8	2	3	1
M2	14	15	13	12	4	11	10	9	7	6	8	5	2	3	1
M3	14	13	12	15	4	11	10	9	8	7	6	2	3	1	5
M4	14	13	15	4	11	9	10	8	12	7	6	5	3	2	1
M5	14	13	15	11	10	9	8	7	2	4	6	3	12	1	5
M6	12	6	9	4	15	14	11	13	8	10	5	7	2	1	3
M7	14	11	13	4	15	10	5	12	7	9	6	8	2	1	3
M8	14	10	11	2	9	8	7	5	4	3	1	12	6	15	13

For material codes see Table 1 . Methods: M1- MOORA [2] ; M2-MULTIMOORA [2] ; M3- RPA [2] ; M4- FLA [2] ; M5- Wpm [15] ; M6 -CVIKOR [4] ; M7 - VIKOR [4] ; M8 - GTM [10] .

Table 7. Massey matrix for the material selection problem

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	r^M	v^M
01	10	-1	1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	-1.07	-15
02	-1	9	0	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	-0.73	-11
03	-1	0	9	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	-0.51	-9
04	-1	-1	-1	12	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0.59	10
05	-1	-1	-1	-1	14	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1.67	-25
06	-1	-1	-1	0	-1	8	-1	0	0	0	0	0	-1	-1	-1	1.07	8
07	-1	-1	-1	0	-1	-1	8	0	0	0	0	0	-1	-1	-1	1.29	10
08	0	0	0	-1	-1	0	0	9	-1	-1	-1	-1	-1	-1	-1	-1.31	-13
09	0	0	0	-1	-1	0	0	-1	7	0	0	-1	-1	-1	-1	-0.59	-6
10	0	0	0	-1	-1	0	0	-1	0	8	-1	-1	-1	-1	-1	-0.94	-9
11	0	0	0	-1	-1	0	0	-1	0	-1	8	-1	-1	-1	-1	-0.38	-4
12	-1	-1	-1	-1	-1	0	0	-1	-1	-1	-1	12	-1	-1	-1	1.05	16
13	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	14	-1	-1	1.20	18
14	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	13	0	1.23	18
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.77	0

For material codes see Table 1 .

Table 8. Colley matrix for the material selection problem

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	r^C	v^C
01	12	-1	-1	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	0.24	-3.0
02	-1	11	0	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	0.36	-1.5
03	-1	0	11	-1	-1	-1	-1	0	0	0	0	-1	-1	-1	-1	0.36	-1.5
04	-1	-1	-1	14	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-1	0.60	3.0
05	-1	-1	-1	-1	16	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0.09	-6.0
06	-1	-1	-1	0	-1	10	-1	0	0	0	0	0	-1	-1	-1	0.71	3.0
07	-1	-1	-1	0	-1	-1	10	0	0	0	0	0	-1	-1	-1	0.80	4.0
08	0	0	0	-1	-1	0	0	11	-1	-1	-1	-1	-1	-1	-1	0.21	-2.5
09	0	0	0	-1	-1	0	0	-1	9	0	0	-1	-1	-1	-1	0.38	-0.5
10	0	0	0	-1	-1	0	0	-1	0	10	-1	-1	-1	-1	-1	0.33	-1.0
11	0	0	0	-1	-1	0	0	-1	0	-1	10	-1	-1	-1	-1	0.42	0.0
12	-1	-1	-1	-1	-1	0	0	-1	-1	-1	-1	14	-1	-1	-1	0.73	5.0
13	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	16	-1	-1	0.79	6.0
14	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	15	0	0.84	6.5
15	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	15	0.64	3.5

For material codes see Table 1 .

Table 9. Keener matrix for the material selection problem

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	r^K
01	0.00	0.21	0.15	0.21	0.79	0.21	0.21	0.50	0.50	0.50	0.50	0.21	0.21	0.21	0.21	0.16
02	0.79	0.00	0.50	0.21	0.79	0.21	0.21	0.50	0.50	0.50	0.50	0.21	0.21	0.21	0.21	0.19
03	0.85	0.50	0.00	0.28	0.79	0.21	0.21	0.50	0.50	0.50	0.50	0.21	0.21	0.21	0.21	0.19
04	0.79	0.79	0.72	0.00	0.85	0.50	0.50	0.79	0.79	0.79	0.79	0.21	0.28	0.28	0.28	0.28
05	0.21	0.21	0.21	0.15	0.00	0.15	0.15	0.21	0.21	0.21	0.21	0.21	0.28	0.28	0.28	0.12
06	0.79	0.79	0.79	0.50	0.85	0.00	0.21	0.50	0.50	0.50	0.50	0.50	0.72	0.28	0.72	0.29
07	0.79	0.79	0.79	0.50	0.85	0.79	0.00	0.50	0.50	0.50	0.50	0.50	0.72	0.28	0.72	0.32
08	0.50	0.50	0.50	0.21	0.79	0.50	0.50	0.00	0.15	0.21	0.15	0.21	0.21	0.21	0.21	0.18
09	0.50	0.50	0.50	0.21	0.79	0.50	0.50	0.85	0.00	0.50	0.50	0.21	0.21	0.21	0.21	0.22
10	0.50	0.50	0.50	0.21	0.79	0.50	0.50	0.79	0.50	0.00	0.15	0.21	0.21	0.21	0.21	0.20
11	0.50	0.50	0.50	0.21	0.79	0.50	0.50	0.85	0.50	0.85	0.00	0.21	0.21	0.21	0.21	0.23
12	0.79	0.79	0.79	0.79	0.79	0.50	0.50	0.79	0.79	0.79	0.79	0.00	0.21	0.28	0.72	0.33
13	0.79	0.79	0.79	0.72	0.72	0.28	0.28	0.79	0.79	0.79	0.79	0.79	0.00	0.72	0.72	0.35
14	0.79	0.79	0.79	0.72	0.72	0.72	0.72	0.79	0.79	0.79	0.79	0.72	0.28	0.00	0.50	0.36
15	0.79	0.79	0.79	0.72	0.72	0.79	0.28	0.79	0.79	0.79	0.79	0.28	0.28	0.50	0.00	0.30

For material codes see Table 1 .

Table 10. Offense, defense, aggregate offense-defense, authority and hub rating vectors for the material selection problem

	r^o	r^d	r^{ad}	r^a	r^h
01	1.066181	1.003908	1.062031	0.274749	0.257197
02	1.047487	1.001655	1.045756	0.269976	0.257809
03	1.038508	1.000955	1.037517	0.267661	0.258001
04	0.953815	0.995223	0.958393	0.245834	0.259428
05	1.114829	1.012164	1.101431	0.287351	0.254921
06	0.965823	0.998960	0.966829	0.248928	0.258516
07	0.957289	0.998704	0.958531	0.246729	0.258564
08	1.058207	1.004154	1.053830	0.272741	0.257153
09	1.025972	1.000692	1.025262	0.264428	0.258072
10	1.039747	1.002103	1.037565	0.267980	0.257702
11	1.016681	0.999666	1.017021	0.262033	0.258338
12	0.927938	0.994487	0.933082	0.239168	0.259554
13	0.921163	0.996091	0.924778	0.237424	0.259139
14	0.920613	0.995406	0.924862	0.237281	0.259295
15	0.946041	0.995834	0.949999	0.243831	0.259253

For material codes see Table 1 .

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