STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **8**, June 2020, pp 481–498.
Published online in International Academic Press (www.IAPress.org)

Progressively Type-II Right Censored Order Statistics from Hjorth Distribution and Related Inference

Narinder Pushkarna, Jagdish Saran, Kanika Verma*

Department of Statistics, Faculty of Mathematical Sciences,University of Delhi, India

Abstract In this paper some recurrence relations satisfied by single and product moments of progressively Type-II right censored order statistics from Hjorth distribution have been obtained. Then we use these results to compute the moments for all sample sizes and all censoring schemes $(R_1, R_2, ..., R_m), m \leq n$, which allow us to obtain BLUEs of location and scale parameters based on progressively Type-II right censored samples. The best linear unbiased predictors of censored failure times are then discussed briefly. Finally, a numerical example with real data is presented to illustrate the inferential method developed here.

Keywords Progressively Type-II censored order statistics, Hjorth distribution, single moments, product moments, recurrence relations, best linear unbiased estimators, best linear unbiased predictors.

AMS 2010 subject classifications 62G30

DOI: 10.19139/soic-2310-5070-751

1. Introduction

The scheme of progressive Type-II censoring is of importance in reliability and life-testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment which may lead to a saving of costs and of time (see Cohen [12] and Sen [29]). In such a random experiment, a group of n independent and identical experimental units is put on a life test at time zero with continuous, identically distributed failure times $X_1, X_2, ..., X_n$. After the j^{th} failure, a prespecified number $R_j \geq 0$ of the $n - j - \sum_{i=0}^{j-1} R_i$ remaining (or surviving) units are randomly withdrawn from the experiment, $1 \leq j \leq m, m \leq n,\ R_0 = 0$. Removed units thus become right censored at the time of failure of other units. This progressive censoring leads to m ordered observed failure times denoted by $X_{1:m:n}^{(R_1,R_2,...,R_m)}$, $X_{2:m:n}^{(R_1,R_2,...,R_m)}$, ..., $X_{m:m:n}^{(R_1,R_2,...,R_m)}$, and these are called progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$. Thus, in this type of sampling, m failures are observed,$\sum_{j=1}^{m} R_j$ units are progressively censored and $n = m + \sum_{j=1}^{m} R_j$ denotes the number of units in the life test. The withdrawal of units may be seen as a model describing drop-outs of units due to failures which have causes other than the specific one under study. In this sense, progressive censoring schemes are applied in clinical trials as well. Here, the drop-outs of patients may be caused by migration, lack of interest or by personal or ethical decisions, and they are regarded as random withdrawals. For a detailed discussion of progressive censoring and the relevant developments in this area, one may refer to Sen [29] and Balakrishnan and Aggarwala [4].
The situation with no censoring corresponds to the special case with $m = n$ and $R_1 = R_2 = ... = R_m = 0$, whereas the situation with ordinary Type-II right censoring at a given order statistic corresponds to the special case with

*Correspondence to: Kanika verma (Email: kanikaverma29apr@gmail.com). Department of Statistics, University of Delhi, Delhi - 110007, India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

$m < n$, $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$.
If the failure times of the n items originally on test are from a continuous population with c.d.f. F(x) and p.d.f. f(x), then the joint p.d.f. of $X_{1:m:n}^{(R_1,R_2,...,R_m)}$, $X_{2:m:n}^{(R_1,R_2,...,R_m)}$, ..., $X_{m:m:n}^{(R_1,R_2,...,R_m)}$ is given by (cf. Balakrishnan and Sandhu [11] and Saran and Pushkarna [27])

$$f_{X_{1:m:n},...,X_{m:m:n}}(x_1, x_2, ..., x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},\ 0 \le x_1 < x_2 < ... < x_m < \infty, \quad (1)$$

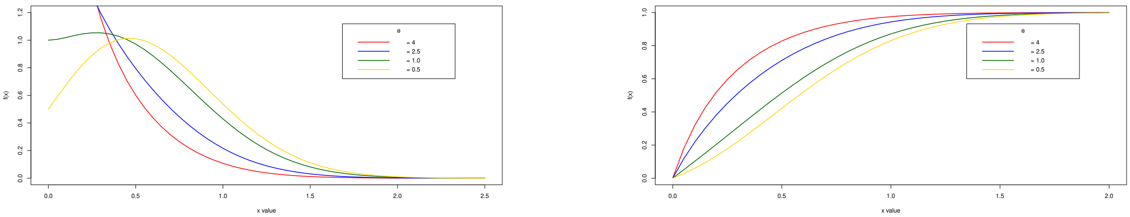
where $A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$.
Here, note that all the factors in $A(n, m-1)$ are positive integers. Also it may be observed that the different factors in $A(n, m-1)$ represent the number of units still on test immediately preceding the first, second, ..., m^{th} observed failures, respectively. Similarly, for convenience in notation, let us define

$$A(p, q) = p(p - R_1 - 1)(p - R_1 - R_2 - 2)...(p - R_1 - R_2 - ... - R_q - q),$$

for $q = 0, 1, ..., p-1$, with all the factors being positive integers.
Progressive censoring and associated inferential procedures have been extensively studied in the literature for a number of distributions by several authors. Cohen ([12], [13], [14], [15] and [16]), Mann ([21], [22]), Cohen and Whitten [17], Viveros and Balakrishnan [30], Balakrishnan and Sandhu [11], Aggarwala and Balakrishnan [1] and Balakrishnan and Aggarwala [4] have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms.
Saran and Pande [26], Saran and Pushkarna ([27], [28]), Saran et al. [25] and Pushkarna et al. [24] have derived recurrence relations for single and product moments of the corresponding progressively Type-II right censored order statistics from half logistic, Burr, left truncated logistic, Frechet and a general class of doubly truncated continuous distributions.
Mahmoud et al. [20] derived some new recurrence relations for single and product moments of progressively Type-II right censored order statistics from the linear exponential distribution and also obtained maximum likelihood estimators (MLEs) of the location and scale parameters. Balakrishnan et al. [5] and Balakrishnan and Saleh ([7], [8], [9], [10]) have established several recurrence relations for single and product moments of progressively Type-II right censored order statistics from logistic, half-logistic, log-logistic, generalized half logistic and generalized logistic distributions and utilized them to derive the best linear unbiased estimators of the location and scale parameters.
In this paper, we derive some recurrence relations satisfied by the single and product moments of progressively Type-II right censored order statistics from Hjorth distribution. These relations enable the recursive computation of moments for all sample sizes and all possible progressive censoring schemes. They generalize the corresponding results for exponential distribution due to Aggarwala and Balakrishnan [1]. Then we use these results to compute the means, variances and covariances of progressively Type-II right censored order statistics for some specific values of the parameters, which will be utilized to derive the best linear unbiased estimators (BLUEs) of location and scale parameters of the location-scale Hjorth distribution as well as their variances and covariances. Tables of these quantities are presented for different sample sizes up to n = 8 and some selected progressive censoring schemes, corresponding to particular values of the parameters. Further, for the special case $R_1 = R_2 = ... = R_m = 0$, the derived results would reduce to the general recurrence relations for the usual order statistics from the Hjorth distribution. Also, we briefly discuss the best linear unbiased predictors (BLUPs) of the censored failure times by making use of the results developed on the BLUEs. Finally, one numerical example on real data is presented to illustrate all the methods of inference developed here.

2. Hjorth distribution

Hjorth distribution is a reliability distribution with increasing, decreasing, constant and bathtub shaped failure rates as its special cases. This distribution is also known as IDB distribution (cf. Hjorth [19]). Its p.d.f. f(x), c.d.f. F(x)



(a) p.d.f. of Hjorth distribution (b) c.d.f. of Hjorth distribution

Figure 2.1

and characterizing differential equation, respectively, are given by:

$$f(x) = \frac{[(1+\beta x)\delta x + \theta] e^{-\frac{\delta x^2}{2}}}{(1+\beta x)^{1+\frac{\theta}{\beta}}}, \quad x \geq 0, \quad \beta,\ \delta,\ \theta > 0, \tag{2}$$

$$F(x) = 1 - \frac{e^{-\delta x^2/2}}{(1+\beta x)^{\frac{\theta}{\beta}}}, \tag{3}$$

$$(1+\beta x) f(x) = [(1+\beta x)\delta x + \theta](1 - F(x)). \tag{4}$$

The failure rate of this distribution is readily seen to be

$$z(t) = \delta t + \frac{\theta}{1+\beta t}. \tag{5}$$

Special cases of the Hjorth distribution are:

$\theta = 0$: the Rayleigh distribution (a Weibull distribution),

$\delta = \beta = 0$: the exponential distribution (a Weibull distribution),

$\delta = 0$: decreasing failure rate,

$\delta \geq \theta\beta$: increasing failure rate,

$0 < \delta \leq \theta\beta$: bathtub curve.

More details on this distribution can be found in Hjorth [19]. The graphs of the p.d.f. and c.d.f. of Hjorth distribution as given in (2) and (3) for $\beta = 2,\ \delta = 3$ and for different values of $\theta = 4.0,\ 2.5,\ 1.0$ and 0.50 are shown in Figures 2.1(a) and 2.1(b), respectively.

The c.d.f. of the location-scale parameter Hjorth distribution is given by

$$F(x) = 1 - \frac{e^{-\delta\left(\frac{x-\mu}{\sigma}\right)^2/2}}{\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{\theta}{\beta}}}, \ x \geq \mu,\ \mu \geq 0,\ \beta,\ \delta,\ \theta,\ \sigma > 0. \tag{6}$$

3. Recurrence relations for single moments

In this section, we shall establish several recurrence relations for single moments of progressively Type-II right censored order statistics from Hjorth distribution satisfying the characterizing differential equation (4). Using (1), we have

$$\begin{aligned} \mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= E[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}]^k \\ &= A(n,m-1) \int \underset{0 \leq x_1 < x_2 < \ldots < x_m < \infty}{\int \cdots \int} \int x_r^k \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t. \end{aligned} \tag{7}$$

Theorem 1
For $k \geq 0$,

$$\mu_{1:1:1}^{(0)}{}^{(k+3)} = \frac{k+3}{\delta\beta}\left[\mu_{1:1:1}^{(0)}{}^{(k)} + (\beta - \frac{\theta}{k+1})\mu_{1:1:1}^{(0)}{}^{(k+1)} - \frac{\delta}{k+2}\mu_{1:1:1}^{(0)}{}^{(k+2)}\right]. \tag{8}$$

Proof
From (7), for $n = m = r = 1$, we obtain

$$\mu_{1:1:1}^{(0)}{}^{(k)} + \beta\mu_{1:1:1}^{(0)}{}^{(k+1)} = A(1,0)\int_{x_1} x_1{}^k(1+\beta x_1)f(x_1)dx_1,$$

using (4), we have

$$\begin{aligned}\mu_{1:1:1}^{(0)}{}^{(k)} + \beta\mu_{1:1:1}^{(0)}{}^{(k+1)} &= \int_{x_1} x_1{}^k(\delta x_1 + \beta\delta x_1^2 + \theta)(1-F(x_1))dx_1 \\ &= \delta\int_{x_1} x_1{}^{k+1}(1-F(x_1))dx_1 \\ &\quad + \delta\beta\int_{x_1} x_1{}^{k+2}(1-F(x_1))dx_1 \\ &\quad + \theta\int_{x_1} x_1{}^{k}(1-F(x_1))dx_1.\end{aligned}$$

Integrating by parts all integrals on the R.H.S. of the above equation by taking $(1-F(x_1))$ for differentiation and the rest of the integrand for integration, and then after some simplification, it leads to the required result (8).

□

Theorem 2
For $n \geq 2$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)(k+3)} = \frac{k+3}{\delta\beta}\left[\frac{1}{n}\mu_{1:1:n}^{(n-1)(k)} + \left(\frac{\beta}{n} - \frac{\theta}{k+1}\right)\mu_{1:1:n}^{(n-1)(k+1)} - \frac{\delta}{k+2}\mu_{1:1:n}^{(n-1)(k+2)}\right]. \tag{9}$$

Proof
Proceeding in a similar manner as in Theorem 1, we can easily establish the relation (9). □

Theorem 3
For $2 \leq m \leq n-1,\ k \geq 0$ and $R_1 \geq 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k)} + \beta\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} &= \theta\left[\frac{(n-R_1-1)}{k+1}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(k+1)}\right. \\ &\quad \left.+ \frac{(R_1+1)}{k+1}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)}\right] \\ &\quad + \delta\left[\frac{(n-R_1-1)}{k+2}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(k+2)}\right. \\ &\quad \left.+ \frac{(R_1+1)}{k+2}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+2)}\right] \\ &\quad + \delta\beta\left[\frac{(n-R_1-1)}{k+3}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(k+3)}\right. \\ &\quad \left.+ \frac{(R_1+1)}{k+3}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+3)}\right].\end{aligned} \tag{10}$$

Proof
The relation in (10) may be proved by following exactly the same steps as those used in proving Theorem 4, which is presented next. □

Theorem 4
For $2 \leq r \leq n-1,\ m < n,\ k \geq 0$ and $R_r \geq 0$,

$$\begin{aligned}
\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} + \beta\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} = \theta \Bigg[& \frac{(n-S_r-r)}{k+1}\mu_{r:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+1)}} \\
& - \frac{(n-S_{r-1}-r+1)}{k+1}\mu_{r-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+1)}} \\
& + \frac{(R_r+1)}{k+1}\mu_{r:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+1)}} \Bigg] \\
& + \delta \Bigg[\frac{(n-S_r-r)}{k+2}\mu_{r:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+2)}} \\
& - \frac{(n-S_{r-1}-r+1)}{k+2}\mu_{r-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+2)}} \\
& + \frac{(R_r+1)}{k+2}\mu_{r:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+2)}} \Bigg] \\
& + \delta\beta \Bigg[\frac{(n-S_r-r)}{k+3}\mu_{r:m-1:n}^{(R_1,R_2,\ldots,R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+3)}} \\
& - \frac{(n-S_{r-1}-r+1)}{k+3}\mu_{r-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+3)}} \\
& + \frac{(R_r+1)}{k+3}\mu_{r:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+3)}} \Bigg], \qquad (11)
\end{aligned}$$

where $S_i = R_1 + R_2 + \ldots + R_i,\ \ 1 \leq i \leq m.$

Proof
Using (7) and (4), we have

$$\begin{aligned}
& \mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} + \beta\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} \\
& = A(n,m-1)\int \underset{0\leq x_1<x_2<\ldots<x_{r-1}<x_{r+1}<\ldots<x_m<\infty}{\int\cdots\int} I(x_{r-1},x_{r+1}) \prod_{u=1,u\neq r}^{m} [1-F(x_u)]^{R_u} f(x_u)dx_u, \qquad (12)
\end{aligned}$$

where

$$I(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^k(1+\beta x_r)[1-F(x_r)]^{R_r} f(x_r)dx_r. \qquad (13)$$

Using the characterizing differential equation (4), we have

$$\begin{aligned}
I(x_{r-1},x_{r+1}) &= \int_{x_{r-1}}^{x_{r+1}} x_r^k(\delta x_r+\delta\beta x_r^2+\theta)[1-F(x_r)]^{R_r+1}dx_r \\
&= \delta\int_{x_{r-1}}^{x_{r+1}} x_r^{k+1}[1-F(x_r)]^{R_r+1}dx_r+\delta\beta\int_{x_{r-1}}^{x_{r+1}} x_r^{k+2}[1-F(x_r)]^{R_r+1}dx_r \\
&\quad +\theta\int_{x_{r-1}}^{x_{r+1}} x_r^{k}[1-F(x_r)]^{R_r+1}dx_r \\
&= \delta I_1(x_{r-1},x_{r+1})+\delta\beta I_2(x_{r-1},x_{r+1})+\theta I_0(x_{r-1},x_{r+1}),
\end{aligned} \tag{14}$$

where

$$I_a(x_{r-1},x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} x_r^{k+a}[1-F(x_r)]^{R_r+1}dx_r;\ \ a=0,\ 1,\ 2.$$

Integration by parts yields,

$$\begin{aligned}
I_a(x_{r-1},x_{r+1}) =& \frac{x_r^{k+a+1}}{k+a+1}[1-F(x_r)]^{R_r+1}\ |_{x_{r-1}}^{x_{r+1}} - \int(R_r+1)[1-F(x_r)]^{R_r}(-f(x_r))\frac{x_r^{k+a+1}}{k+a+1}dx_r \\
=& \frac{1}{(k+a+1)}\Big[x_{r+1}^{k+a+1}[1-F(x_{r+1})]^{R_r+1}-x_{r-1}^{k+a+1}[1-F(x_{r-1})]^{R_r+1} \\
&+(R_r+1)\int[1-F(x_r)]^{R_r}x_{k+a+1}f(x_r)dx_r\Big].
\end{aligned} \tag{15}$$

Upon substituting for $I_0(x_{r-1},x_{r+1}),\ I_1(x_{r-1},x_{r+1})$ and $I_2(x_{r-1},x_{r+1})$ from equation (15) in (14) and then substituting the resultant expression for $I(x_{r-1},x_{r+1})$ in (12) and simplifying, it leads to Theorem 4. □

Next, we state another result on single moments which can easily be established on similar lines.

Theorem 5
For $2\le m\le n,\ k\ge 0$ and $R_m\ge 0$,

$$\begin{aligned}
\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}+\beta\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} =& \theta\Big[-\frac{(n-S_{m-1}-m+1)}{k+1}\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} \\
&+\frac{(R_m+1)}{k+1}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}}\Big] \\
&+\delta\Big[-\frac{(n-S_{m-1}-m+1)}{k+2}\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+2)}} \\
&+\frac{(R_m+1)}{k+2}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+2)}}\Big] \\
&+\delta\beta\Big[-\frac{(n-S_{m-1}-m+1)}{k+3}\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+3)}} \\
&+\frac{(R_m+1)}{k+3}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+3)}}\Big].
\end{aligned} \tag{16}$$

Remark 1. It may be mentioned that if $R_1=R_2=...=R_{k-1}=0$, i.e. there is no censoring before the time of the k^{th} failure, then the first k progressively Type-II right censored order statistics are simply the first k usual order statistics. Thus, for the special case $R_1 = R_2 = ... = R_m = 0$, so that $m = n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$, the recurrence relations established in Section 3 would reduce to the corresponding recurrence relations for the single moments of usual order statistics from the Hjorth distribution satisfying the characterizing differential equation (4).

Remark 2. Setting $\delta = \beta = 0$ and $\theta = 1$, we observe that (4) reduces to $f(x) = 1 - F(x)$, which is the characterizing differential equation for the $exp(1)$ distribution with p.d.f. $f(x) = e^{-x},\ x > 0$, the recurrence relations in Section 3 will reduce to and verify the corresponding recurrence relations established by Aggarwala and Balakrishnan [1] for the progressively Type-II right censored order statistics from exponential distribution.
It may be mentioned that one can derive similar recurrence relations for progressively Type-II right censored order statistics by taking different values of parameters as special cases of Hjorth distribution as given in Section 2.

4. Recurrence relations for product moments

Using (1) we can write the product moments of progressively Type-II right censored order statistics as follows:

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k_1,k_2)}} = E\left[\left\{X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^{k_1}\left\{X_{s:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^{k_2}\right]$$
$$= A(n,m-1)\int\int\cdots\int_{0\le x_1<x_2<\ldots<x_m<\infty}\int x_r^{k_1}x_s^{k_2}\prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \tag{17}$$

where $1 \le r < s \le m \le n$ and $k_1,\ k_2 \ge 0$.
In this Section, we shall derive various recurrence relations for the product moments of progressively Type-II right censored order statistics from Hjorth distribution with p.d.f. f(x) and c.d.f. F(x) satisfying the characterizing differential equation (4).

Theorem 6
For $2 \le s \le m \le n - R_1,\ k,t \ge 0$ and $R_1 \ge 0$,

$$\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t)}} + \beta\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}} = \frac{\theta}{k+1}\left[(n-R_1-1)\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,R_4,\ldots,R_m)^{(k+1,t)}} + (R_1+1)\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}}\right]$$
$$+ \frac{\delta}{k+2}\left[(n-R_1-1)\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,R_4,\ldots,R_m)^{(k+2,t)}} + (R_1+1)\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+2,t)}}\right]$$
$$+ \frac{\delta\beta}{k+3}\left[(n-R_1-1)\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,R_4,\ldots,R_m)^{(k+3,t)}} + (R_1+1)\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+3,t)}}\right]. \tag{18}$$

Proof
The relation in (18) may be proved by following exactly the same steps as those used in proving Theorem 7. □

Theorem 7
For $2 \le r < s \le m < n,\ \ k,t \ge 0$ and $R_r \ge 0$,

$$
\begin{aligned}
&\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t)}} + \beta\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}} \\
&= \frac{\theta}{(k+1)}\Bigg[(n-S_r-r)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+1,t)}} \\
&\qquad -(n-S_{r-1}-r+1)\mu_{r-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+1,t)}} \\
&\qquad +(R_r+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+1,t)}}\Bigg] \\
&+ \frac{\delta}{(k+2)}\Bigg[(n-S_r-r)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+2,t)}} \\
&\qquad -(n-S_{r-1}-r+1)\mu_{r-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+2,t)}} \\
&\qquad +(R_r+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+2,t)}}\Bigg] \\
&+ \frac{\delta\beta}{(k+3)}\Bigg[(n-S_r-r)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(k+3,t)}} \\
&\qquad -(n-S_{r-1}-r+1)\mu_{r-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(k+3,t)}} \\
&\qquad +(R_r+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_r,\ldots,R_m)^{(k+3,t)}}\Bigg]. \qquad (19)
\end{aligned}
$$

Proof
From (17), let us consider for $2 \le r < s \le m < n,\ k,t \ge 0$ and $R_r \ge 0$,

$$
\begin{aligned}
&\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t)}} + \beta\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}} \\
&= A(n,m-1)\int \underset{0\le x_1<x_2<\ldots<x_{r-1}<x_{r+1}<\ldots x_m<\infty}{\int\cdots\int} x_s^k I(x_{r-1},x_{r+1}) \prod_{u=1,u\neq r}^{m} f(x_u)[1-F(x_u)]^{R_u}dx_u, \qquad (20)
\end{aligned}
$$

where $I(x_{r-1},x_{r+1})$ is the same as given in equation (13), or equivalently, in equations (14) and (15). Now upon using (15) in (14) and then putting the value of $I(x_{r-1},x_{r+1})$, so obtained, into the equation (20) and then simplifying, it leads to (19). □

Theorem 8
For $1 \le r < s < m < n,\ k, t \ge 0$ and $R_s \ge 0$,

$$\begin{aligned}
\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t)}} + \beta\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}} = \frac{\theta}{(t+1)}\Big[&(n-S_s-s)\mu_{r,s:m-1:n}^{(R_1,R_2,\ldots,R_{s-1},R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(k,t+1)}}\\
&-(n-S_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-2},R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(k,t+1)}}\\
&+(R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}}\Big]\\
&+\frac{\delta}{(t+2)}\Big[(n-S_s-s)\mu_{r,s:m-1:n}^{(R_1,R_2,\ldots,R_{s-1},R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(k,t+2)}}\\
&-(n-S_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-2},R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(k,t+2)}}\\
&+(R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+2)}}\Big]\\
&+\frac{\delta\beta}{(t+3)}\Big[(n-S_s-s)\mu_{r,s:m-1:n}^{(R_1,R_2,\ldots,R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(k,t+3)}}\\
&-(n-S_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-2},R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(k,t+3)}}\\
&+(R_s+1)\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+3)}}\Big]. \qquad (21)
\end{aligned}$$

Proof
The relation in (21) may be proved by following the similar steps as those used in proving (19). □

Next, we state another result on product moments which can easily be established on similar lines.

Theorem 9
For $1 \le r < m < n,\ k,\ t \ge 0$ and $R_m \ge 0$,

$$\begin{aligned}
&\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t)}} + \beta\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}}\\
&= \frac{\theta}{(t+1)}\Big[-(n-S_{m-1}-m+1)\mu_{r,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k,t+1)}}\\
&\qquad+(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}}\Big]\\
&\quad+\frac{\delta}{(t+2)}\Big[-(n-S_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k,t+2)}}\\
&\qquad+(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+2)}}\Big]\\
&\quad+\frac{\delta\beta}{(t+3)}\Big[-(n-S_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k,t+3)}}\\
&\qquad+(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+3)}}\Big]. \qquad (22)
\end{aligned}$$

Remark 3. For the special case $R_1 = R_2 = \ldots = R_m = 0$, the recurrence relations established in Section 4 reduce to the corresponding recurrence relations for the product moments of usual order statistics from the Hjorth distribution satisfying the characterizing differential equation (4).

Remark 4. Setting $\delta = \beta = 0$ and $\theta = 1$, we observe that (4) reduces to $f(x) = 1 - F(x)$, which is the characterizing differential equation for $exp(1)$ distribution with p.d.f. $f(x) = e^{-x},\ x > 0$, the recurrence relations in Section 4 will reduce to and verify the corresponding recurrence relations established by Aggarwala and

Balakrishnan [1] for the product moments of progressively Type-II right censored order statistics from exponential distribution.
It may be mentioned that one can derive similar recurrence relations for product moments of progressively Type-II right censored order statistics by taking different values of parameters as special cases of Hjorth distribution as given in Section 2.

5. Numerical Results

The recurrence relations obtained in the preceding Sections 3 and 4 allow us to evaluate the means, variances and covariances of progressively Type-II right censored order statistics from Hjorth distribution for all sample sizes 'n' and all censoring schemes $(R_1,\ R_2,\ ...,\ R_m),\ m < n$. These quantities can be used for various inferential purposes; for example, they are useful in determining BLUEs of location/scale parameters and BLUPs of censored failure times. In this section, we compute means, variances and covariances of the progressively Type-II right censored order statistics from Hjorth distribution for some specific values of parameters, viz. $\beta = 2,\ \delta = 3,\ \theta = 4$ for sample sizes up to 8 and for different choices of m and progressive censoring schemes $(R_1,\ R_2,\ ...,\ R_m),\ m < n$. These values are presented in the following Tables 1 and 2.

Table 1. First four single moments of progressively Type-II right censored order statistics from Hjorth distribution

S. No.	m	n	$(R_1, ..., R_m)$	$\mu_{r:m:n}^{(R_1,R_2,...,R_m)^{(k)}}$, $r = 1, 2, ..., m$ and $k = 1$				
1	2	4	1,1	0.066369	0.206697			
2	2	5	1,2	0.052615	0.145046			
3	2	5	2,1	0.052615	0.192479			
4	2	5	0, 3	0.052615	0.121387			
5	2	5	3, 0	0.052615	0.330690			
6	2	6	2, 2	0.043547	0.135628			
7	2	7	3, 2	0.037129	0.128931			
8	3	4	0, 0, 1	0.066369	0.159242	0.301608		
9	3	5	0, 0, 2	0.052615	0.121387	0.216024		
10	3	5	0, 1, 1	0.052615	0.121387	0.263571		
11	3	5	2, 0, 0	0.052615	0.192479	0.468901		
12	3	6	1, 1, 1	0.043547	0.112015	0.254038		
13	3	6	1, 2, 0	0.043547	0.112014	0.391169		
14	3	6	1, 0, 2	0.043547	0.112014	0.206469		
15	3	7	0, 2, 2	0.037129	0.082054	0.175809		
16	3	7	1, 2, 1	0.037129	0.091330	0.232938		
17	4	5	0, 0, 0, 1	0.052615	0.121387	0.216024	0.358664	
18	4	5	1, 0, 0, 0	0.052615	0.145046	0.287344	0.559680	
19	4	6	1, 0, 1, 0	0.043547	0.112015	0.206469	0.483520	
20	4	7	1, 2, 0, 0	0.037129	0.091330	0.232938	0.508229	
21	4	8	1, 1, 2, 0	0.032352	0.077131	0.146898	0.426065	
22	4	8	1, 3, 0, 0	0.032352	0.077131	0.218325	0.494222	
23	4	8	2, 0, 0, 2	0.032352	0.086387	0.156387	0.251806	
24	5	6	0*4, 1	0.043547	0.097956	0.168248	0.263799	0.406097
25	5	8	1*3, 0*2	0.032352	0.077131	0.146898	0.289752	0.562379
S. No.	m	n	$(R_1, ..., R_m)$	$\mu_{r:m:n}^{(R_1,R_2,...,R_m)^{(k)}}$, $r = 1, 2, ..., m$ and $k = 2$				
1	2	4	1,1	0.009192	0.067926			
2	2	5	1,2	0.005760	0.033252			
3	2	5	2,1	0.005760	0.060425			

4	2	5	0, 3	0.005760	0.022921			
5	2	5	3, 0	0.005760	0.184343			
6	2	6	2, 2	0.003933	0.029581			
7	2	7	3, 2	0.002850	0.027187			
8	3	4	0, 0, 1	0.009192	0.039451	0.124874		
9	3	5	0, 0, 2	0.005760	0.022921	0.064248		
10	3	5	0, 1, 1	0.005760	0.022921	0.097929		
11	3	5	2, 0, 0	0.005760	0.060425	0.308261		
12	3	6	1, 1, 1	0.003933	0.019711	0.092033		
13	3	6	1, 2, 0	0.003933	0.019711	0.230183		
14	3	6	1, 0, 2	0.003933	0.019711	0.059191		
15	3	7	0, 2, 2	0.002850	0.010429	0.043946		
16	3	7	1, 2, 1	0.002850	0.013036	0.079381		
17	4	5	0, 0, 0, 1	0.005760	0.022921	0.064248	0.165292	
18	4	5	1, 0, 0, 0	0.005760	0.033252	0.114770	0.405006	
19	4	6	1, 0, 1, 0	0.003933	0.019711	0.059191	0.315679	
20	4	7	1, 2, 0, 0	0.002850	0.013036	0.079381	0.346407	
21	4	8	1,3,0,0	0.000031	0.000322	0.017726	0.277668	
22	4	8	2,0,0,2	0.000030	0.000542	0.003307	0.015794	
23	4	8	1,1,2,0	0.000031	0.000322	0.002642	0.196049	
24	5	6	0*4,1	0.003933	0.014898	0.038967	0.089529	0.203174
25	5	8	1*3, 0*2	0.002158	0.009257	0.030166	0.112655	0.404858

S. No.	m	n	$(R_1, ..., R_m)$	$\mu_{r:m:n}^{(R_1,R_2,...,R_m)^{(k)}}$, $r = 1, 2, ..., m$ and $k = 3$				
1	2	4	1,1	0.001959	0.030602			
2	2	5	1,2	0.000973	0.010507			
3	2	5	2,1	0.000973	0.026486			
4	2	5	0, 3	0.000973	0.005906			
5	2	5	3, 0	0.000973	0.141576			
6	2	6	2, 2	0.000548	0.009025			
7	2	7	3, 2	0.000337	0.008123			
8	3	4	0, 0, 1	0.001959	0.013268	0.065272		
9	3	5	0, 0, 2	0.000973	0.005906	0.024311		
10	3	5	0, 1, 1	0.000973	0.005906	0.047067		
11	3	5	2, 0, 0	0.000973	0.026486	0.256666		
12	3	6	1, 1, 1	0.000548	0.004782	0.043513		
13	3	6	1, 2, 0	0.000548	0.004782	0.1797883		
14	3	6	1, 0, 2	0.000548	0.004782	0.021753		
15	3	7	0, 2, 2	0.000337	0.001812	0.014435		
16	3	7	1, 2, 1	0.000337	0.002562	0.036056		
17	4	5	0, 0, 0, 1	0.000973	0.005906	0.024311	0.092579	
18	4	5	1, 0, 0, 0	0.000973	0.010507	0.058445	0.355776	
19	4	6	1, 0, 1, 0	0.000548	0.004782	0.021754	0.258806	
20	4	7	1, 2, 0, 0	0.000337	0.002562	0.036056	0.293434	
21	4	8	1, 1, 2, 0	0.002158	0.009257	0.030166	0.258757	
22	4	8	1, 3, 0, 0	0.002158	0.009257	0.071410	0.331807	
23	4	8	2, 0, 0, 2	0.002158	0.011764	0.034077	0.082350	
24	5	6	0*4,1	0.000548	0.003097	0.011523	0.037099	0.120319
25	5	8	1*3, 0*2	0.000221	0.001527	0.008038	0.055338	0.353311

S. No.	m	n	$(R_1, ..., R_m)$	$\mu_{r:m:n}^{(R_1,R_2,...,R_m)^{(k)}}$, $r = 1, 2, ..., m$ and $k = 4$				
1	2	4	1,1	0.000563	0.017339			

2	2	5	1,2	0.000223	0.004222			
3	2	5	2,1	0.000223	0.014770			
4	2	5	0, 3	0.000223	0.001925			
5	2	5	3, 0	0.000223	0.134627			
6	2	6	2, 2	0.000104	0.003541			
7	2	7	3, 2	0.000054	0.003148			
8	3	4	0, 0, 1	0.000563	0.005600	0.040819		
9	3	5	0, 0, 2	0.000223	0.001925	0.011112		
10	3	5	0, 1, 1	0.000223	0.001925	0.027615		
11	3	5	2, 0, 0	0.000223	0.014770	0.254485		
12	3	6	1, 1, 1	0.000104	0.001482	0.025268		
13	3	6	1,2,0	0.000104	0.001482	0.171873		
14	3	6	1,0,2	0.000104	0.001482	0.009717		
15	3	7	0, 2, 2	0.000054	0.000401	0.005897		
16	3	7	1, 2, 1	0.000054	0.000644	0.020421		
17	4	5	0, 0, 0, 1	0.000223	0.001925	0.011112	0.060623	
18	4	5	1, 0, 0, 0	0.000223	0.004222	0.035867	0.363793	
19	4	6	1, 0, 1, 0	0.000104	0.001482	0.009717	0.252951	
20	4	7	1, 2, 0, 0	0.000054	0.000644	0.020421	0.29349	
21	4	8	1, 1, 2, 0	0.000031	0.000322	0.017726	0.277668	
22	4	8	1, 3, 0, 0	0.000030	0.000542	0.003307	0.015794	
23	4	8	2, 0, 0, 2	0.000031	0.000322	0.002642	0.196049	
24	5	6	0*4,1	0.000104	0.000818	0.004139	0.018084	0.081892
25	5	8	1*3, 0*2	0.000031	0.000322	0.002642	0.032810	0.359287

Table 2. Variances and covariances of progressively Type-II right censored order statistics from Hjorth distribution

m	s	r	$\sigma^{(1,1)}_{r,s:m:4}$	$\sigma^{(2,1)}_{r,s:m:5}$	$\sigma^{(1,2)}_{r,s:m:5}$	$\sigma^{(0,3)}_{r,s:m:5}$	$\sigma^{(3,0)}_{r,s:m:5}$	$\sigma^{(2,2)}_{r,s:m:6}$	$\sigma^{(3,2)}_{r,s:m:7}$
2	1	1	0.004787	0.002992	0.002992	0.002992	0.002992	0.002036	0.001471
	2	1	0.004923	0.003106	0.003098	0.003085	0.003054	0.002120	0.001538
		2	0.004923	0.003106	0.003098	0.003085	0.003054	0.002120	0.001538
m	s	r	$\sigma^{(0,0,1)}_{r,s:m:4}$	$\sigma^{(0,0,2)}_{r,s:m:5}$	$\sigma^{(0,1,1)}_{r,s:m:5}$	$\sigma^{(2,0,0)}_{r,s:m:5}$	$\sigma^{(1,1,1)}_{r,s:m:6}$	$\sigma^{(1,2,0)}_{r,s:m:6}$	$\sigma^{(1,0,2)}_{r,s:m:6}$
3	1	1	0.004787	0.002992	0.002992	0.002992	0.002036	0.002036	0.002036
	2	1	0.004921	0.003085	0.003085	0.003106	0.002108	0.002108	0.002108
		2	0.014093	0.008186	0.008186	0.023377	0.007164	0.007164	0.007164
	3	1	0.004926	0.003137	0.003127	0.003001	0.002150	0.002097	0.002154
		2	0.014066	0.008310	0.008280	0.022222	0.007269	0.007077	0.00729
		3	0.033907	0.017581	0.028460	0.088392	0.027497	0.077169	0.016562
	s	r	$\sigma^{(0,2,2)}_{r,s:m:7}$	$\sigma^{(1,2,1)}_{r,s:m:7}$					
	1	1	0.001471	0.001471					
	2	1	0.001515	0.001521					
		2	0.003696	0.004694					
	3	1	0.001561	0.001564					
		2	0.003805	0.004811					
		3	0.013037	0.025120					
m	s	r	$\sigma^{(0,0,0,1)}_{r,s:m:5}$	$\sigma^{(1,0,0,0)}_{r,s:m:5}$	$\sigma^{(1,0,1,0)}_{r,s:m:6}$	$\sigma^{(1,2,0,0)}_{r,s:m:7}$	$\sigma^{(1,3,0,0)}_{r,s:m:8}$	$\sigma^{(1,1,2,0)}_{r,s:m:8}$	$\sigma^{(2,0,0,2)}_{r,s:m:8}$
4	1	1	0.002992	0.002992	0.002036	0.001471	0.001111	0.001111	0.001111
	2	1	0.003085	0.003098	0.002108	0.001521	0.001146	0.001146	0.001151
		2	0.008186	0.012214	0.007164	0.004694	0.003308	0.003308	0.004301

3	1	0.003137	0.003122	0.002154	0.001564	0.001187	0.001180	0.001183
	2	0.008310	0.012234	0.007291	0.004811	0.003416	0.003400	0.004406
	3	0.017581	0.032203	0.016562	0.025120	0.023745	0.008587	0.009620
4	1	0.003107	0.002941	0.002068	0.001498	0.001144	0.001160	0.001198
	2	0.008219	0.011465	0.006970	0.004594	0.003284	0.003333	0.004451
	3	0.017347	0.030049	0.015772	0.023660	0.022425	0.008369	0.009699
	4	0.036652	0.091765	0.081887	0.088111	0.087552	0.077225	0.018944

6. BLUEs of μ and σ

Suppose we obtain a progressively Type-II censored data from the location-scale parameter Hjorth distribution with c.d.f. as given in (6).

In this section, we make use of means, variances and covariances of progressively Type-II right censored order statistics as determined by using the recurrence relations given in Sections 3 and 4 for deriving the BLUEs of the location and scale parameters μ and σ as well as the variances and covariance of these estimates.

Let $Y_{1:m:n} \le Y_{2:m:n} \le ... \le Y_{m:m:n}$ be a progressively Type-II right censored sample from the location-scale parameter Hjorth distribution (6), and let $X_{i:m:n} = \frac{(Y_{i:m:n}-\mu)}{\sigma}$, $i = 1, 2, ..., m$, be the corresponding progressively Type-II right censored order statistics from the location-scale parameter Hjorth distribution.

Let us denote $E(X_{i:m:n})$ by μ_i, $Var(X_{i:m:n})$ by $\sigma_{i,i}$ and $Cov(X_{i:m:n}, X_{j:m:n})$ by $\sigma_{i,j}$; furthermore, let

$$Y = (Y_{1:m::n}, Y_{2:m:n}, ..., Y_{m:m:n})^T,$$

$$\mu = (\mu_1, \mu_2, ..., \mu_m)^T,$$

$$\mathbf{1} = (\underbrace{1, 1, ..., 1})^T$$

and

$$\sum = (\sigma_{r,s:n}),\ 1 \le r,\ s \le m.$$

Then, the BLUEs of μ and σ are obtained by minimizing the generalized variance $Q(\delta) = (Y - A\delta)^T \sum^T (Y - A\delta)$ with respect to δ, where $\delta = (\mu, \sigma)^T$, A is $m \times 2$ matrix $(\mathbf{1}, \mu)$, $\mathbf{1}$ is $m \times 1$ vector with components all 1's, μ is the mean vector of **X**, and $\sum$ is the variance-covariance matrix of **X**. The minimization leads to the expressions for the BLUES's of μ and σ as (see Anrold et al. [2] and Balakrishnan and Cohen [6])

$$\mu^* = \left\{ \frac{\mu^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1} - \mu^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} Y \ = \sum_{r=1}^{m} a_r Y_{r:m:n} \tag{23}$$

and

$$\sigma^* = \left\{ \frac{1^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1} - \mathbf{1}^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} Y \ = \sum_{r=1}^{m} b_r Y_{r:m:n}, \tag{24}$$

and the variances and covariance of these BLUEs are given by

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\mu^T \sum^{-1} \mu}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1, \tag{25}$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2 \tag{26}$$

and

$$Cov(\mu^*,\ \sigma^*) = \sigma^2 \left\{ \frac{-\mu^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3. \tag{27}$$

The coefficients of the BLUEs in (23) and (24) satisfy the conditions $\sum_{r=1}^{m} a_r = 1$ and $\sum_{r=1}^{m} b_r = 0$ respectively. The coefficients of the BLUEs for μ and σ, and variances and covariance of these estimates are presented in Tables 3, 4 and 5, respectively, for various sample sizes up to n = 8 and for different choices of m and progressive censoring schemes.

Table 3. Coefficients of the BLUEs of location parameter

S.No.	m	n	$(R_1, R_2, ..., R_m)$	$a_i, i = 1, 2, ..., m$				
1	2	4	1, 1	1.472961	-0.472961			
2	2	5	1, 2	1.569237	-0.569237			
3	2	5	2, 1	1.376189	-0.376189			
4	2	5	0, 3	1.765070	-0.765070			
5	2	5	3, 0	1.189212	-0.189212			
6	2	6	2, 2	1.472919	-0.472919			
7	2	7	3, 2	1.404446	-0.404446			
8	3	4	0, 0, 1	1.352880	-0.116892	-0.235988		
9	3	5	0, 0, 2	1.381140	-0.102141	-0.278998		
10	3	5	0, 1, 1	1.377761	-0.190427	-0.187334		
11	3	5	2, 0, 0	1.178155	-0.077954	-0.100201		
12	3	6	1, 2, 0	1.299918	-0.217482	-0.082436		
13	3	6	1, 0, 2	1.316218	-0.084399	-0.231819		
14	3	6	1, 1, 1	1.313292	-0.157708	-0.155584		
15	3	7	0, 2, 2	1.413979	-0.216323	-0.197656		
16	3	7	1, 2, 1	1.338522	-0.205897	-0.132625		
17	4	5	0, 0, 0, 1	1.252672	-0.069726	-0.057280	-0.125666	
18	4	5	1, 0, 0, 0	1.214988	-0.130246	-0.035790	-0.048953	
19	4	6	1, 0, 1, 0	1.202378	-0.055281	-0.090079	-0.057017	
20	4	7	1, 2, 0, 0	1.214988	-0.130246	-0.035790	-0.048953	
21	4	8	1, 1, 2, 0	1.231181	-0.084092	-0.105042	-0.042047	
22	4	8	1, 3, 0, 0	1.226968	-0.1531338	-0.03119558	-0.04263828	
23	4	8	2, 0, 0, 2	1.198464	-0.04572552	-0.03812323	-0.1146151	
24	5	6	0*4, 1	1.199748	-0.047806	-0.039169	-0.033903	-0.078870
25	5	6	1*3, 0*2	1.172194	-0.063154	-0.052897	-0.023079	-0.033063

Table 4. Coefficients of the BLUEs of scale parameter

S.No.	m	n	$(R_1, R_2, ..., R_m)$	$b_i, i = 1, 2, ..., m$				
1	2	4	1, 1	-7.126182	7.126182			
2	2	5	1, 2	-10.818890	10.818890			
3	2	5	2, 1	-7.149825	7.149825			
4	2	5	0, 3	-14.540870	14.540870			
5	2	5	3, 0	-3.596154	3.596154			
6	2	6	2,2	-10.859990	10.859990			
7	2	7	3, 2	-10.892980	10.892980			
8	3	4	0, 0, 1	-5.204608	1.575683	3.628925		
9	3	5	0, 0, 2	-7.137124	1.756899	5.380225		
10	3	5	0, 1, 1	-7.037568	3.408359	3.629209		
11	3	5	2, 0, 0	-3.303420	1.357226	1.946194		
12	3	6	1, 1, 1	-7.069212	3.436102	3.633110		
13	3	6	1, 2, 0	-6.700802	4.762049	1.938753		
14	3	6	1, 0, 2	-7.167823	1.776460	5.391363		
15	8	0	0, 2, 2	-11.014400	5.626087	5.388309		
16	3	7	1, 2, 1	-8.971029	5.342983	3.628045		
17	4	5	0, 0, 0, 1	-4.590840	1.114410	0.985681	2.490749	
18	4	5	1, 0, 0, 0	-5.498248	3.216274	0.905817	1.376158	
19	4	6	1, 0, 1, 0	-4.429882	1.076152	1.982419	1.371311	
20	4	7	1, 2, 0, 0	-5.498248	3.216274	0.905817	1.376158	
21	4	8	1, 1, 2, 0	-6.881663	2.382165	3.145720	1.353778	
22	4	8	1, 3, 0, 0	-6.686736	4.405297	0.9097395	1.371700	
23	4	8	2, 0, 0, 2	-5.978192	1.259320	1.085963	3.632910	
24	5	6	0*4, 1	-4.293554	0.882489	0.765931	0.720468	1.924666
25	5	6	1*3, 0*2	-4.978081	1.706113	1.500884	0.686065	1.085019

7. Best Linear Unbiased Predictors (BLUPs)

Based on observations on m progressively Type-II right censored order statistics $Y_{1:m:n}^{(R_1,\dots,R_m)}, ..., Y_{m:m:n}^{(R_1,\dots,R_m)}$, we discuss the prediction of times to failure of the last $R_m(\geq 1)$ units still surviving at the observation $Y_{m:m:n}^{(R_1,\dots,R_m)}$. Of course, one can discuss the prediction of other censored failure times in a similar manner as well. Doganaksoy and Balakrishnan [18] established that the BLUEs remain unchanged if the BLUPs of future failures are treated as observed values.

The BLUP of $Y_{m+1:m+1:n}^{(R_1,\dots,R_{m-1},0,R_m-1)}$ from any location-scale family of distributions is given by

$$Y_{m+1:m+1:n}^{(R_1,\dots,R_{m-1},0,R_m-1)^*} = \mu^* + \mu_{m+1:m+1:n}\sigma^* + w^T\Sigma^{-1}\left(Y - \mu^*\mathbf{1} - \sigma^*\mu\right)$$

and its variance is given by

$$\sigma^2\left\{\sigma_{m+1,m+1:m+1:n} - \omega^T\Sigma^{-1}\omega + \lambda_1^2 1^T\Sigma^{-1}1 + \lambda_2^2\mu^T\Sigma^{-1}\mu + 2\lambda_1\lambda_2\mu^T\Sigma^{-1}1\right\},$$

where

$$Y = \left(Y_{1:m+1:n}^{(R_1,\dots,R_{m-1},0,R_m-1)}, ..., Y_{m:m+1:n}^{(R_1,\dots,R_{m-1},0,R_m-1)}\right)^T,$$

$$E(Y) = \mu 1 + \sigma\mu = \left(\mu + \sigma\mu_{1:m+1:n}, ..., \mu + \sigma\mu_{m:m+1:n}\right)^T,$$

$$Var(Y) = \sigma^2\Sigma = \sigma^2\begin{pmatrix} \sigma_{1,1:m+1:n} & \cdots & \sigma_{1,m:m+1:n} \\ \cdots & \cdots & \cdots \\ \sigma_{m,1:m+1:n} & \cdots & \sigma_{m,m:m+1:n} \end{pmatrix},$$

Table 5. Variances and covariance of the BLUEs when $\mu = 0$ and $\sigma = 1$

S.No.	m	n	$(R_1, R_2, ..., R_m)$	$Var(\mu^*)$	$Var(\sigma^*)$	$Cov(\mu^*, \sigma^*)$
1	2	4	1,1	0.009165	1.022933	-0.066925
2	2	5	1,2	0.005791	1.054638	-0.054344
3	2	5	2, 1	0.005781	1.058957	-0.054367
4	2	5	0, 3	0.005781	1.058957	-0.054367
5	2	5	3, 0	0.005541	0.929466	-0.048682
6	2	6	2, 2	0.003967	1.059481	-0.045231
7	2	7	3, 2	0.002883	1.063107	-0.038747
8	3	4	0, 0, 1	0.006917	0.505254	-0.032818
9	3	5	0, 0, 2	0.004347	0.525413	-0.026700
10	3	5	0, 1, 1	0.004333	0.515230	-0.026301
11	3	5	2, 0, 0	0.004250	0.461464	-0.024107
12	3	6	1, 1, 1	0.002961	0.517949	-0.021895
13	3	6	1, 2, 0	0.002916	0.485591	-0.020685
14	3	6	1, 0, 2	0.002971	0.528151	-0.022239
15	3	7	0, 2, 2	0.002150	0.531686	-0.019013
16	3	7	1, 2, 1	0.002146	0.521983	-0.018780
17	4	5	0, 0, 0, 1	0.003864	0.335724	-0.017129
18	4	5	1, 0, 0, 0	0.003822	0.306044	-0.015956
19	4	6	1, 0, 1, 0	0.002615	0.321713	-0.013656
20	4	7	1, 2, 0, 0	0.001886	0.316162	-0.011459
21	4	8	1, 1, 2, 0	0.001434	0.331363	-0.010355
22	4	8	1, 3, 0, 0	0.002916	0.485591	-0.020685
23	4	8	2, 0, 0, 2	0.002971	0.528151	-0.022239
24	5	6	0*4, 1	0.002475	0.250814	-0.010516
25	5	8	1*3, 0*2	0.001345	0.238305	-0.007473

$E(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}) = \mu + \sigma\mu_{m+1:m+1:n}$,
$Var\left(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}\right) = \sigma^2\sigma_{m+1,m+1:m+1:n}$,
$Cov\left(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}, Y\right) = \sigma^2\omega = \sigma^2\left(\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n}\right)^T$,
$\lambda_1 = \frac{\mu^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}\mu w^T\Sigma^{-1}1 - \mu_{m+1:m+1:n}\mu^T\Sigma^{-1}1 + \mu^T\Sigma^{-1}1 w^T\Sigma^{-1}\mu}{\Delta}$
and
$\lambda_2 = \frac{-\mu^T\Sigma^{-1}1 + \mu^T\Sigma^{-1}1 w^T\Sigma^{-1}1 + \mu_{m+1:m+1:n}1^T\Sigma^{-1}1 + 1^T\Sigma^{-1}1 w^T\Sigma^{-1}\mu}{\Delta}$

with $\Delta = \left(\mu^T\Sigma^{-1}\mu\right)\left(1^T\Sigma^{-1}1\right) - \left(\mu^T\Sigma^{-1}1\right)^2$. Also, $\mu_{i:m+1:n}$ and $\sigma_{i,j:m+1:n}$ denote respectively the mean and covariance of the progressively Type-II right censored order statistics from the standard $(\mu = 0, \sigma = 1)$ distribution, and μ^* and σ^* are the BLUEs of μ and σ based on the progressively Type-II censoed sample Y. The BLUPs and their variances can therefore be readily computed from the means, variances and covariances of the progressively Type-II right censoed order statistics produced in Section 5. It is also illustrated in the next section with a numerical example using a real data set.

8. Illustrative Example

Consider the following data which represent failure times of air conditioning equipment in a boeing 720 airplane(Proschan [23]), arranged in increasing order of magnitude: 12, 21, 26, 27, 29, 29, 48, 57, 59, 70, 74, 153, 326, 386, 502. Take a random sample of size 7 from this data as: 21, 26, 27, 29, 29, 48, 57 and assuming that this sample data follows Hjorth distribution in (6) and then producing a progressively censored data from above sample data, we have

m	Scheme	$y_{i:m:n}$
3	1,2,1	21, 26, 29

Before carrying out the inferential analysis for these data, let us verify model assumption. Specifically, the progressively Type-II censored data $y_{i:3:7}$ were plotted against the values $\mu_{i:3:7}$ for $i = 1, 2, 3$ determined in Section 5 (as 0.037129, 0.09133, 0.232938) and this indicates a very high correlation (correlation coefficient is 0.922773) which suggest that the Hjorth model is a good fit model for these data.
In this case, we have $n = 7$, and based on progressively Type-II right censored sample $y_{1:3:7}, y_{2:3:7}, y_{3:3:7}$ presented above, we find BLUEs of μ and σ to be

$$\begin{aligned}\mu^* &= (1.338522 \times 21) + (-0.205897 \times 26) + (-0.132625 \times 29)\\ &= 18.9095\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= (-8.971029 \times 21) + (5.342983 \times 26) + (3.68045 \times 29)\\ &= 57.258999\end{aligned}$$

respectively, and their standard errors to be $SE(\mu^*) = \sigma^* \sqrt{\frac{\mu^T \Sigma^{-1} \mu}{\delta}} = 2.652661$ and $SE(\sigma^*) = \sigma^* \sqrt{\frac{1^T \Sigma^{-1} 1}{\delta}} = 41.36875765$.
We obtain the BLUP of $y_{4:4:7}$ to be $y^*_{4:4:7} = 37.40697$ and its standard error to be $SE(y^*_{4:4:7}) = 0.623606$ (by taking $w = (\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n})^T$ and progressive censoring scheme $(R_1, ..., R_{m-1}, 0, R_m - 1)$).

9. Conclusion

In this paper, we have established several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from Hjorth distribution. With the help of these relations and using R software, we have computed all the means, variances and covariances of progressively Type-II right censored order statistics for different sample sizes and all possible censoring schemes. These moments have then been used to obtain the best linear unbiased estimators (BLUEs) of location and scale parameters of location-scale Hjorth distribution (6), as well as the best linear unbiased predictors (BLUPs) of the times to failure of the surviving units in the experiment. Finally, a numerical example has been presented to illustrate all the inferential methods developed here using a real data set.

Acknowledgements

Authors are grateful to the Managing Editor and the learned referee for giving valuable comments, which led to an improvement in the presentation of the paper.

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