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A Generalized Modification of the Kumaraswamy Distribution for Modeling and Analyzing Real-Life Data

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Abstract In this paper, a generalized modification of the Kumaraswamy distribution is proposed, and its distributional and characterizing properties are studied. This distribution is closed under scaling and exponentiation, and has some well-known distributions as special cases, such as the generalized uniform, triangular, beta, power function, Minimax, and some other Kumaraswamy related distributions. Moment generating function, Lorenz and Bonferroni curves, with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted, L, and trimmed L moments, are derived. The maximum likelihood estimation method is used for estimating its parameters and applied to six different simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes. Finally, four real-life data sets are used to illustrate the usefulness and the flexibility of this distribution in application to real-life data.

Keywords Generalized Modification of the Kumaraswamy Distribution, Maximum Likelihood Estimator, Moments, Simulation Study, Applications.

AMS 2010 subject classifications 60E05, 62E15, 65C05.

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1. Introduction

There are a lot of popular statistical distributions that used for modeling and analyzing real data, and so did their generalization by mixing these distributions with each other. In particular, the generalization of some well-known statistical distributions provides flexible and effective modeling in the case when the standard distribution cannot statistically fit the data. Even these generalization distributions may have some limitations in modeling and analyzing some real data, especially, non-standard and abnormal data. New modifications need to be considered and studied in order to use the modeling and analysis such as non-standard data real-life data.
The Kumaraswamy distribution was first introduced in 1980, by Kumaraswamy [49] as a probability density function for double-bounded random processes, then later so many researchers continued further studied about the distribution. Garg [27], Nadarajah [59], Jones [45], Mitnik [56], Gholizadeh et al [29], and Mitnik [57], developed further theoretical research on the distribution. In particular, Gholizadeh et al [29] considered classical and Bayesian point and interval estimators for the shape parameter of the Kumaraswamy distribution using Monte-Carlo simulation. Hussian [36] used the maximum likelihood estimation and Bayesian estimation methods to estimate the parameters of the

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Kumaraswamy distribution using both simple random sampling and ranked set sampling techniques, and used a simulation study to compare the resultant estimators in terms of their biases and mean square errors.

Tables 1a and 1b show lists of distributions that are modified and/or generalized version of the Kumaraswamy distribution. All researchers showed in Tables 1a and 1b studied some of its structural, mathematical, statistical properties of their distribution, as well as some proposed parameters estimates methods. In particular, de Pascoa et al [21] introduced the Kumaraswamy generalized gamma distribution, as a capable distribution of modeling bathtub-shaped hazard rate functions, study some of its properties, adopted the method of maximum likelihood and a Bayesian procedure to estimate its parameters, and illustrated its usefulness on two real data.

Cordeiro and de Castro [14] introduced the Kumaraswamy generalized distribution, as a distribution that has flexible different shapes for the hazard function to be used in modeling survival data. Lemonte et al [50] proposed the exponentiated Kumaraswamy distribution as a generalization of the Kumaraswamy distribution and studied some of its properties, then showed that it can be used effectively in analyzing lifetime data. Bourguignon et al [10] introduced the Kumaraswamy Pareto distribution, studied some of its structural properties, estimating its parameters using the method of maximum likelihood, and used a real data set to compare this distribution with other well-known distributions. Çkmakyapan and Kadılar [11] studied some mathematical properties of Kumaraswamy Lindley distribution, and parameter estimation using the method of maximum likelihood and concluded that this distribution is a useful tool to analyze customer lifetime duration in marketing research. Sharma and Chakrabarty [74] introduced the size-biased form of Kumaraswamy distribution and studied some of its distributional and characterizing properties, and used the maximum likelihood and the matching quantiles methods to estimate its parameters in order to fit four simulated data sets, as well as, a given real life data set, representing the measurements of tensile strength of 30 polyester fibers.

Table 1a: List of Modified and/or Generalized Version of the Kumaraswamy Distribution

Year	Name	Reference
1997	The Marshall-Olkin Kumaraswamy distribution	Marshall et al [53]
2011	The Kumaraswamy generalized distribution	Cordeiro and de Castro [14]
	The Kumaraswamy generalized gamma distribution	De Pascoa et al [21]
2012	The Kumaraswamy normal distribution	Correa et al [19]
	The Kumaraswamy log-logistic distribution	De Santana et al [22]
	The Kumaraswamy Gumbel distribution	Cordeiro et al [15]
	The Kumaraswamy modified Weibull distribution	Cordeiro et al [16]
	The Kumaraswamy half -normal generalized distribution	Cordeiro et al [17]
	The Kumaraswamy inverse Weibull distribution	Shahbaz et al [71]
2013	The exponentiated Kumaraswamy distribution	Lemonte et al [50]
	The Kumaraswamy Pareto distribution	Bourguignon et al [10]
	The Kumaraswamy generalized Pareto distribution	Nadarajah and Eljabri [60]
	The Kumaraswamy-generalized exponentiated Pareto distribution	Shams [72]
	The Kumaraswamy-generalized Lomax distribution	Shams [73]
	Kumaraswamy-power series distribution.	Bidram and Nekoukhou [8]
	The Kumaraswamy GP Distribution	Nadarajah and Eljabri [59]
	The Kumaraswamy Burr XII distribution	Paranałba et al [65]
	Kumaraswamy linear exponential distribution	Elbatal [23]

2014	The Kumaraswamy-geometric distribution	Akinsete et al [4]
	The Kumaraswamy Kumaraswamy distribution	El-Sherpieny et al [25]
	The Kumaraswamy-inverse exponential distribution	Oguntunde et al [64]
	The Kumaraswamy Lindley distribution	Çakmakyapan and Kadılar [11]
	The Kumaraswamy-half-Cauchy distribution	Ghosh [30]
	The Kumaraswamy inverse Rayleigh distribution	Hussian and Amin [37]
	The Kumaraswamy generalized Rayleigh distribution	Gomes et al [32]
	The Exponentiated Kumaraswamy–Dagum distribution	Huang and Oluyede [35]
2015	The exp-Kumaraswamy distributions	Javanshiri et al [42]
	The Kumaraswamy Kumaraswamy distribution	Mahmoud et al [52]
	The Kumaraswamy-G Poisson distribution	Ramos et al [66]
	The exponentiated Kumaraswamy-exponential distribution	Rodrigues et al [68]

Table 1b: List of Modified and/or Generalized Version of the Kumaraswamy Distribution

Year	Name	Reference
2016	The Kumaraswamy generalized power Weibull distribution	Selim et al [70]
	The Kumaraswamy exponential-Weibull distribution	Cordeiro et al [18]
	The Kumaraswamy- Laplace Distribution	Nassar [62]
	The size-biased Kumaraswamy distribution	Sharma and Chakrabarty [74]
	The transmuted Kumaraswamy distribution	King et al [48]
	The Kumaraswamy Gompertz Makeham Distribution	Chukwu and Ogunde [13]
2017	The generalized inverted Kumaraswamy distribution	Iqbal et al [39]
	The Marshall-Olkin-Kumaraswamy-G distribution	Handique, et al [34]
	The inverted Kumaraswamy distribution	Abd AL-Fattah et al [1]
	The Marshall-Olkin Kumaraswamy distribution	George and Thobias [28]
2018	The exponentiated generalized Kumaraswamy distribution	Elgarhy et al [24]
	Kumaraswamy odd Burr G distribution	Nasir et al [61]
	The Kumaraswamy exponentiated U-quadratic distribution	Muhammad et al [58]
	The Gamma–Kumaraswamy distribution	Ghosh and Hamedani [31]
	The Kumaraswamy extension exponential distribution	Ibrahim et al [38]
	The odd generalized exponential Kumaraswamy distribution	Kaile et al [46]
	The Weibull- Kumaraswamy distribution	Aminu et al [5]
	The Kumaraswamy Marshall–Olkin log-logistic distribution	Cakmakyapan et al [12]
	Cubic rank transmuted Kumaraswamy distribution	Saraçoğlu and Tanis [69]
	The Kumaraswamy generalized Kappa distribution	Nawaz et al [63]
	The Marshall-Olkin extended inverted Kumaraswamy distribution	Usman and Ahsan ul Haq [77]
	The beta Kumaraswamy Burr Type X Distribution	Madaki et al [51]
	The exponentiated Inverse Kumaraswamy Distribution	Kawsar et al [47]
2019	The exponentiated Kumaraswamy-G class of distributions	Silva et al [75]
	The gamma Kumaraswamy-G family of distributions	Arshad et al [6]
	The Kumaraswamy log-logistic Weibull distribution	Mdlongwa et al [55]
	The Topp Leone generalized inverted Kumaraswamy distribution	Reyad et al [67]
	The Generalized inverted Kumaraswamy	Jamal et al [41]
	Inflated Kumaraswamy distributions	Cribari-Neto and Santos [20]
	Truncated Inverted Kumaraswamy Generated Family of Distributions	Bantan et al [7]
	Generalized Transmuted Kumaraswamy distribution	Ishaq et al [40]

From Tables 1a and 1b above, we can see that through the large number of amendments on or modifications of the Kumaraswamy distribution that have been studied over the past twenty years, we can know the importance of this distribution on the one hand, and on the other hand, the multiplicity of

generalization and modifications of this distribution, as these were among the most important reasons that prompted us to do this study.
In this paper, a generalized modification of the Kumaraswamy distribution is proposed in Section 2, and its properties consisting of boundaries, limits, mode, quantities, reliability and hazard functions, and Renyi entropy, are studied in Section 3. In Section 4, we considered distributions related to this distribution as special cases. In Section 5, order statistics distribution is derived. In Section 6, its mean deviations, moment generating function, and with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted, L, and the trimmed L moments and its Lorenz and Bonferroni curves are obtained. The maximum likelihood estimation method for estimating its parameters was used in Section 7 and its results were applied in Section 8 to six different models simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes. Finally, in Section 9, four real-life data sets are used in order to show the usefulness and the flexibility of this distribution in application to real-life data.

2. The Kumaraswamy Distribution and its Generalized Modification of

Definition 1 (Kumaraswamy, [49]):
The rv X having a probability density function (pdf), f is given by;

$$f(x) = \begin{cases} abx^{a-1}\left[1-x^{a}\right]^{b-1}, & 0<x<1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where a and b are non-negative numbers, is said to have the Kumaraswamy distribution with parameters a and b.
We note that the domain of the function f given by (1) is $[0,1]$, and the parameters a and b are shape parameters. A generalized modification of (1) is given below;
Let $0<a,b,c,\alpha,\beta<\infty$, such that$\alpha<\beta$, and define the function$f$by:

$$f(x;a,b,c,\alpha,\beta) = \begin{cases} \frac{bc}{a}(\beta-\alpha)^{-c}\left(\frac{x}{a}\right)^{b-1}\left[\beta-\left(\frac{x}{a}\right)^{b}\right]^{c-1}, & a\alpha^{\frac{1}{b}}<x<a\beta^{\frac{1}{b}} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Let us write $f(x)$ instead of $f(x;a,b,c,\alpha,\beta)$ for simplicity.
Now, we have the following proposition
Proposition 1:
The function f defined by (2) is a pdf and its cumulative distribution function (CDF) F given by;

$$F_X(x;a,b,c,\alpha,\beta) = \begin{cases} 0, & x\le a\alpha^{\frac{1}{b}} \\ 1-\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c}, & a\alpha^{\frac{1}{b}}<x<a\beta^{\frac{1}{b}} \\ 1, & x\ge a\beta^{\frac{1}{b}} \end{cases} \tag{3}$$

Proof:
Since $0<a,b,c,\alpha,\beta<\infty$, $\alpha<\beta$, and $a\alpha^{\frac{1}{b}}<x<\alpha^{\frac{1}{b}}$, then $\alpha<\left(\frac{x}{a}\right)^{b}<\beta$, hence $\beta-\left(\frac{x}{a}\right)^{b}>0$, implying that f given in (2) is non negative. Now; let

$$\left(\frac{x}{a}\right)^{b} = \beta w \tag{4}$$

then $x = \alpha\beta^{\frac{1}{b}} w^{\frac{1}{b}}$, and $dx = \frac{\alpha\beta}{b}\beta^{\frac{1}{b}-1} w^{\frac{1}{b}-1} dw$, therefore;

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{a\alpha^{\frac{1}{b}}}^{a\beta^{\frac{1}{b}}} \frac{bc}{a}(\beta-\alpha)^{-c}\left(\frac{x}{a}\right)^{b-1}\left[\beta-\left(\frac{x}{a}\right)^{b}\right]^{c-1} dx$$

$$= c\beta^{c}(\beta-\alpha)^{-c}\int_{\frac{\alpha}{\beta}}^{1}(1-w)^{c-1}dw = -\beta^{c}(\beta-\alpha)^{-c}(1-w)^{c}\Big|_{\frac{\alpha}{\beta}}^{1} = 1.$$

It follows that, for any x such that, $a\alpha^{\frac{1}{b}} < x < a\beta\frac{1}{b}$;

$$F(x) = \int_{-\infty}^{x} f(y)dy = -\beta^{c}(\beta-\alpha)^{-c}(1-w)^{c}\Big|_{\frac{\alpha}{\beta}}^{\frac{1}{\beta}\left(\frac{x}{a}\right)^{b}} = 1-\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c}$$

Note that for, $a\alpha^{\frac{1}{b}} < x < \alpha\beta\frac{1}{b}$, we have that; $\frac{\partial}{\partial x}F_X(x;a,b,c,\alpha,\beta) = f(x;a,b,c,\alpha,\beta)$.

Definition 2:
The rv X is said to have a generalized modification of the Kumaraswamy distribution (GMKD) with parameters a, b, c, αandβ, written as $X \sim GMKD(a,b,c,\alpha,\beta)$, if its pdf is given by (2), or equivalently, its CDF is given by (3).
Figure 1 shows some plots of the pdf of the $GMKD$ for some of its parameter's values, inducting that this distribution has a lot of various different flexible shapes.

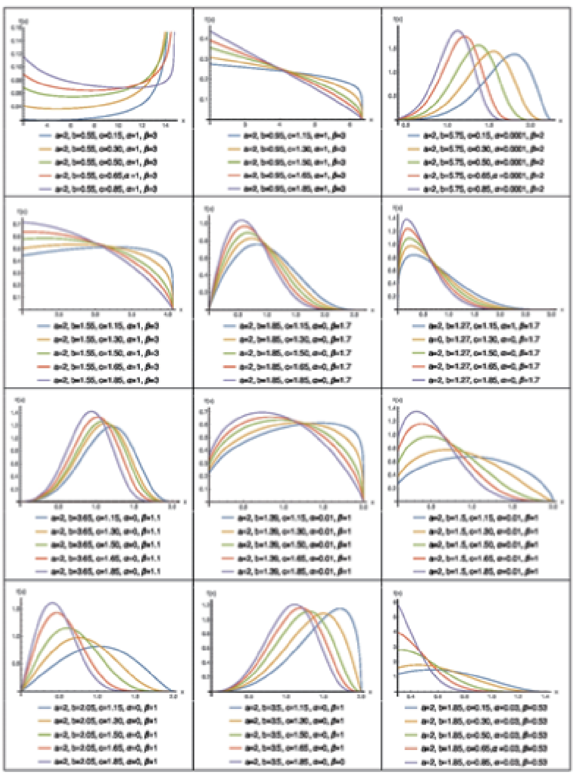

Figure 1. **Different pdf Plots of the GMKD Models**

We first note that, $f(x;1,b,c,1,1)$ is the pdf of Kumaraswamy distribution with parameters b and c, given by (1).
Furthermore, we have the following represents the method or the transformation that can be used for modifying the Kumaraswamy pdf given in (1) in order to get what we have called the GMKD.
Proposition 2:
Let the rv Y be a rv having the Kumaraswamy distribution with parameters b and c, and the rv $X = a\left[\alpha + (\beta-\alpha)Y^b\right]^{\frac{1}{b}}$, where $0 < a, \alpha, \beta < \infty$,such that$\alpha < \beta$, then $X \sim GMKD(a,b,c,\alpha,\beta)$.
Proof:
First, it is easy to find that the CDF of Y is given, for $0 < y < 1$, by;

$$F_Y(y) = 1 - \left[1 - y^b\right]^c$$

Since $0 < a, b, c, \alpha, \beta < \infty$and$\alpha < \beta$, then $0 < y < 1$ is equivalent to $a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$, therefore; for $a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$, we have that;

$$\begin{aligned}
F_X(x) &= P(X \leq x)\\
&= P\left(a\left[\alpha + (\beta-\alpha)Y^b\right]^{\frac{1}{b}} \leq x\right)\\
&= P\left(Y \leq \left[\frac{\left(\frac{x}{a}\right)^b - \alpha}{\beta-\alpha}\right]^{\frac{1}{b}}\right)\\
&= F_Y\left(\left[\frac{\left(\frac{x}{a}\right)^b - \alpha}{\beta-\alpha}\right]^{\frac{1}{b}}\right) = 1 - \left[1 - \left(\left[\frac{\left(\frac{x}{a}\right)^b - \alpha}{\beta-\alpha}\right]^{\frac{1}{b}}\right)^b\right]^c = 1 - \left[\frac{\beta - \left(\frac{x}{a}\right)^b}{\beta-\alpha}\right]^c.
\end{aligned}$$

3. Some Properties of the GMKD

3.1. Boundaries and Some Limits of the pdf

Let us study the behavior of the pdf of the $GMKD(a,b,c,\alpha,\beta)$. At the boundary's points, we have from (2) that;

$$f\left(a\alpha^{\frac{1}{b}}\right) = \frac{bc}{a}\frac{a^{1-\frac{1}{b}}}{(\beta-\alpha)}$$

and;

$$f\left(a\beta^{\frac{1}{b}}\right) = 0,$$

Therefore;

$$\lim_{a\to 0^+} f\left(a\alpha^{\frac{1}{b}}\right) = 0,$$

and

$$\lim_{b\to 1} f\left(a\alpha^{\frac{1}{b}}\right) = \frac{c}{a(\beta-\alpha)}$$

3.2. *The Mode:*

We have for $a\alpha^{\frac{1}{b}} < x < a\beta\frac{1}{b}$ that;

$$\frac{\partial}{\partial x}f(x)=\frac{bc}{a}(\beta-\alpha)^{-c}\left(\frac{x}{a}\right)^{b-2}\left[\beta-\left(\frac{x}{a}\right)^{b}\right]^{c-2}\left\{\left(\frac{b-1}{a}\right)\beta+\left(\frac{1-bc}{a}\right)\left(\frac{x}{a}\right)^{b}\right\}$$

Hence, if $bc \neq 1$, provided that $\frac{\alpha}{\beta} < \frac{1-b}{1-bc} < 1$, which is satisfied if $b > 1$ and $c > 1$, then $\frac{\partial}{\partial x}f(x)=0$ has a root given by;

$$x_0=a\left[\left(\frac{1-b}{1-bc}\right)\beta\right]^{\frac{1}{b}}$$

And that;

$$\frac{\partial^2}{\partial x^2}f(x_0)=\frac{cb^{c-1}(1-bc)}{a^3(\beta-\alpha)^c}\beta^{\frac{2b-3}{b}+c-2}\left(\frac{1-b}{1-bc}\right)^{\frac{2b-3}{b}}\left(\frac{1-c}{1-bc}\right)^{c-2},$$

Therefore, if $b > 1$ and $c > 1$, then $bc > 1$, $0 < \frac{1-b}{1-bc} < 1$ and $0 < \frac{1-c}{1-bc} < 1$, hence $\frac{\partial^2}{\partial x^2}f(x_0)$ is negative, implying that a single mode exists. Note that the condition $\frac{\alpha}{\beta} < \frac{1-b}{1-bc} < 1$ implies that $a\alpha^{\frac{1}{b}} < x_0 < \alpha\beta^{\frac{1}{b}}$. Therefore, we have proved the following.

3.3. *Proposition 3:*

If $b > 1$ and $c > 1$, then a single-mode exists for the $GMKD(a, b, c, \alpha, \beta)$.

3.4. *Quantile Function*

Let $0 < p < 1$, then the quantile function of the rv $X \sim GMKD(a,\ b, c, \alpha, \beta)$, Q, defined by;

$$Q(u) = \inf\{x \in \mathbb{R}; p \leq F(x)\}$$

can be found using (3), to be;

$$Q(u) = a\left[\beta-(\beta-a)(1-u)^{\frac{1}{c}}\right]^{\frac{1}{b}} \tag{5}$$

In particular, the median of X, $Med(X)$; is given by;

$$Med(X) = a\left[\beta-\frac{(\beta-a)}{2^{\frac{1}{c}}}\right]^{\frac{1}{b}}$$

3.5. *Reliability Function*

The survival function of $X \sim GMKD(a, b, c, \alpha, \beta)$, using (3), is given by;

$$\overline{F}(x) = 1 - F(x) = \begin{cases} 1, & x \leq a\alpha^{\frac{1}{b}} \\ \left[\frac{\beta-\left(\frac{x}{a}\right)^b}{\beta-a}\right]^c, & aa^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}} \\ 0, & x \geq a\beta^{\frac{1}{b}} \end{cases} \tag{6}$$

3.6. Hazard Function

The hazard function of the rv $X \sim GMKD(a,b,c,\alpha,\beta)$, $h(x)$, using (2) and (6), is given for $a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$, by;

$$h(x) = \frac{\text{f(x)}}{1 - F(x)} = \frac{bc}{a} \frac{\left(\frac{x}{a}\right)^{b-1}}{\beta - \left(\frac{x}{a}\right)^{b}}$$

If $b \geq 1$, then $h(x)$ is increasing; and if $b < 1$, then $h(x)$ is increasing for $x{>}a[(1-b)\beta]^{\frac{1}{b}}$, =0 for $x{=}a[(1-b)\beta]^{\frac{1}{b}}$, and decreasing for x $<a[(1-b)\beta]^{\frac{1}{b}}$.

3.7. Renyi Entropy

Let us compute the Renyi entropy as a measure of variation of the uncertainty of the rv X $X \sim GMKD(a,b,c,\alpha,\beta)$. For $\theta > 0$ such that $\theta \neq 1$, we have for the rv $X \sim GMKD(a,b,c,\alpha,\beta)$ that;

$$I_X(\theta) = \frac{1}{1-\theta} log \int_{-\infty}^{+\infty} [f(x)]^{\theta} dx \tag{7}$$

$$I_X(\theta) = \frac{1}{1-\theta} log \left[\frac{bc}{a(\beta - a)} \right]^{\theta} \int_{a\alpha^{\frac{1}{b}}}^{a\beta^{\frac{1}{b}}} \frac{bc}{a} \left(\frac{x}{a}\right)^{(b-1)\theta} \left[\frac{\beta - \left(\frac{x}{a}\right)^{b}}{\beta - a} \right]^{(c-1)\theta} dx$$

Using the transformation given by (4), we have that;

$$I_X(\theta) = \log(a) - \log(b) + \frac{\theta}{1-\theta}\log(c) + \frac{1}{1-\theta} \left\{ \left[\left(c - \frac{1}{b}\right)\theta + \frac{1}{b} - 1 \right] \log(\beta) - c\theta \log(\beta - \alpha) + \log\left[B^{*}\left(\left(1 - \frac{1}{b}\right)(\theta - 1) - 1, (c-1)\theta - 1; \frac{\alpha}{\beta} \right) \right] \right\} \tag{8}$$

where $B^{*}(a,b;c)$ is defined by;

$$B^{*}(a,b;z) = \int_{z}^{1} x^{a-1}(1-x)^{b-1} dx \tag{9}$$

Or equivalently;

$$B^{*}(a,b;z) = B(a,b) - B_z(a,b)$$

where B and B_y are, respectively, the beta and the incomplete beta functions, Abramowitz and Stegun ([3], p. 258), defied by;

$$B(a,b) = \int_{0}^{1} x^{a-1}(1-x)^{b-1} dx \tag{10}$$

and

$$B_z(a,b) = \int_{0}^{z} x^{a-1}(1-x)^{b-1} dx \tag{11}$$

We may call $B^{*}(\ a,\ b;z)$ given by (9), “the upper beta function at z”.

4. Distributions Related to GMKD

4.1. Special Cases of GMKD

4.1.1. $GMKD(1,\ a,\ b,\ 0,\ 1)$ is the Kumaraswamy distribution, Kumaraswamy [49], with pdf;

$$f(x)=ab\ x^{a-1}\left[\beta - x^{a}\right]^{b-1}, \qquad 0<x<1$$

4.1.2. $GMKD(1,\ 1,\ 1,\ 0,\ 1)$ is the standard uniform distribution with pdf;

$$f(x)=1, \qquad 0<x<1$$

4.1.3. $GMKD(1,\ 2,\ 1,\ 0,\ 1)$ is the triangular distribution with pdf;

$$f(x)=2\ x, \qquad 0<x<1$$

4.1.4. $GMKD(1,\ b,\ 1,\ 0,\ 1)$ is the beta distribution with parameters 1 and b, with pdf;

$$f(x)=b\ x^{b-1}, \qquad 0<x<1$$

4.1.5. $GMKD(1,\ 1,\ c,\ 0,\ 1)$ is the beta distribution with parameters c and 1, with pdf;

$$f(x)=c\ [1-x]^{c-1}, \qquad 0<x<1$$

4.1.6. $GMKD(1,\ 1, \delta, \alpha, \beta)$ is the power function distribution with pdf;

$$f(x)=\frac{\delta}{(\beta-\alpha)}\left[\frac{\beta-x}{\beta-\alpha}\right]^{\delta-1}, \qquad a<x<\beta$$

4.1.7. $GMKD(\lambda, a\theta, b, 0, 1)$ is the Kumaraswamy power function distribution, Abdul-Moniem [2], with pdf;

$$f(x)=\frac{ab\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1}, \qquad 0<x<\lambda$$

4.1.8.$GMKD(1, \beta, \gamma,\ 0,\ 1)$ is the Minimax distribution, McDonald [54], with pdf;

$$f(x)=\beta\gamma x^{\beta-1}\left[1-x^{\beta}\right]^{\gamma-1}, \qquad 0<x<1$$

4.1.9. $GMKD(\mathrm{c},\ -\mathrm{a},\ \theta,\ 0,\ 1)$ is the exponentiated Pareto distribution, Gupta et al [33], with pdf;

$$f(x)=\theta a c^{\alpha} x^{-(a+1)}\left[1-\left(\frac{x}{c}\right)^{-a}\right]^{\theta-1}, \qquad 0<c<x$$

4.1.10. $GMKD(\alpha, \beta, 1,\ 0,\ 1)$ is the generalized uniform distribution, Tiwari et al [76], with pdf;

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}, \qquad 0<x<a$$

4.2. Exponentiation Property

Proposition 4: GMKD is closed under exponentiation.
Let the rv $X \sim GMKD(a,b,c,\alpha,\beta)$ and the rv $Y = X^{k}$, where $0<k<\infty$, then $Y \sim GMKD(a^{k},\ \frac{b}{k},\ c, \alpha, \beta)$.
Proof:

$$F_{Y}(y) = P(Y \le y) = P\left(X^{k} \le y\right) = P\left(X \le y^{\frac{1}{k}}\right) = F_{X}\left(y^{\frac{1}{k}}; a, b, c, \alpha,\ \beta\right)$$

Therefore;

$$F_Y(y) = 1 - \left[\frac{\beta - \left(\frac{y^{\frac{1}{k}}}{a}\right)^b}{\beta - a}\right]^c = 1 - \left[\frac{\beta - \left(\frac{y}{a^k}\right)^{\frac{b}{k}}}{\beta - a}\right]^c = F_X\left(y; a^k,\ \frac{b}{k},\ c, \alpha, \beta\right)$$

4.3. Scaling Property

Proposition 5: GMKD is closed under scaling.
Let the rv $X \sim GMKD(a, b, c, \alpha, \beta)$ and the rv $Y = kX$, where $0 < k < \infty$, then $Y \sim GMKD(ka,\ b,\ c, \alpha, \beta)$.
Proof:

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(kX \le y) \\
&= P\left(X \le \frac{y}{k}\right) = F_X\left(\frac{y}{k}; a,\ b,\ c, \alpha,\ \beta\right) \\
&= 1 - \left[\frac{\beta - \left(\frac{y}{ka}\right)^b}{\beta - a}\right]^c \\
&= F_X(y; ka,\ b,\ c, \alpha, \beta)
\end{aligned}$$

4.4. Related Distributions of GMKD

Lemma 1:
Let the rv $X \sim GMKD(1,\ b,\ 1,\ 0,\ 1)$ then;
(1.1) The rv Y defined by $Y = \theta - b^2 log(X)$ has an exponential distribution with parameters θ and b, Johnson et al [43] p. 494, with CDF given by;

$$F_Y(y) = 1 - e^{-(\frac{y-\theta}{b})}, \qquad x > \theta$$

(1.2) The rv Y defined by $Y = \mu - \beta log\left[-b log(X^b)\right]$ has a Gumbel (generalized extreme value type-I) distribution with parameters μ andβ, Forbes et al [26] p. 98 , with CDF given by;

$$F_Y(y) = e^{-e^{-(\frac{y-\mu}{\beta})}}$$

(1.3) The rv Y defined by $Y = a + \beta log(\frac{X^b}{1-X^b})$ has a logistic distribution with parameters α and β, Johnson et al [44] p. 115, with CDF given by;

$$F_Y(y) = \left[1 + e^{-(\frac{y-\alpha}{\beta})}\right]^{-1}$$

(1.4) The rv Y defined by $Y = \frac{k}{X}$, where k is a positive constant, has a Pareto distribution with parameters k and b, Johnson et al [43] p. 574, with CDF given by;

$$F_Y(y) = 1 - (\frac{k}{y})^b$$

(1.5) The rv Y defined by $Y = \xi + a\left[-log(X^b)\right]^{\frac{1}{\theta}}$ has a Weibull distribution with parameters ξ, α, and θ, Johnson et al [43] p. 629, with its CDF given by;

$$F_Y(y) = 1 - e^{-(\frac{y-\xi}{a})^\theta}$$

Proof;
(1.1) Consider the distribution of the rv $Y = \frac{1}{X}$;

$$\begin{aligned} F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) \\ &= P\left(\frac{1}{y} \le X\right) \\ &= 1 - P\left(X < \frac{1}{y}\right) \\ &= 1 - F_X\left(\frac{1}{y}; 1, \beta, \ 1, \ 0, \ 1\right) \\ &= 1 - \left\{1 - \left[1 - \left(\frac{1}{y}\right)^{\beta}\right]\right\} \\ &= 1 - (y)^{-\beta} \end{aligned}$$

Therefore, the rv Y has a Pareto distribution with parameters γnd0.
Proof of **(1.2)** through **(1.5)** can be shown on the same lines as the proof of **(1.1)**.
Lemma 2:
(2.1) Let the rv X ~ GMKD(1, 1, 1, 0, 1) then the rv Y defined by Y $=(\frac{1-X}{X})^{\frac{1}{\delta}}.e^{-\frac{\gamma}{\delta}}$ has a log-logistic distribution with parameters δ and γ, Johnson et al [42] p. 151, with CDF given by;

$$F_Y(y) = 1 - \left[1 + y^{\delta} e^{\gamma}\right]^{-1}, \qquad y \ge 0$$

(2.2) Let the rv X ~ GMKD(a, b, c, α, β) then the rv Y defined by Y $=\beta - \left(\frac{x}{a}\right)^{b}$ has the generalized uniform distribution, Tiwari et al [76], with CDF given by;

$$F_Y(y) = \left[\frac{y}{\beta - a}\right]^{c}, \qquad 0 \le y \le \beta - a$$

(2.3) Let the rv X ~ GMKD(a, b, c, α, β) then the rv Y defined by Y $=(\beta - a)^{-1}\left[\beta - \left(\frac{x}{a}\right)^{b}\right]$ has the beta distribution with parameters 1 and c, with CDF given by;

$$F_Y(y) = y^{c}, \qquad 0 \le y \le 1$$

(2.4) Let the rv U has the standard uniform distribution, U(0, 1), then the rv X defined by $X = a\left[\beta - (\beta - a)(1 - u)^{\frac{1}{c}}\right]^{\frac{1}{b}}$, then X ~ GMKD$(a, b, c, \alpha, \beta)$.
Proof;
On the same lines as the proof of Lemma 1.

4.5. Generate GMKD random variates

Using result Lemma 2(4), we can generate $GMKD(a, b, c, \alpha, \beta)$ random variates as follows;

1. Generate $u \sim U(0, 1)$.
2. Set $x = a\left[\beta - (\beta - a)(1 - u)^{\frac{1}{c}}\right]^{\frac{1}{b}}$.

5. Order Statistics

Let X_1 , X_2 , ..., X_n be a random sample of size n from $GMKD(a,b,c,\alpha,\beta)$, and let $X_{1:n}$, $X_{2:n}$, ..., $X_{n:n}$ be their order statistics, then for $i = 1, 2, 3, \ldots, n$, the pdf of i-th order statistics $X_{i:n}$, is given for by;

$$f_{i:n}(x;a,\ b,\ c,\alpha,\ \beta)=\begin{cases} \frac{n!}{(i-1)!(n-i)!}\, f(x)\,[F(x)]^{i-1}\,[1-F(x)]^{n-i}, & a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}} \\ 0 & otherwise, \end{cases}$$

Hence, for $a\alpha^{\frac{1}{b}} < x < a\beta^{\frac{1}{b}}$, we have from (2) and (3) that;

$$f_{i:n}(x;a,\ b,\ c,\alpha,\ \beta)=\frac{n!}{(i-1)!\,(n-i)!}\,\frac{bc}{a\,(\beta-a)}\left(\frac{x}{a}\right)^{b-1}\left[1-\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c}\right]^{i-1}\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-a}\right]^{c(n-i)+c-1} \tag{12}$$

Since $i = 1, 2, 3, \ldots, n$, we have that;

$$\left[1-\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c}\right]^{i-1} = \sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{cj} \tag{13}$$

Hence, using (13), we can write (12) as;

$$f_{i:n}(x;a,\ b,\ c,\alpha,\ \beta)=\frac{n!}{(i-1)!\,(n-i)!}\,\frac{bc}{a\,(\beta-a)}\left(\frac{x}{a}\right)^{b-1}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c(n-i)+c+cj-1}$$

Let for $0 \le j \le i \le n$;

$$A(i,j;n) = (-1)^{j}\frac{n!}{(n-i)!\,(i-j-1)!\,j!}$$

And using the fact that the pdf of the rv X, f, given by (2), satisfies that;

$$\frac{a(\beta-a)}{bc}f(x;a,\ b,\ c,\alpha,\ \beta)=\left(\frac{x}{a}\right)^{b-1}\left[\frac{\beta-\left(\frac{x}{a}\right)^{b}}{\beta-\alpha}\right]^{c-1}$$

Then the pdf of the rv $X_{i:n}$, $f_{i:n}$, given by (12), can be written as;

$$f_{i:n}(x;a,\ b,\ c,\alpha,\ \beta)=\sum_{j=0}^{i-1}\frac{A(i,j;n)}{(n-i+j+1)}f(x;a,\ b,\ (n-i+j+1)c,\ \alpha,\ \beta) \tag{14}$$

6. Moments

6.1. Moments about the Origin

Let $k = 1, 2, 3, \ldots$, then the moment of the rv $X \sim GMKD(a,b,c,\alpha,\beta)$, of order k about zero is given by;

$$E(X^k) = \int_{a\alpha^{\frac{1}{b}}}^{a\beta^{\frac{1}{b}}} x^k\,\frac{bc}{a}\,(\beta-a)^{-c}\left(\frac{x}{a}\right)^{b-1}\left[\beta-\left(\frac{x}{a}\right)^{b}\right]^{c-1}dx$$

Using the transformation given by (4), we have that;

$$E\left(X^k\right) = ca^k\beta^{\frac{k}{b}}\left(\frac{\beta}{\beta-\alpha}\right)^c\int_{\frac{\alpha}{\beta}}^{1} w^{\frac{k}{b}}(1-w)^{c-1}dw \tag{15}$$

$$= ca^k\beta^{\frac{k}{b}}\left(\frac{\beta}{\beta-\alpha}\right)^c B^*\left(\frac{k}{b}+1,c;\frac{a}{\beta}\right), \tag{16}$$

where $B^*(\ ,\ ;z)$ is "the upper beta function at z" given by (9).
Now expand $(1-\mathrm{w})^{c-1}$ in the integral of (15), using the binomial series expansion, Abramowitz and Stegun [3] p.14, to get that;

$$E\left(X^k\right) = ca^k\beta^{\frac{k}{b}}\left(\frac{\beta}{\beta-\alpha}\right)^c\sum_{i=0}^{\infty}\binom{c-1}{i}\frac{(-1)^i}{\left(\frac{k}{b}+i+1\right)}\left[1-\left(\frac{\alpha}{\beta}\right)^{\frac{k}{b}+i+1}\right] \tag{17}$$

Therefore, an interesting relation can be seen from (16) and (17), for the upper beta function, is given by;

$$B^*\left(\frac{k}{b}+1,c;\frac{\alpha}{\beta}\right) = \sum_{i=0}^{\infty}\binom{c-1}{i}\frac{(-1)^i}{(\frac{k}{b}+i+1)}\left[1-\left(\frac{\alpha}{\beta}\right)^{\frac{k}{b}+i+1}\right]$$

or equivalently, in a general form, given in the following proposition;
Proposition 6:

$$B^*(x,c;y) = \sum_{i=0}^{\infty}\binom{c-1}{i}\frac{(-1)^i}{(x+i)}\left[1-y^{x+i}\right]$$

6.2. *Mean and Variance*

Using (16), the mean of $X \sim GMKD(a,b,c,\alpha,\beta)$ is given by;

$$E(X) = ca\beta^{\frac{1}{b}}\left(\frac{\beta}{\beta-\alpha}\right)^c B^*\left(\frac{1}{b}+1,c;\frac{\alpha}{\beta}\right) \tag{18}$$

And hence the variance;

$$Var(X) = ca^2\beta^{\frac{2}{b}}\left\{B^*\left(\frac{2}{b}+1,c;\frac{\alpha}{\beta}\right)-c\left(\frac{\beta}{\beta-\alpha}\right)^c\left[B^*\left(\frac{1}{b}+1,c;\frac{a}{\beta}\right)\right]^2\right\}$$

In particular, if $\alpha=0$ and $\beta=1$,then

$$E(X) = ca\frac{\Gamma(\frac{1}{b}+1)\Gamma(c)}{\Gamma(\frac{1}{b}+c+1)}$$

$$V(X) = ca^2\left[\frac{\Gamma(\frac{2}{b}+1)\Gamma(c)}{\Gamma(\frac{2}{b}+c+1)}-c\left(\frac{\Gamma(\frac{1}{b}+1)\Gamma(c)}{\Gamma(\frac{1}{b}+c+1)}\right)^2\right]$$

and;

$$E(X^k) = ca^k\frac{\Gamma(\frac{k}{b}+1)\Gamma(c)}{\Gamma(\frac{k}{b}+c+1)}$$

6.3. The Moment Generating Function

Similarly, the moment generating function of the rv $X \sim GMKD(a, b, c, \alpha, \beta)$, $M_X(t)$, can be found to be;

$$M_X(t) = E\left(e^{tX}\right) = c\left(\frac{\beta}{\beta-\alpha}\right)^c \sum_{i=0}^{\infty} \frac{\left(at\beta^{\frac{1}{b}}\right)^i}{i!} B^*\left(\frac{i}{b}+1, c; \frac{\alpha}{\beta}\right)$$

6.4. Harmonic Mean

The harmonic mean of $X \sim GMKD(a, b, c, \alpha, \beta)$, on the same lines as that of the moment of X, is given by;

$$E(\frac{1}{X}) = \frac{c}{\alpha\beta^{\frac{1}{b}}}(\frac{\beta}{\beta-\alpha})^c B^*\left(1-\frac{1}{b}, c; \frac{\alpha}{\beta}\right)$$

6.5. Incomplete Moments

The k-th incomplete moment of $X \sim GMKD(a, b, c, \alpha, \beta)$, $I(z, k)$, is defined by;

$$I(z,k) = \int_{-\infty}^{z} x^k f(x)\,dx$$

$$= \int_{aa^{\frac{1}{b}}}^{z} x^k \frac{bc}{a} (\beta-\alpha)^{-c}\left(\frac{x}{a}\right)^{b-1}\left[\beta-\left(\frac{x}{a}\right)^b\right]^{c-1} dx$$

$$= ca^k\beta^{\frac{k}{b}}(\frac{\beta}{\beta-\alpha})^c\left[B_{\frac{1}{\beta}(\frac{z}{a})^b}\left(\frac{k}{b}+1, c\right) - B_{\frac{\alpha}{\beta}}(\frac{k}{b}+1, c)\right] \tag{19}$$

6.6. Mean Deviations

The mean deviation of X about its mean $\mu = E(X)$, $MD(\mu)$, is given by;

$$MD(\mu) = E\left|X-\mu\right|$$

Which can be found, Cordeiro et al [15], to be;

$$MD(\mu) = 2\left[\mu F(\mu) - I(\mu-1)\right]$$

Hence, using (3), (18) and (19), for the rev $X \sim GMKD(a, b, c, \alpha, \beta)$, we have that;

$$MD(\mu) = 2c\alpha\beta^{\frac{1}{b}}(\frac{\beta}{\beta-\alpha})^c\left\{B^*\left(\frac{1}{b}+1, c; \frac{1}{\beta}\left(\frac{\mu}{a}\right)^b\right) - B^*\left(\frac{1}{b}+1, c; \frac{\alpha}{\beta}\right)\left[\frac{\beta-\left(\frac{\mu}{a}\right)^b}{\beta-\alpha}\right]^c\right\}$$

Similarly, the mean deviation of X about its median m, $MD(m)$, is given by;

$$MD(m) = \mu - 2I(m, 1)$$

$$MD(m) = \mathrm{ca}\beta^{\frac{1}{b}}\left(\frac{\beta}{\beta-\alpha}\right)^c\left[B\left(\frac{1}{b}+1, c\right) + B_{\frac{\alpha}{\beta}}\left(\frac{1}{b}+1, c\right) - 2B_{\frac{1}{\beta}(\frac{m}{a})^b}\left(\frac{1}{b}+1, c\right)\right]$$

6.7. *Probability Weighted Moments*

The probability weighted moments of order s and r of $X \sim GMKD(a,b,c,\alpha,\beta)$, $\rho_{s,r}$, is given by;

$$\rho_{s,r} = E(X^s[f(X)]^r)$$

Using the transformation given in (4), we have that;

$$\rho_{s,r} = a^{s-r+1} b^{r-1} c^r \frac{\beta^{\frac{s}{b}+r(c-1)+1}}{(\beta-\alpha)^{rc}} B^* \left(\frac{s}{b}+1, r(c-1)+1; \frac{a}{\beta}\right)$$

6.8. *Moments of Order Statistics*

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $GMKD(a,b,c,\alpha,\beta)$, and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be their order statistics, then for $i = 1, 2, 3, \ldots, n$, we have that;

$$E(X_{i:n}^m) = \int_{a\alpha^{\frac{1}{b}}}^{\alpha\beta^{\frac{1}{b}}} x^m f_{i:n}(x; a, b, c, \alpha, \beta)\, dx$$

Hence, using (14) we have that;

$$E(X_{i:n}^m) = \sum_{j=0}^{i-1} \frac{A(i,j;n)}{(n-i+j+1)} \int_{a\alpha^{\frac{1}{b}}}^{\alpha\beta^{\frac{1}{b}}} x^m f(x; a, b, (n-i+j+1)c, \alpha, \beta)\, dx$$

Therefore, using (16), we have that;

$$E(X_{i:n}^m) = ca^m \beta^{\frac{m}{b}} \sum_{j=0}^{i-1} A(i,j;n) B^* \left(\frac{m}{b}+1, (n-i+j+1)c; \frac{a}{\beta}\right) \left(\frac{\beta}{\beta-\alpha}\right)^{(n-i+j+1)c} \tag{20}$$

6.9. *L-Moments*

Let $r = 1, 2, 3, \ldots$ then the r-th L-moment γ_r of $X \sim GMKD(a,b,c,\alpha,\beta)$, is given by;

$$\gamma_r = \frac{1}{r} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} E(X_{r-i:r})$$

which is, with the use of (20), can be found to be;

$$\gamma_r = \frac{c\alpha\beta^{\frac{1}{b}}}{r} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \sum_{j=0}^{i-1} A(r-i,j;r) B^* \left(\frac{1}{b}+1, (i+j+1)c; \frac{a}{\beta}\right) \left(\frac{\beta}{\beta-\alpha}\right)^{(i+j+1)c}$$

In particular, the first L-moments are given by;

$$\gamma_1 = ca\beta^{\frac{1}{b}} \left(\frac{\beta}{\beta-\alpha}\right)^c B^* \left(\frac{1}{b}+1, c; \frac{a}{\beta}\right)$$

And;

$$\gamma_2 = ca\beta^{\frac{1}{b}} \left(\frac{\beta}{\beta-\alpha}\right)^c \left[1 - 2\left(\frac{\beta}{\beta-\alpha}\right)^c\right] B^* \left(\frac{1}{b}+1, 2c; \frac{a}{\beta}\right)$$

Hence, the L-coefficient of variation τ, Bilkova[9], defined by;

$$\tau = \frac{\gamma_2}{\gamma_1}$$

is given by;

$$\tau = \left[1 - 2\left(\frac{\beta}{\beta-\alpha}\right)^c\right] \frac{B^*\left(\frac{1}{b}+1, 2c; \frac{\alpha}{\beta}\right)}{B^*\left(\frac{1}{b}+1, c; \frac{\alpha}{\beta}\right)}$$

6.10. *The L and the Trimmed L-Moments*

Let $r = 1, 2, 3, \ldots$ then the r-th trimmed L-moments (TL-moments) of the rv $X \sim GMKD(a, b, c, \alpha, \beta)$, $\gamma_r^{(s,t)}$, is given by;

$$\gamma_r^{(s,t)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+s-k:r+s+t})$$

which is, with the use of (20), can be found to be;

$$\gamma_r^{(s,t)} = \frac{ca\beta^{\frac{1}{b}}}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}$$
$$\sum_{j=0}^{r+s-k-1} A\left(r+s-k, j; r+s+t\right) B^*\left(\frac{1}{b}+1, (t+k+j+1)c; \frac{\alpha}{\beta}\right)\left(\frac{\beta}{\beta-\alpha}\right)^{(t+k+j+1)c}$$

Hence, the r-th L-moments of the rv X, γ_r, can be found, since the L-moments is a special case of the TL-moments, namely, when $s = t = 0$, that is $\gamma_r = \gamma_r^{(0,0)}$.

6.11. *Lorenz and Bonferroni Curves*

For $0 < \pi < 1$, the Lorenz curve, $L(\pi)$, and Bonferroni curves, $B(\pi)$, for the rv $X \sim GMKD(a, b, c, \alpha, \beta)$, are given by;

$$L(\pi) = \frac{I(Q(\pi), 1)}{\mu}$$

and

$$B(\pi) = \frac{I(Q(\pi), 1)}{\pi\mu}$$

where $Q(\pi)$ is the quantile function of the rv X at π, and $I(z, k)$ is the incomplete moment of the rv X given by (19). Therefore, using (16) and (19), we have that;

$$L(\pi) = \frac{B_{\frac{1}{\beta}\left(\frac{Q(\pi)}{a}\right)^b}\left(\frac{1}{b}+1, c\right) - B_{\frac{\alpha}{\beta}}\left(\frac{1}{b}+1, c\right)}{B^*\left(\frac{1}{b}+1, c; \frac{\alpha}{\beta}\right)}$$

And similarly, that;

$$B(\pi) = \frac{B_{\frac{1}{\beta}\left(\frac{Q(\pi)}{a}\right)^b}\left(\frac{1}{b}+1, c\right) - B_{\frac{\alpha}{\beta}}\left(\frac{1}{b}+1, c\right)}{\pi B^*\left(\frac{1}{b}+1, c; \frac{\alpha}{\beta}\right)}$$

7. Parameters Estimation of the GMKD

We will use the maximum likelihood estimation (MLE) method for estimating the parameters of the GMKD. Let $x_1, x_2, \ldots, x_n$ be a random sample from $GMKD(a, b, c, \alpha, \beta)$, as given by (2), then the likelihood function$L = L(a, b, c, \alpha, \beta; x_1, x_2, \ldots, x_n)$ can be written as,

$$L = \prod_{i=1}^{n} \left\{ \frac{bc}{a} (\beta - \alpha)^{-c} \left(\frac{x_i}{a} \right)^{b-1} \left[\beta - \left(\frac{x_i}{a} \right)^b \right]^{c-1} \right\}$$

$$= \left[\frac{bc}{a^b (\beta - \alpha)^c} \right]^n \prod_{i=1}^{n} \left\{ \left(\frac{x_i}{a} \right)^{b-1} \left[\beta - \left(\frac{x_i}{a} \right)^b \right]^{c-1} \right\}$$

Hence;

$$logL = nlog(b) + nlog(c) - nblog(a) - nclog(\beta - \alpha) + (b-1) \sum_{i=1}^{n} log(x_i) + (c-1) \sum_{i=1}^{n} log\left[\beta - \left(\frac{x_i}{a} \right)^b \right]$$

Therefore;

$$\frac{\partial}{\partial \text{a}} logL = -\frac{nb}{a} + \frac{b}{a}(c-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i}{a} \right)^b}{\beta - \left(\frac{x_i}{a} \right)^b} \tag{21}$$

Since, $\frac{\partial^2}{\partial a^2} logL$ can be shown to be not in a simple form, therefore a local maximum of L at a has to be explicitly examined. Now;

$$\frac{\partial}{\partial b} logL = \frac{n}{b} - nlog(a) + \sum_{i=1}^{n} log(x_i) - (c-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i}{a} \right)^b log\left(\frac{x_i}{a} \right)}{\beta - \left(\frac{x_i}{a} \right)^b} \tag{22}$$

Hence also, $\frac{\partial^2}{\partial b^2} logL$ can be shown to be not in a simple form, therefore a local maximum of L at b has to be explicitly examined. Now;

$$\frac{\partial}{\partial c} logL = \frac{n}{c} - nlog(\beta - \alpha) + \sum_{i=1}^{n} log\left[\beta - \left(\frac{x_i}{a} \right)^b \right] \tag{23}$$

Hence;

$$\frac{\partial^2}{\partial c^2} logL = -\frac{n}{c^2}$$

Therefore, $\frac{\partial^2}{\partial c^2} logL < 0$, which indicates that L has a local maximum at c. Similarly,

$$\frac{\partial}{\partial \beta} logL = -\frac{nc}{\beta - \alpha} + (c-1) \sum_{i=1}^{n} \frac{1}{\beta - \left(\frac{x_i}{a} \right)^b} \tag{24}$$

Hence;

$$\frac{\partial^2}{\partial \beta^2} \log L = \frac{nc}{(\beta - \alpha)^2} - (c-1) \sum_{i=1}^{n} \frac{1}{\left[\beta - \left(\frac{x_i}{a} \right)^b \right]^2}$$

Or equivalently;

$$\frac{\partial^2}{\partial \beta^2} logL = (a_1 - a_2)c + a_1$$

where;

$$a_1 = \frac{n}{(\beta-\alpha)^2}$$

and

$$a_2 = \sum_{i=1}^{n} \frac{1}{\left[\beta - \left(\frac{x_i}{a}\right)^b\right]^2}$$

Therefore, if $a_1 > a_2$ and $c < \frac{a_2}{a_1-a_2}$, then $\frac{\partial^2}{\partial\beta^2} logL < 0$, or if $a_1 < a_2$ and $c > \frac{a_2}{a_1-a_2}$, then $\frac{\partial^2}{\partial\beta^2} logL < 0$, and therefore, a local maximum of L at β has to be explicitly examined. Finally;

$$\frac{\partial}{\partial\alpha} logL = \frac{nc}{\beta-\alpha}$$

and that;

$$\frac{\partial^2}{\partial\alpha^2} logL = \frac{nc}{(\beta-\alpha)^2}$$

implying that $\frac{\partial}{\partial\alpha} logL > 0$ and that $\frac{\partial^2}{\partial\alpha^2} logL > 0$, then an alternative way to find the MLE of α has to be considered. Since $a\alpha^{\frac{1}{b}} < x < a\beta\frac{1}{b}$, then the MLE of $a\alpha^{\frac{1}{b}}$ and $a\beta^{\frac{1}{b}}$ are; respectively, $x_{1:n}$ and $x_{n:n}$, and hence;

$$\frac{a\alpha^{\frac{1}{b}}}{a\beta^{\frac{1}{b}}} = \frac{x_{1:n}}{x_{n:n}}$$

Or equivalently;

$$\frac{a}{\beta} = \left(\frac{x_{1:n}}{x_{n:n}}\right)^b$$

And hence;

$$\hat{\alpha} = \left(\frac{x_{1:n}}{x_{n:n}}\right)^{\hat{b}} \hat{\beta}$$

Now, letting $\frac{\partial}{\partial a} logL = 0$, we have from (21) that;

$$(c-1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{a}\right)^b}{\beta - \left(\frac{x_i}{a}\right)^b} - n = 0$$

And letting $\frac{\partial}{\partial b} logL = 0$, then from (22) we have that;

$$\frac{n}{b} - nlog(a) + \sum_{i=1}^{n} log(x_i) - (c-1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{a}\right)^b log\left(\frac{x_i}{a}\right)}{\beta - \left(\frac{x_i}{a}\right)^b} = 0$$

Similarly, letting $\frac{\partial}{\partial c} logL = 0$, then from (23) we have that;

$$\frac{n}{c} - nlog(\beta-\alpha) + \sum_{i=1}^{n} log\left[\beta - \left(\frac{x_i}{a}\right)^b\right] = 0 \tag{25}$$

Hence, from (25) we have that;

$$c = \frac{n}{n log(\beta - \alpha) - \sum_{i=1}^{n} log\left[\beta - \left(\frac{x_i}{a}\right)^b\right]}$$

And finally, letting $\frac{\partial}{\partial \beta} log L = 0$, then from (24), we have that;

$$(c-1)\sum_{i=1}^{n} \frac{1}{\beta - \left(\frac{x_i}{a}\right)^b} - \frac{nc}{\beta - \alpha} = 0$$

Then the MLE of the parameters a, b, c, α and β, can be found by solving the following equations;

$$\mathrm{A}_1(\hat{\mathrm{a}}) = 0$$

$$\mathrm{A}_2\left(\hat{\mathrm{b}}\right) = 0$$

$$\hat{c} = \frac{n}{n log\left(\hat{\beta} - \hat{a}\right) - \sum_{i=1}^{n} log\left[\hat{\beta} - \left(\frac{x_i}{\hat{a}}\right)^b\right]}$$

$$\hat{\alpha} = \left(\frac{x_{1:n}}{x_{n:n}}\right)^{\hat{b}} \hat{\beta}$$

and,

$$\mathrm{A}_3\left(\hat{\beta}\right) = 0$$

using any numerical procedure, say, Newton Rapson method, where;

$$\mathrm{A}_1(\hat{\mathrm{a}}) = (\hat{c} - 1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\hat{\mathrm{a}}}\right)^b}{\hat{\beta} - \left(\frac{x_i}{\hat{\mathrm{a}}}\right)^b} - n$$

$$\mathrm{A}_2(\hat{\mathrm{b}}) = \frac{n}{\hat{\mathrm{b}}} - n\log(\hat{\mathrm{a}}) + \sum_{i=1}^{n} \log(x_i) - (\hat{c} - 1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\hat{\mathrm{a}}}\right)^{\hat{\mathrm{b}}} \log\left(\frac{x_i}{\hat{\mathrm{a}}}\right)}{\hat{\beta} - \left(\frac{x_i}{\hat{\mathrm{a}}}\right)^{\hat{\mathrm{b}}}}$$

$$\mathrm{A}_3\left(\hat{\beta}\right) = (\hat{c} - 1)\sum_{i=1}^{n} \frac{1}{\hat{\beta} - \left(\frac{x_i}{\hat{\mathrm{a}}}\right)^{\hat{\mathrm{b}}}} - \frac{n\hat{c}}{\hat{\beta} - \hat{\alpha}}$$

8. A Simulation Study

Using the results given in Section 7 and the Absoft Pro Fortran compiler for computing, different GMKD models data sets were simulated, in order to check the performance of the MLEs of the parameters of each model through their mean squares errors (MSE) computed from different simulated sample sizes. The steps are given below;

1. Six different GMKD models are considered, that have different pdf's shapes and variable ranges.
2. Six sample sizes, namely; 15, 30, 50, 100, 200, and 300 are used.
3. For each sample size, 5,000 random variates are generated from each of the given GMKD models.

4. For each sample size and for each GMKD model, the parameters are estimated using the MLE method given in Section 9.
5. The means, standard deviation (SD), bias, and MSE for each of the parameters are computed for each random sample for each sample size of the given GMKD models.

Table 2 shows the actual and the MLEs parameters values of the different simulated GMKD data sets, and Figure 2 shows their corresponding pdf's plots, while Tables 3a and 3b present the bias of the parameters of the different simulated GMKD data sets for each sample size, while Tables 4a and 4b present the MSE of the parameters of the different simulated GMKD data sets for each sample size, while.

Table 2: Actual and MLE Parameters Values of the Simulated GMKD Data Sets

Data Set	Value	Parameters					Variable Range	
		a	b	c	α	β	Minimum	Maximum
1	Actual	1.3	1.3	2	0.2	1	0.377	1.3
	MLE	1.280123	1.333333	2.02672	0.198141	1.012141	0.38	1.292
2	Actual	1.95	2.27	2.5	0.1	3	0.71	3.164
	MLE	1.93213	2.26713	2.38939	0.1041	2.9968	0.71	3.135
3	Actual	2	0.8	2.75	0	2	0	4.757
	MLE	2.15889	0.837778	2.66798	0.002103	1.888474	0.001	4.611
4	Actual	2	3.5	1.25	0.015	2.5	0.602	2.599
	MLE	1.98913	3.49911	1.25318	0.015111	2.511031	0.6	2.588
5	Actual	2.3	1.65	1.35	0	0.9	0	2.158
	MLE	2.29789	1.65356	1.34917	0.001041	0.89256	0.033	2.145
6	Actual	2.5	0.15	0.5	0	1	0	2.5
	MLE	2.51023	0.151713	0.497749	0.000675	0.999351	0	2.5

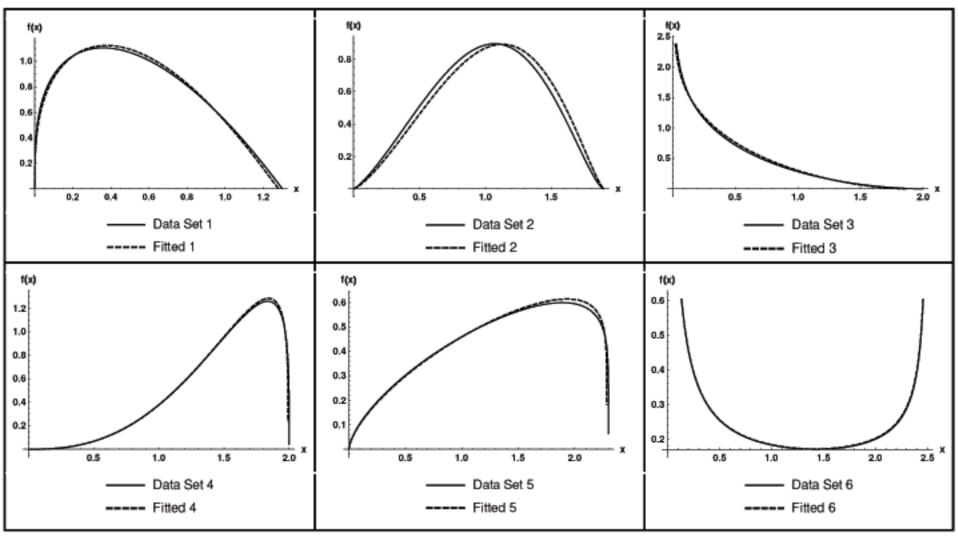

Figure 2. **Plots of the Actual and Simulated GMKD pdf's**

Table 3a: The Bias of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

n	Actual Value					Bias				
	a	b	c	α	β	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$
15	1.3	1.3	2	0.2	1	-0.1948	0.2595	0.3009	0.3104	-0.1980
	1.9	2.27	2.5	0.1	3	0.2299	0.2842	-0.1470	0.2589	0.2652
	2	0.8	2.75	0	2	-0.2609	-0.1484	-0.0944	-0.1561	0.2371
	2	3.5	1.25	0.015	2.5	-0.1223	0.1301	0.1463	-0.0931	-0.1260
	2.3	1.65	1.35	0	0.9	0.1087	-0.0251	0.1200	-0.0831	-0.0349
	2.5	0.15	0.5	0	1	-0.0305	-0.0011	0.0948	0.0848	0.0979
30	1.3	1.3	2	0.2	1	0.1480	-0.2120	-0.1652	0.2914	0.2573
	1.9	2.27	2.5	0.1	3	0.2665	0.2583	-0.1793	0.2861	-0.1910
	2	0.8	2.75	0	2	0.2105	0.1820	0.2049	-0.1290	0.2389
	2	3.5	1.25	0.015	2.5	0.1664	-0.0780	0.1726	0.1503	-0.1206
	2.3	1.65	1.35	0	0.9	-0.0634	0.1075	-0.0905	-0.0865	-0.0587
	2.5	0.15	0.5	0	1	0.1327	0.0805	-0.0540	0.1176	-0.0206
50	1.3	1.3	2	0.2	1	-0.1963	0.2016	-0.1718	-0.2336	-0.2231
	1.9	2.27	2.5	0.1	3	0.2288	-0.2059	-0.1856	0.2614	-0.1791
	2	0.8	2.75	0	2	0.1706	0.2169	0.1568	0.2171	0.2257
	2	3.5	1.25	0.015	2.5	-0.1336	-0.0931	-0.0896	0.1991	0.1689
	2.3	1.65	1.35	0	0.9	-0.0409	-0.0241	0.0886	0.1401	-0.0767
	2.5	0.15	0.5	0	1	0.0681	0.0604	0.0671	-0.0093	0.0596

Table 3b: The Bias of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

n	Actual Value					MSE				
	a	b	c	α	β	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{a}$	$\hat{\beta}$
100	1.3	1.3	2	0.2	1	-0.1591	-0.1926	0.2185	0.2924	0.2937
	1.9	2.27	2.5	0.1	3	-0.1836	-0.1966	0.2219	0.2488	-0.1921
	2	0.8	2.75	0	2	0.1560	0.2203	0.2160	-0.1744	0.2287
	2	3.5	1.25	0.015	2.5	0.1747	-0.0712	-0.1015	-0.1005	0.1209
	2.3	1.65	1.35	0	0.9	-0.0021	0.1052	-0.0552	0.0944	-0.0498
	2.5	0.15	0.5	0	1	0.0607	0.0696	-0.0038	-0.0011	0.0265
200	1.3	1.3	2	0.2	1	-0.1197	0.1319	-0.2100	-0.2343	-0.2318
	1.9	2.27	2.5	0.1	3	0.1657	0.1599	0.1153	0.1573	0.1419
	2	0.8	2.75	0	2	-0.1904	0.2026	-0.1896	-0.1259	0.2010
	2	3.5	1.25	0.015	2.5	-0.1436	-0.1249	-0.0565	-0.0798	-0.0641
	2.3	1.65	1.35	0	0.9	0.0087	0.0354	0.0409	0.0984	-0.0127
	2.5	0.15	0.5	0	1	0.0559	0.0472	0.0194	0.0220	-0.0294
500	1.3	1.3	2	0.2	1	-0.1500	-0.1952	0.1998	0.1018	0.2689
	1.9	2.27	2.5	0.1	3	-0.1544	-0.1525	-0.1094	0.1985	-0.1877
	2	0.8	2.75	0	2	0.1748	-0.1082	-0.0719	-0.1424	-0.1077
	2	3.5	1.25	0.015	2.5	0.1622	0.1086	0.1142	-0.0723	-0.0960
	2.3	1.65	1.35	0	0.9	0.1093	-0.0363	0.0684	0.0586	-0.0154
	2.5	0.15	0.5	0	1	-0.0486	0.0414	0.0361	0.0642	-0.0175

Table 4a: The MSE of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

n	Actual Value					MSE				
	a	b	c	α	β	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{a}$	$\hat{\beta}$
15	1.3	1.3	2	0.2	1	0.5927	1.7371	2.9691	0.3803	1.8725
	1.9	2.27	2.5	0.1	3	2.0704	1.6583	3.0146	0.2945	0.9887
	2	0.8	2.75	0	2	0.9750	0.6742	3.2541	0.3980	1.0615
	2	3.5	1.25	0.015	2.5	2.4903	1.1561	0.9519	0.3141	1.1174
	2.3	1.65	1.35	0	0.9	1.0894	1.7070	1.0354	0.1967	1.0994
	2.5	0.15	0.5	0	1	1.3065	0.7012	1.0601	0.3277	1.0420
30	1.3	1.3	2	0.2	1	0.5789	1.3874	2.8764	0.2388	1.6527
	1.9	2.27	2.5	0.1	3	1.4840	1.5209	2.7058	0.2361	0.7022
	2	0.8	2.75	0	2	0.7488	0.5465	2.8353	0.2015	0.7482
	2	3.5	1.25	0.015	2.5	2.4378	0.9590	0.7315	0.2884	0.9814
	2.3	1.65	1.35	0	0.9	0.9746	1.6750	0.9971	0.1424	1.0558
	2.5	0.15	0.5	0	1	1.1496	0.5271	1.0367	0.1885	0.9625
50	1.3	1.3	2	0.2	1	0.5409	0.9923	2.5963	0.1884	0.8690
	1.9	2.27	2.5	0.1	3	1.4806	1.4228	2.6926	0.2551	0.6916
	2	0.8	2.75	0	2	0.6871	0.3610	2.6592	0.1720	0.6247
	2	3.5	1.25	0.015	2.5	2.3722	0.8314	0.6624	0.2335	0.9708
	2.3	1.65	1.35	0	0.9	0.6705	1.2943	0.9945	0.1326	0.9960
	2.5	0.15	0.5	0	1	1.1326	0.4787	0.7650	0.1711	0.9073

Table 4b: The MSE of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

n	Actual Value					MSE				
	a	b	c	α	β	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{a}$	$\hat{\beta}$
100	1.3	1.3	2	0.2	1	0.2661	0.1654	2.4133	0.1658	0.5843
	1.9	2.27	2.5	0.1	3	1.1363	1.0940	2.3125	0.2453	0.3996
	2	0.8	2.75	0	2	0.4507	0.1933	2.6239	0.1514	0.1601
	2	3.5	1.25	0.015	2.5	2.1658	0.2596	0.1633	0.1886	0.7211
	2.3	1.65	1.35	0	0.9	0.4748	0.3919	0.4946	0.1194	0.2988
	2.5	0.15	0.5	0	1	0.4727	0.2051	0.4871	0.1368	0.4452
200	1.3	1.3	2	0.2	1	0.2307	0.0728	2.0742	0.1318	0.3668
	1.9	2.27	2.5	0.1	3	1.1023	1.0374	2.1287	0.2103	0.1809
	2	0.8	2.75	0	2	0.2873	0.1468	2.1122	0.1197	0.0866
	2	3.5	1.25	0.015	2.5	2.1013	0.1805	0.1208	0.1442	0.5059
	2.3	1.65	1.35	0	0.9	0.3564	0.3594	0.1107	0.0397	0.2369
	2.5	0.15	0.5	0	1	0.2310	0.0373	0.3756	0.0637	0.0890
500	1.3	1.3	2	0.2	1	0.1296	0.0680	2.0558	0.1303	0.2105
	1.9	2.27	2.5	0.1	3	1.0568	1.0234	2.0443	0.1810	0.1500
	2	0.8	2.75	0	2	0.1345	0.0454	2.0936	0.0729	0.0492
	2	3.5	1.25	0.015	2.5	2.0158	0.1547	0.0577	0.0182	0.2845
	2.3	1.65	1.35	0	0.9	0.1519	0.2200	0.0601	0.0373	0.0368
	2.5	0.15	0.5	0	1	0.2031	0.0157	0.0559	0.0430	0.0773

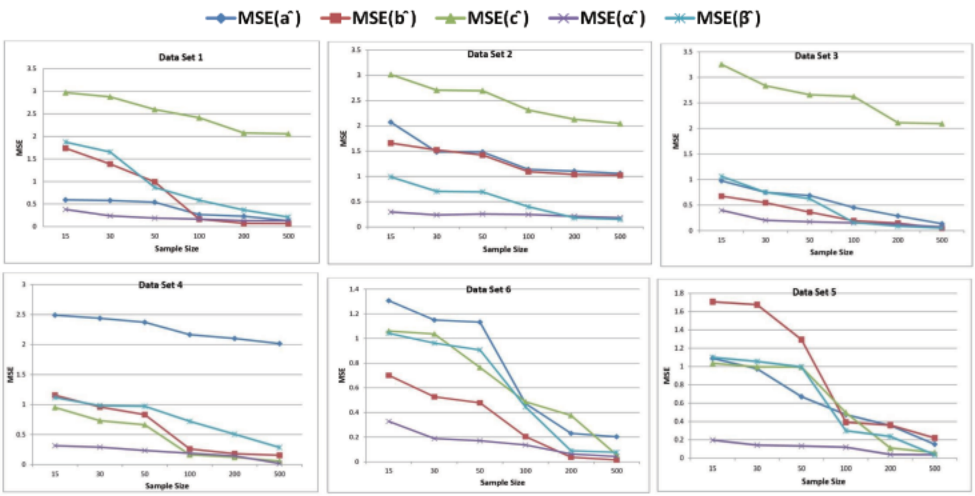

Figure 3. **Behaviour of the MSE Plots of the Estimated Parameters for the GMKD Simulated Data Sets**

Figure 3 shows the behaviour of the plots of the MSE of each of the model estimated parameters for the different GMKD simulated data set, which are decreasing as the sample size increases.

9. Application of Fitting GMKD Model to Real-Life Data

We consider four real-life data sets in order to show the usefulness of the proposed estimation procedure to estimate and fit the GMKD model to these real-life data sets. The data sets are;
***Data Set 1*:** Represents the first-semester mathematics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Fall 2018.
***Data Set 2*:** Represents the second-semester mathematics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Fall 2018.
***Data Set 3*:** Represents the first-semester introductory statistics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Spring 2018.
***Data Set 4*:** Represents the second-semester introductory statistics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Spring 2018.
Table 5 shows some statistics for the actual grades data sets, Table 6a shows the actual and predicted first and second-semester Mathematics course examinations grades frequencies with model parameters estimates and the chi-squares goodness of fit for the proposed GMKD and the Kumaraswamy power function distribution (KPFD), which is close to the GMKD (see Section 4.1), and Table 6b shows the actual and predicted first and second-semester Introductory Statistics course examinations grades frequencies with model parameters estimates and the chi-squares goodness of fit for the proposed GMKD and the KPFD. The p-values of the Chi-Squares of for each grade four data set using the GMKD model inducting very good estimates statistically, as well as, better than all the KPFD models. These results can be seen visually also from Figure 4, illustrating the histograms and the fitted pdfs for each of the grade data sets.

Table 5: Some Statistics for Actual Grades Data Sets

Statistics	Examination			
	Math 1	Math 2	Stat 1	Stat 2
N	592	592	509	509
Mean	9.032095	10.456081	10.368369	11.885069
Std. Error of Mean	0.204036	0.202118	0.208087	0.192046
Median	8.5	10.5	10.5	12.5
Mode	6.5*	9.5*	9.5*	13.5
Variance	24.645330	24.184193	22.039726	18.772696
Skewness	0.243412	-0.004408	0.040214	-0.211136
Kurtosis	-0.877241	-0.932818	-0.890339	-0.832699
Std. Error of Kurtosis	0.200505	0.200505	0.216089	0.216089
Minimum	0.0	0.0	0.0	1
Maximum	20.0	20.0	20.0	20.0

* Multiple modes exist. The smallest value is shown

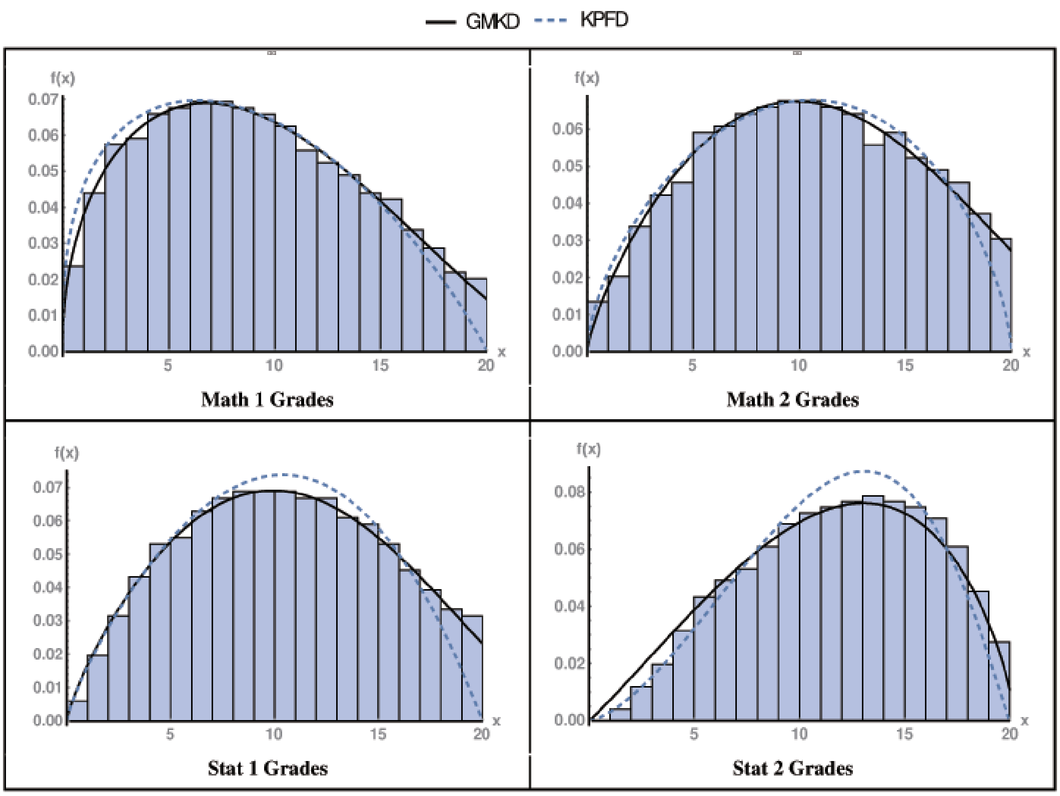

Figure 4. **Histograms and the Fitted pdfs for the Grades Data Sets**

Table 6a: Actual and Predicted Mathematics Exams 1 and 2 Grades Frequencies with Model Parameters Estimates and Goodness of Fit

Grades Range	Frequencies					
	Stat Exam 1			Stat Exam 2		
	Actual	Predicted GMKB	Predicted KPFD	Actual	Predicted GMKB	Predicted KPFD
0.0 — 1.0	14	17	22	8	6	8
1.1 — 2.0	26	27	31	12	14	16
2.1 — 3.0	34	33	36	20	20	22
3.1 — 4.0	35	36	38	25	26	26
4.1 — 5.0	39	39	40	27	30	30
5.1 — 6.0	40	40	41	35	33	33
6.1 — 7.0	41	41	41	36	36	36
7.1 — 8.0	41	41	41	38	38	38
8.1 — 9.0	40	40	40	39	39	39
9.1 — 10.0	39	38	39	40	40	40
10.1 — 11.0	37	37	37	40	40	40
11.1 — 12.0	33	35	35	39	39	40
12.1 — 13.0	31	32	32	38	38	39
13.1 — 14.0	29	29	29	33	36	38
14.1 — 15.0	26	26	26	35	34	36
15.1 — 16.0	25	23	22	31	31	33
16.1 — 17.0	20	20	18	29	28	29
17.1 — 18.0	17	16	14	27	25	24
18.1 — 19.0	13	13	9	22	21	18
19.1 — 20.0	12	10	1	18	18	7
Total	592	592	592	592	592	592
Estimated Model Parameters	a	21.111111	20	a	20.318890	20
	b	1.449899	1.33345	b	1.751212	1.610260
	c	2.683171	1.911199	c	3.111119	1.680140
	α	0.010120		α	0.036670	
	β	1.233333		β	1.677778	
Goodness of Fit	χ^2	1.43279416	28.032592	χ^2	1.93479971	21.048621
	df	14	15*	df	14	16
	p-value	0.9999897	0.0213659	p-value	0.99993212	0.17665

* The number of internals were adjusted in order to make the expected number of observations in each interval equal to or greater than 5, which is in tern effected the number of the degree of the freedom.

Table 6b: Actual and Predicted Statistics Exams 1 and 2 Grades Frequencies with Model Parameters Estimates and Goodness of Fit

Grades Range	Frequencies					
	Stat Exam 1			Stat Exam 2		
	Actual	Predicted GMKB	Predicted KPFD	Actual	Predicted GMKB	Predicted KPFD
0.0 — 1.0	2	5	4	0	1	1
1.1 — 2.0	10	12	11	2	5	3
2.1 — 3.0	16	17	17	6	8	6
3.1 — 4.0	22	22	22	10	12	10
4.1 — 5.0	27	26	26	16	16	14
5.1 — 6.0	28	29	29	22	20	19
6.1 — 7.0	32	31	32	25	24	23
7.1 — 8.0	34	33	35	27	27	28
8.1 — 9.0	36	35	36	31	31	33
9.1 — 10.0	37	35	37	35	34	37
10.1 — 11.0	37	35	38	37	36	40
11.1 — 12.0	35	34	37	38	38	43
12.1 — 13.0	34	33	36	39	39	44
13.1 — 14.0	31	31	34	40	40	44
14.1 — 15.0	30	29	31	39	39	43
15.1 — 16.0	27	27	28	38	38	39
16.1 — 17.0	22	24	23	36	35	34
17.1 — 18.0	21	20	18	31	30	27
18.1 — 19.0	16	17	11	23	23	17
19.1 — 20.0	12	14	4	14	13	4
Total	509	509	509	509	509	509
Estimated Model Parameters	a	20.999999	20	a	20.05333	20
	b	1.785556	1.799513	b	2.172103	2.499990
	c	2.891111	1.997890	c	1.771321	2.137350
	α	0.021012		α	0.000111	
	β	1.388889		β	1.02115	
Goodness of Fit	χ^2	3.2402072	13.148697	χ^2	4.47101738	16.635000
	df	14	14*	df	13*	13*
	p-value	0.99856701	0.5148497	p-value	0.98507207	0.216531

* The number of internals were adjusted in order to make the expected number of observations in each interval equal to or greater than 5, which is in tern effected the number of the degree of the freedom.

10. Summary

A new generalized modification of the Kumaraswamy distribution is introduced, and its properties consisting of boundaries, limits, mode, quantities, reliability and hazard functions, and Renyi entropy, are studied, and some of its different various shapes are given to show its flexibility. This distribution is closed under scaling and exponentiation. Some of the well-known distributions, such as the generalized uniform, triangular, beta, power function, Minimax, and some other Kumaraswamy related distributions, are special cases of this distribution. Its order statistics, mean deviations, moment generating function, and Lorenz and Bonferroni curves, with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted, L, and the trimmed L moments are derived. We used the maximum likelihood estimation method for estimating its parameters, and are applied to six different models, having different pdf's shapes, simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes, which are shown to be decreasing as the sample size increases. Finally, four real-life data sets of students grades' at Ahmed Bin Mohammed Military College, Doha-Qatar, the first two sets, representing the first and second semester mathematics course examination grades for the academic semesters Spring 2011 till Fall 2018, while the third and fourth sets representing the first and second semester introductory statistics course examination grades for the academic semesters Spring 2011 till Spring 2018, are used in order to show the usefulness and the flexibility of this distribution in application to real-life data sets, as well as our examinations grades data sets. We also used the Kumaraswamy power function distribution models, which is close to our proposed distribution. The results are very good, statistically, via the chi-squares goodness of fit tests, and visually, via the histograms and the fitted pdfs for each of our examinations grade data sets, and even better than all the Kumaraswamy power function distribution models.

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