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Smoothness and Gaussian Density Estimates for Stochastic Functional Differential Equations with Fractional Noise

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Abstract In this paper, we study the density of the solution to a class of stochastic functional differential equations driven by fractional Brownian motion. Based on the techniques of Malliavin calculus, we prove the smoothness and establish upper and lower Gaussian estimates for the density.

Keywords Stochastic functional differential equations, Density estimates, Malliavin calculus, fractional Brownian motion

AMS 2010 subject classifications 34K50, 60H07, 60H10

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1. Introduction

In the last decade, Gaussian density estimates for the solutions of various stochastic equations have been intensively studied. Particularly, the class of stochastic equations with fractional noise has been discussed by several authors, see [1, 2, 4, 8] and references therein.

We recall that fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = (B_t^H)_{t\in\mathbb{R}_+}$ with covariance function

$$R_H(t,s) := E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H > \frac{1}{2}$, B_t^H admits the so-called Volterra representation

$$B_t^H = \int_0^t K_H(t,s)dB_s, \tag{1}$$

where $(B_t)_{t\in_+}$ is a standard Brownian motion, the kernel K_H is defined by

$$K_H(t,s) = C_H s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad s \le t$$

with $C_H := \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}}$, where β is the Beta function.

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In this paper, we consider stochastic functional differential equations of the form

$$\begin{cases} X_t = \eta(0) + \int\limits_0^t \left[\int\limits_{-r}^0 \rho(u) g(X_{u+s}) du + a(s, X_s) \right] ds + \int\limits_0^t \sigma(s, X_s) dB_s^H, \quad t \in [0,T], \\ X_t = \eta(t), \quad t \in [-r, 0], \end{cases} \tag{2}$$

where $r > 0$ is delay time, the kernel ρ and initial condition η are deterministic functions on $[-r, 0]$. The stochastic integral is interpreted as a pathwise Riemann-Stieltjes integral, which has been frequently used in the studies related to fBm. We refer the reader to [12] for a detailed presentation of this integral.

The density of solutions to the equation (2) has been discussed in some special cases. When $H = \frac{1}{2}$, B^H reduces to standard Brownian motion and in this case, the existence and smoothness of the probability density of solutions were proved by Takeuchi in [11]. When $H > \frac{1}{2}$, Gaussian density estimates were obtained by Dung et al. in [6] for the equation (2) with $g = 0$. However, the case of $g \neq 0$ has not investigated yet. Thus, in the present paper, our aim is to establish analogue results for the equation (2) with $g \neq 0$ and $H > \frac{1}{2}$. More specifically, we obtain the following properties:

(i) the existence and Gaussian estimates for the density of solutions,

(ii) the smoothness of the density of solutions.

It should be noted that the information about the density will be very useful in practical studies, see e.g. [7]. In a spirit close to [6, 11], the main tools of this paper are the techniques of Malliavin calculus. However, we would like to emphasize that the complexity of stochastic integrals with respect to fBm and the appearance of delayed integral term in (2) require a fine analysis for proving the properties (i) and (ii). The rest of this article is organized as follows. In Section 2, we recall some fundamental concepts of Malliavin calculus and a general Gaussian estimate for the density of Malliavin differentiable random variables. The main results of the paper are stated and proved in Section 3. The conclusion is given in Section 4.

2. Preliminaries

Let us recall some elements of Malliavin calculus with respect to Brownian motion B, where B is used to present B_t^H as in (1) (for more details see [9]). We suppose that $(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filtration generated by the Brownian motion B. For $h \in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int\limits_0^T h(t) dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(B(h_1), ..., B(h_n)), \tag{3}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(^n)$, $h_1, ..., h_n \in L^2[0,T]$. If F has the form (3), we define its Malliavin derivative as the process $DF := \{D_t F, t \in [0,T]\}$ given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1), ..., B(h_n)) h_k(t).$$

More generally, we can define the kth order derivative $D^k F$ by iterating the derivative operator k times, i.e. $D^k_{t_1,...,t_k} F = D_{t_k} ... D_{t_1} F$. For any integer k and any $p \geq 1$, we denote by $D^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p := E|F|^p + E\left[\int\limits_0^T |D_{u_1} F|^p du_1\right] + E\left[\int\limits_0^T ... \int\limits_0^T |D^k_{u_1,...,u_k} F|^p du_1 .... du_k\right].$$

A random variable F is said to be Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$.

In order to obtain Gaussian density estimates for solutions to the equation (2), we will use a general criterion established recently by Nourdin and Viens in [10]. We recall here [6, Theorem 2.4] for a convenient version which can be of interest for the readers who are not used to working with the Ornstein-Uhlenbeck operator.

Proposition 1
Let F be in $\mathbb{D}^{1,2}$ with mean zero. If there exist positive constants c, C such that, for all $x \in$, almost surely

$$c \leq \int_0^T D_r F E[D_r F|\mathcal{F}_r] dr \leq C,$$

then the density ρ_F of F exists and satisfies, for almost all $x \in$

$$\frac{E|F|}{2C} \exp\left(-\frac{x^2}{2c}\right) \leq \rho_F(x) \leq \frac{E|F|}{2c} \exp\left(-\frac{x^2}{2C}\right). \tag{4}$$

3. The main results

In the whole this section, we consider the equation (2) with the following fundamental assumptions. Note that the conditions on a and σ are similar to that required in Section 5 of [6].

(A_1) The coefficients $a, g, \sigma \in \mathcal{C}_b^{1,1}([0,T]\times)$, and there exists a constant $m > 0$ so that $|\sigma(t,x)| \geq m$, for all $(t,x) \in [0,T]\times$.

(A_2) The kernel $\rho : [0,T] \to$ satisfies:

$$\int_0^T |\rho(s)| ds < \infty.$$

Let us first give a short discussion about the existence and uniqueness of solutions. We denote by $\mathcal{C}_b^{1,1}([0,T]\times)$ the space of bounded functions $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with bounded partial derivatives of the first order and we write

$$f_1'(s,x) = \frac{\partial f(s,x)}{\partial s}, \qquad f_2'(s,x) = \frac{\partial f(s,x)}{\partial x}.$$

We define the function

$$F(t,x) := \int_0^x \frac{1}{\sigma(t,u)} du, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

For $(t,z) \in [0,T] \times \mathbb{R}$, consider the function $\Phi(t,z) := F(t,z) - x$, where $x \in \mathbb{R}$ is fixed. Since $\Phi_2'(t,z) = \sigma(t,z)^{-1} \neq 0$, by the Implicit Function Theorem, there exists a function $\bar{G}(t,x)$ such as $\Phi(t, \bar{G}(t,x)) = 0$, i.e. $F(t, \bar{G}(t,x)) = x$. Moreover, we have

$$F_2'(t,x) = \sigma(t,x)^{-1} \quad \text{and} \quad F_1'(t,x) = -\int_0^x \frac{\sigma_1'(t,u)}{(\sigma(t,u))^2} du,$$

$$\bar{G}_2'(t,x) = (F_2'(t, \bar{G}(t,x)))^{-1} = \sigma(t, \bar{G}(t,x)), \tag{5}$$

$$\bar{G}_1'(t,x) = -\frac{F_1'(t,\bar{G}(t,x))}{F_2'(t,\bar{G}(t,x))} = -F_1'(t,\bar{G}(t,x))\sigma(t,\bar{G}(t,x)). \tag{6}$$

Set $G(t,x)$ defined by

$$\begin{cases} G(t,x) := \bar{G}(t,x) & t \in [0,T] \\ G(t,x) = \eta(t) & t \in [-r,0] \end{cases}$$

We consider stochastic functional differential equation with additive noise

$$Y_t = y_0 + \int_0^t \left(A(s, Y_s) + \frac{1}{\sigma(s, G(s, Y_s))} \int_{s-r}^{s} \rho(u-s) g(u, G(u, Y_u)) du \right) ds + B_t^H, \tag{7}$$

where $y_0 := F(0, x_0)$, and $A(y, s) = F_1'(s, G(s, y)) + \frac{a(s, G(s,y)}{\sigma(s, G(s,y))}$.

It was already pointed out in [6] that $A(y, s)$ is Lipschitz and has linear growth. On the other hand, under Assumptions (A_1) and (A_2), we can check that the functions $\frac{1}{\sigma(s,G(s,y))}$ and $g(s, G(s, y))$ are also Lipschitz and have linear growth. Hence, by repeating the computations presented in the proof of Proposition 3.1 in [3], we can infer that the equation (7) admits a unique strong solution $(Y_t)_{t\in[0,T]}$.

Based on the properties of $(Y_t)_{t\in[0,T]}$, we have the following propositions.

Proposition 2
Let Assumptions (A_1) and (A_2) hold. Then, the equation (2) has a unique strong solution given by $X_t = G(t, Y_t)$, $-r \le t \le T$. This solution is an $\mathcal{F}_t$-adapted process and, for all $\varepsilon \in (0, H)$, whose trajectories are Hölder continuous of order $H - \varepsilon$ on $[0, T]$.

Proof
The proof is similar to that of Lemma 5.1 in [6]. So we omit it. □

Proposition 3
Under the Assumptions (A_1) and (A_2), the unique strong solution $(X_t)_{t\in[-r,T]}$ to the equation (2) is Malliavin differentiable and satisfies $D_\theta X_t = 0$ for $\theta > t$ or $t \in [-r, 0]$, and for all $0 \le \theta \le t \le T$,

$$D_\theta X_t = \sigma(t, X_t) \left(\int_\theta^t N(s, X_s) D_\theta X_s ds + \int_\theta^t \frac{1}{\sigma(s, X_s)} \int_{s-r}^{s} \rho(u-s) g_2'(u, X_u) D_\theta X_u du ds + K_H(t, \theta) \right), \tag{8}$$

where

$$N(s, X_s) := -\frac{\sigma_1'(s, X_s)}{\sigma^2(s, X_s)} - \frac{\sigma_2'(s, X_s)}{\sigma^2(s, X_s)} \int_{s-r}^{s} \rho(u-s) g(u, X_u) du + \frac{a_2'(s, X_s)\sigma(s, X_s) - a(s, X_s)\sigma_2'(s, X_s)}{\sigma^2(s, X_s)}.$$

Proof
Let $(Y_t)_{t\in[0,T]}$ be the unique strong solution to (7). By using the same argument as in the proof of Lemma 5.3 in [6], we have $Y_t \in \mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$\begin{aligned} D_\theta Y_t = \mathbf{1}_{[0,t]}(\theta) \Bigg(& \int_\theta^t M(s, Y_s) D_\theta Y_s ds + K_H(t, \theta) \\ & + \int_\theta^t \frac{1}{\sigma(s, G(s, Y_s))} \int_{s-r}^{s} \rho(u-s) g_2'(u, G(u, Y_u)) \sigma(u, G(u, Y_u)) D_\theta Y_u du ds \Bigg), \quad 0 \le t \le T, \end{aligned} \tag{9}$$

where

$$\begin{aligned} M(s, Y_s) := & -\frac{\sigma_1'(s, G(s, Y_s))}{\sigma(s, G(s, Y_s))} - \frac{\sigma_2'(s, G(s, Y_s))}{\sigma(s, G(s, Y_s))} \int_{s-r}^{s} \rho(u-s) g(u, G(u, Y_u)) du \\ & + \frac{a_2'(s, G(s, Y_s))\sigma(s, G(s, Y_s)) - a(s, G(s, Y_s))\sigma_2'(s, G(s, Y_s))}{\sigma(s, G(s, Y_s))}. \end{aligned}$$

From the relation $X_t = G(t, Y_t)$ and the chain rule of Malliavin derivatives (see Proposition 1.2.3 in [9]), we have $X_t \in \mathbb{D}^{1,2}$, and

$$D_\theta X_t = G_2'(t, Y_t) D_\theta Y_t = \sigma(t, X_t) D_\theta Y_t, \qquad 0 \le \theta \le t \le T. \tag{10}$$

This, combined with (9), gives us (8). So the proof of Proposition is complete. □

From now on, we will use the symbol C to denote a generic constant, whose value may change from one line to another.

Proposition 4
Let $(X_t)_{t\in[-r,T]}$ be the solution to the equation (2). Assume that (A_1), (A_2) hold. Then there exists a finite constant $C > 0$ such that:

$$|D_\theta X_t| \le C K_H(t, \theta), \quad a.s.$$

for all $0 \le \theta \le t \le T$.

Proof
From (9), (A_1), (A_2) and the boundedness of $A(t, Y_t)$, we have

$$\begin{aligned} |D_\theta Y_t| &\le \int_\theta^t |M(s, Y_s)||D_\theta Y_s| ds + C \int_\theta^t |D_\theta Y_s| ds + K_H(t, \theta) \\ &\le C \int_\theta^t |D_\theta Y_s| ds + K_H(t, \theta), \qquad \text{for all } 0 \le \theta \le t \le T. \end{aligned} \tag{11}$$

An application of Gronwall's inequality now gives that

$$|D_\theta Y_t| \le K_H(t, \theta) + C \int_\theta^t K_H(s, \theta) \exp\{C(t-s)\} ds \le K_H(t, \theta) + C \int_\theta^t K_H(s, \theta) ds.$$

Since $\partial_1 K_H(t, \theta) \ge 0$, we have

$$|D_\theta Y_t| \le K_H(t, \theta) + C(t-\theta) K_H(t, \theta) \le C K_H(t, \theta). \tag{12}$$

From (10), (12) and the boundedness of $\sigma(t, x)$, we have

$$|D_\theta X_t| \le C|\sigma(t, X_t)| K_H(t, \theta) \le C K_H(t, \theta), \qquad \text{for all } 0 \le \theta \le t \le T. \tag{13}$$

□

Proposition 5
Let $(X_t)_{t\in[-r,T]}$ be the solution to the equation (2). Assume that (A_1), (A_2) hold. Then there exists a finite constant $c > 0$ such that

$$\int_0^t D_\theta X_t E[D_\theta X_t | \mathcal{F}_\theta] d\theta \ge c t^{2H}, \quad a.s$$

for all $t \in (0, T]$.

Proof
It follows from (9) that $(D_\theta Y_t)_{t\in[\theta,T]}$ solves the following ordinary differential equation

$$\begin{aligned} &dD_\theta Y_t \\ &= \left(M(t, Y_t) D_\theta Y_t + \frac{1}{\sigma(t, G(s, Y_t))} \int_{t-r}^t \rho(s-t) g_2'(s, G(s, Y_s)) \sigma(s, G(s, Y_s)) D_\theta Y_s ds \right) dt + \partial_1 K_H(t, \theta) dt, \end{aligned}$$

with the initial condition $D_\theta Y_t|_{t=\theta} = 0$, where $\partial_1 K_H(t,\theta) = \frac{\partial K_H(t,\theta)}{\partial t}$. By the Comparison Theorem and $\partial_1 K_H(t,\theta) \geq 0$, it is easy that $D_\theta Y_t \geq 0$. So we have that $D_\theta X_t \geq 0$ when $\sigma(t,x) > 0$ and $D_\theta X_t \leq 0$ when $\sigma(t,x) < 0$. Hence, for all $0 \leq \theta \leq t \leq T$,

$$D_\theta X_t E[D_\theta X_t|\mathcal{F}_\theta] \geq 0 \quad a.s.$$

Define

$$h(t) := \int_{(1-\varepsilon)t}^{t} D_\theta X_t E[D_\theta X_t|\mathcal{F}_\theta]d\theta = \int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)D_\theta Y_t E[\sigma(t,X_t)D_\theta Y_t|\mathcal{F}_\theta]d\theta, \quad 0 < \varepsilon \leq 1.$$

From the equation (9), we have

$$D_\theta Y_t = U(t,\theta) + K_H(t,\theta), \quad \theta \leq t.$$

where

$$U(t,\theta) := \int_\theta^t M(s,Y_s)D_\theta Y_s ds + \int_\theta^t \frac{1}{\sigma(s,G(s,Y_s))} \int_{s-r}^{s} \rho(u-s)g_2'(u,G(u,Y_u))\sigma(u,G(u,Y_u))D_\theta Y_u du ds$$

Thus we can rewrite $h(t)$ as follows

$$\begin{aligned} h(t) = &\int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)U(t,\theta)E[\sigma(t,X_t)U(t,\theta)|\mathcal{F}_\theta]d\theta + \int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)U(t,\theta)E[\sigma(t,X_t)K_H(t,\theta)|\mathcal{F}_\theta]d\theta \\ &+ \int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)K_H(t,\theta)E[\sigma(t,X_t)U(t,\theta)|\mathcal{F}_\theta]d\theta + \int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)K_H(t,\theta)E[\sigma(t,X_t)K_H(t,\theta)|\mathcal{F}_\theta]d\theta. \end{aligned}$$

From (12) and $\partial_1 K_H(t,\theta) \geq 0$,

$$|U(t,\theta)| \leq C\int_\theta^t K_H(s,\theta)ds \leq CK_H(t,\theta)(t-\theta), \quad \theta \leq t. \tag{14}$$

On the other hand, for all $0 \leq s \leq t$, we have

$$\begin{aligned} E|B_t^H - B_s^H|^2 =& E\left(\int_0^s [K_H(t,u) - K_H(s,u)]dB_u + \int_s^t K_H(t,u)dB_u\right)^2 \\ &= E\left(\int_0^s [K_H(t,u) - K_H(s,u)]dB_u\right)^2 + E\left(\int_s^t K_H(t,u)dB_u\right)^2 \\ &\geq E\left(\int_s^t K_H(t,u)dB_u\right)^2 = \int_s^t K_H^2(t,u)du. \end{aligned}$$

Using the fact that $E|B_t^H - B_s^H|^2 = |t-s|^{2H}$, we deduce

$$\int_s^t K_H^2(t,u)du \leq |t-s|^{2H}, \ 0 \leq s \leq t. \tag{15}$$

From (14) and (15), we can get the following estimates

$$\begin{aligned}\Big|\int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)U(t,\theta)E[\sigma(t,X_t)K_H(t,\theta)|\mathcal{F}_\theta]d\theta\Big| &\le C\int_{(1-\varepsilon)t}^{t}(t-\theta)K_H^2(t,\theta)d\theta \\ &\le C(\varepsilon t)\int_{(1-\varepsilon)t}^{t}K_H^2(t,\theta)d\theta \\ &\le C(\varepsilon t)(\varepsilon t)^{2H} = C(\varepsilon t)^{2H+1},\end{aligned}$$

$$\Big|\int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)K_H(t,\theta)E[\sigma(t,X_t)U(t,\theta)|\mathcal{F}_\theta]d\theta\Big| \le C(\varepsilon t)^{2H+1},$$

$$\begin{aligned}\Big|\int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)U(t,\theta)E[\sigma(t,X_t)U(t,\theta)|\mathcal{F}_\theta]d\theta\Big| &\le C\int_{(1-\varepsilon)t}^{t}(t-\theta)^2K_H^2(t,\theta)d\theta \\ &\le C(\varepsilon t)^2\int_{(1-\varepsilon)t}^{t}K_H^2(t,\theta)d\theta = C(\varepsilon t)^{2H+2}.\end{aligned}$$

From the definition of $K_H(t,r)$, for all $0<r\le t$, we deduce

$$K_H(t,r)\ge C_H\int_r^t (u-r)^{H-\frac{3}{2}}du = \frac{C_H}{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}}$$

So, for all $0<s\le t$, we have

$$\begin{aligned}\int_s^t K_H^2(t,\theta)d\theta &\ge \left(\frac{C_H}{H-\frac{1}{2}}\right)^2\int_s^t (t-\theta)^{2H-1}d\theta \\ &= \frac{1}{2H}\left(\frac{C_H}{H-\frac{1}{2}}\right)^2(t-s)^{2H} \\ &= C'_H(t-s)^{2H}, \qquad (16)\end{aligned}$$

where $C'_H := \frac{1}{2H}\left(\frac{C_H}{H-\frac{1}{2}}\right)^2$. Making use of the elementary inequality $|a+b|\ge|a|-|b|$ yields

$$\begin{aligned}h(t) &\ge \Big|\int_{(1-\varepsilon)t}^{t} \sigma(t,X_t)K_H(t,\theta)E[\sigma(t,X_t)K_H(t,\theta)|\mathcal{F}_\theta]d\theta\Big| - 2C(\varepsilon t)^{2H+1} - C(\varepsilon t)^{2H+2} \\ &\ge (\varepsilon t)^{2H}\left(C'_H m^2 - 2C(\varepsilon t) - C(\varepsilon t)^2\right).\end{aligned}$$

Now we choose $\varepsilon\in(0,1]$ such that

$$C(\varepsilon T)^2 + 2C(\varepsilon T) \le \frac{C'_H m^2}{2}.$$

Then, we get

$$h(t) \geq \frac{C'_H m^2}{2}(\varepsilon t)^{2H} := ct^{2H}.$$

So we can finish the proof of Proposition because

$$\int_0^t D_\theta X_t E[D_\theta X_t|\mathcal{F}_\theta]d\theta \geq h(t).$$

□

We now are ready to formulate and prove the main results of this paper.

Theorem 1
Asume that (A_1) and (A_2) hold and let $(X_t)_{t\in[-r,T]}$ be the unique strong solution to the equation (2). Then, for each $t \in (0,T]$, the density ρ_{X_t} exists and satisfies the bounds for all $x \in \mathbb{R}$

$$\frac{E|X_t - EX_t|}{2Ct^{2H}} \exp\left(-\frac{(x-EX_t)^2}{2ct^{2H}}\right) \leq \rho_{X_t}(x) \leq \frac{E|X_t - EX_t|}{2ct^{2H}} \exp\left(-\frac{(x-EX_t)^2}{2Ct^{2H}}\right). \tag{17}$$

where c, C are finite positive constants.

Proof
For each $t \in (0,T]$, we consider the random variable $F := X_t - EX_t$. Clearly, F has mean zero and is Malliavin differentiable with $D_\theta F = D_\theta X_t$. Hence, by Propositions 4 and 5, we can get

$$0 < ct^{2H} \leq \int_0^T D_\theta F E[D_\theta F|\mathcal{F}_\theta]d\theta = \int_0^t D_\theta X_t E[D_\theta X_t|\mathcal{F}_\theta]d\theta \leq Ct^{2H},$$

where c, C are some finite positive constants. In view of Proposition 1, we can conclude that the density ρ_F of the random variable F exits and satisfies

$$\frac{E|F|}{2Ct^{2H}} \exp\left(-\frac{x^2}{2ct^{2H}}\right) \leq \rho_F(x) \leq \frac{E|F|}{2ct^{2H}} \exp\left(-\frac{x^2}{2Ct^{2H}}\right), \ x \in \mathbb{R},$$

which gives us (17) because $\rho_{X_t}(x) = \rho_F(x - EX_t)$. □

Theorem 2
Suppose the Assumptions (A_1) and (A_2). Let $(X_t)_{t\in[-r,T]}$ be the solution to the equation (2). In addition, we assume that a, g and σ are infinitely differentiable functions in x with bounded derivatives of all orders. Then, for each $t \in (0,T]$, the random variable X_t has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}$.

Proof
Fix $t \in (0,T]$, thanks to Theorem 2.1.4 in [9], we have to check the following properties

(a) $X_t \in \mathbb{D}^\infty = \bigcap_{i\geq 1}\bigcap_{p\geq 1}\mathbb{D}^{i,p}$,

(b) $\left(\int_0^t |D_\theta X_t|^2 d\theta\right)^{-1} \in \bigcap_{p\geq 1} L^p(\Omega)$.

It is easy to verify that the coefficients of the equation (7) are infinitely differentiable in y with bounded derivatives of all orders. Hence, we can infer that $Y_t \in \mathbb{D}^\infty$. So X_t does. Let us now check the property (b).

By Proposition 3, we have

$$D_\theta X_t = \sigma(t,X_t)\left(\int_\theta^t N(s,X_s)D_\theta X_s ds + \int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^s \rho(u-s)g_2'(u,X_u)D_\theta X_u duds + K_H(t,\theta)\right).$$

Hence,

$$\begin{aligned}&\int_0^t |D_\theta X_t|^2 d\theta \\ &= \int_0^t \left(\sigma(t,X_t)\left(\int_\theta^t N(s,X_s)D_\theta X_s ds + \int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^s \rho(u-s)g_2'(u,X_u)D_\theta X_u duds + K_H(t,\theta)\right)\right)^2 d\theta.\end{aligned}$$

For each $y \geq y_0 := \dfrac{4}{C_H' t^{2H} m^2}$, where $C_H' = \frac{1}{2H}\left(\frac{C_H}{H-\frac{1}{2}}\right)^2$, the real number $\varepsilon := \left(\dfrac{4}{yC_H' m^2 t^{2H}}\right)^{\frac{1}{2H}}$ belongs to $(0,1]$. Using the fundamental inequality $(a+b+c)^2 \geq \frac{a^2}{2} - 2(b^2+c^2)$ and (16) we obtain

$$\begin{aligned}\int_0^t |D_\theta X_t|^2 d\theta \geq& \int_{t(1-\varepsilon)}^t \frac{\sigma^2(t,X_t)K_H^2(t,\theta)}{2} d\theta - 2\int_{t(1-\varepsilon)}^t \left(\sigma(t,X_t)\int_\theta^t N(s,X_s)D_\theta X_s ds\right)^2 d\theta \\ &- 2\int_{t(1-\varepsilon)}^t \left(\sigma(t,X_t)\int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^s \rho(u-s)g_2'(u,X_u)D_\theta X_u duds\right)^2 d\theta, \\ &\geq \frac{C_H' m^2(\varepsilon t)^{2H}}{2} - I_y(t) \\ &= \frac{2}{y} - I_y(t),\end{aligned}$$

where

$$\begin{aligned}I_y(t) =& 2\int_{t(1-\varepsilon)}^t \left(\sigma(t,X_t)\int_\theta^t N(s,X_s)D_\theta X_s ds\right)^2 d\theta \\ &+ 2\int_{t(1-\varepsilon)}^t \left(\sigma(t,X_t)\int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^s \rho(u-s)g_2'(u,X_u)D_\theta X_u duds\right)^2 d\theta\end{aligned}$$

By Markov's inequality, we have

$$P\left(\int_0^t |D_\theta X_t|^2 d\theta \leq \frac{1}{y}\right) \leq P\left(\frac{2}{y} - I_y(t) \leq \frac{1}{y}\right) = P\left(I_y(t) \geq \frac{1}{y}\right) \leq y^{p/2} E\left(|I_y(t)|^{p/2}\right) \ \ \forall\, p \geq 2. \tag{18}$$

Under the assumptions (A_1), (A_2) and the inequality $(|a|+|b|)^{p/2} \leq 2^{p/2-1}(|a|^{p/2}+|b|^{p/2})$, we can get

$$
\begin{aligned}
E|I_y(t)|^{p/2} &= 2^{p/2} E\Big(\int_{t(1-\varepsilon)}^{t} \Big(\sigma(t,X_t)\int_\theta^t N(s,X_s)D_\theta X_s ds\Big)^2 d\theta \\
&+ \int_{t(1-\varepsilon)}^{t} \Big(\sigma(t,X_t)\int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^{s} \rho(u-s) g_2'(u,X_u) D_\theta X_u du ds\Big)^2 d\theta\Big)^{p/2} \\
&\leq 2^{p-1} E\left(\int_{t(1-\varepsilon)}^{t} \Big(\sigma(t,X_t)\int_\theta^t N(s,X_s)D_\theta X_s ds\Big)^2 d\theta\right)^{p/2} \\
&+ 2^{p-1} E\left(\int_{t(1-\varepsilon)}^{t} \Big(\sigma(t,X_t)\int_\theta^t \frac{1}{\sigma(s,X_s)}\int_{s-r}^{s} \rho(u-s) g_2'(u,X_u) D_\theta X_u du ds\Big)^2 d\theta\right)^{p/2} \\
&\leq 2^{p-1} E\left(\Big(C\int_{t(1-\varepsilon)}^{t} \Big(\int_\theta^t D_\theta X_s ds\Big)^2 d\theta\Big)^{p/2} + \Big(C\int_{t(1-\varepsilon)}^{t} \Big(\int_\theta^t\int_{s-r}^{s} \rho(u-s) D_\theta X_u du ds\Big)^2 d\theta\Big)^{p/2}\right) \\
&\leq CE\left(\Big(\int_{t(1-\varepsilon)}^{t}\int_\theta^t |D_\theta X_s|^2 ds d\theta\Big)^{p/2} + \Big(\int_{t(1-\varepsilon)}^{t} \Big(\int_\theta^t\int_{s-r}^{s} \rho(u-s) D_\theta X_u du ds\Big)^2 d\theta\Big)^{p/2}\right),
\end{aligned}
$$

where C is some positive constant. By using Hölder's inequality we obtain

$$
\begin{aligned}
\Big(\int_\theta^t\int_{s-r}^{s} \rho(u-s) D_\theta X_u du ds\Big)^2 &\leq (t-\theta)\int_\theta^t \Big(\int_{s-r}^{s} |\rho(u-s) D_\theta X_u| du\Big)^2 ds \\
&\leq T\int_\theta^t \Big(\int_{s-r}^{s} |\rho(u-s)| du\Big)\Big(\int_{s-r}^{s} |\rho(u-s)||D_\theta X_u|^2 du\Big) ds \\
&\leq T\Big(\int_{-r}^{0} |\rho(u)| du\Big)^2 \int_\theta^t |D_\theta X_u|^2 du.
\end{aligned}
$$

So it holds that

$$
E|I_y(t)|^{\frac{p}{2}} \leq CE\left(\int_{t(1-\varepsilon)}^{t}\int_\theta^t |D_\theta X_s|^2 ds d\theta\right)^{p/2} \quad \forall\ p \geq 2.
$$

Using Proposition 4 and (15), we can verify that

$$
\begin{aligned}
E|I_y(t)|^{p/2} &\le CE\left(\int_{t(1-\varepsilon)}^{t}\int_{\theta}^{t} K_H^2(s,\theta)dsd\theta\right)^{p/2} = CE\left(\int_{t(1-\varepsilon)}^{t}\int_{\theta}^{t} K_H^2(s,\theta)dsd\theta\right)^{p/2} \\
&= CE\left(\int_{t(1-\varepsilon)}^{t}\int_{t(1-\varepsilon)}^{s} K_H^2(s,\theta)d\theta ds\right)^{p/2} \le CE\left(\int_{t(1-\varepsilon)}^{t} (s-t(1-\varepsilon))^{2H}ds\right)^{p/2} \\
&= \frac{C}{2H+1}(\varepsilon t)^{p(2H+1)/2} = C\left(\frac{4}{yC'_H m^2}\right)^{\frac{p(2H+1)}{4H}} = C\left(\frac{4}{ym^2}\right)^{\frac{p(2H+1)}{4H}} \quad \forall\ p\ge 2. \qquad (19)
\end{aligned}
$$

From (18) and (19), we deduce

$$
P\left(\int_0^t |D_\theta X_t|^2 d\theta \le \frac{1}{y}\right) \le Cy^{p/2}\left(\frac{4}{ym^2}\right)^{\frac{p(2H+1)}{4H}} \quad \forall\ p\ge 2.
$$

Now for any $\alpha \ge 1$ and $p > 4H\alpha$, we have the following estimates

$$
\begin{aligned}
E\left(\int_0^t |D_\theta X_t|^2 d\theta\right)^{-\alpha} &= \int_0^\infty \alpha y^{\alpha-1} P\left(\int_0^t |D_\theta X_t|^2 d\theta \le \frac{1}{y}\right) dy \\
&\le \int_0^{y_0} \alpha y^{\alpha-1}dy + \int_{y_0}^\infty \alpha y^{\alpha-1} P\left(\int_0^t |D_\theta X_t|^2 d\theta < \frac{1}{y}\right) dy \\
&\le y_0^\alpha + \alpha C\int_{y_0}^\infty y^{\alpha-1}y^{p/2}\left(\frac{4}{ym^2}\right)^{\frac{p(2H+1)}{4H}} dy \\
&= y_0^\alpha + \alpha C\left(\frac{4}{m^2}\right)^{\frac{p(2H+1)}{4H}} \frac{y_0^{\alpha-\frac{p}{4H}}}{\frac{p}{4H}-\alpha}.
\end{aligned}
$$

Recalling $y_0 = \dfrac{4}{C'_H t^{2H} m^2}$, we conclude that

$$
E\left(\int_0^t |D_\theta X_t|^2 d\theta\right)^{-\alpha} < \infty \ \ \forall\ \alpha \ge 1.
$$

So the property (b) is proved. This finishes the proof of Theorem. □

4. Conclusion

In this paper, we employed the techniques of Malliavin calculus to obtain smoothness and Gaussian density estimates for solutions to a fundamental class of stochastic functional differential equations with fractional noise. Our results develop further the studies initated in [6, 11] and hence, our work partly enriches the knowledge of the theory of stochastic functional differential equations.

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