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Iterative Algorithms for a Generalized System of Mixed Variational-Like Inclusion Problems and Altering Points Problem

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Abstract In this article, we introduce and study a generalized system of mixed variational-like inclusion problems involving $\alpha\beta$-symmetric η-monotone mappings. We use the resolvent operator technique to calculate the approximate common solution of the generalized system of variational-like inclusion problems involving $\alpha\beta$-symmetric η-monotone mappings and a fixed point problem for nonlinear Lipchitz mappings. We study strong convergence analysis of the sequences generated by proposed Mann type iterative algorithms. Moreover, we consider an altering points problem associated with a generalized system of variational-like inclusion problems. To calculate the approximate solution of our system, we proposed a parallel S-iterative algorithm and study the convergence analysis of the sequences generated by proposed parallel S-iterative algorithms by using the technique of altering points problem. The results presented in this paper may be viewed as generalizations and refinements of the results existing in the literature.

Keywords Resolvent operator, $\alpha\beta$-symmetric η-monotone mapping, Altering points, Parallel S-iterative algorithm, Lipschiz mapping.

AMS 2010 subject classifications 49J40, 47J25, 47H10

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1. Introduction

The theory of variational inequality was introduced by Hartmann and Stampacchia [17] in 1966 as a tool to study partial differential equations with applications. It has emerged as a powerful tool for wide class of unrelated problems arises in various branches of physical, engineering, pure and applied sciences in a unified and general framework, see; for example, [3, 4, 5, 8, 9, 10, 21]. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques for their own sake as well as for their applications. In 1989, Parida et al. [26] studied a generalized form of variational inequalities. They called it variational-like inequality problem and established its relationship with mathematical programming problem. Variational-like inequality problem has many important and novel applications in economics and optimizations, see; for example, [7, 32, 35].

There are a substantial number of numerical methods including projection methods, Wiener-Hopf equations techniques, auxiliary principle technique, descent, and Newton methods, see; for example [11, 12, 13, 14, 24, 25] to solve variational and variational-like inequalities. Projection method and its variant forms represent important

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tools for finding the approximate solutions of various type of variational and variational-like inequalities. It is worth mentioning that it is not possible to find the projection of the solution except in very simple cases. In addition, there are classes of variational inequalities, see; for example, [12, 13, 24, 25] for which the projection type methods cannot be applied to study the existence of the solution and to compute the approximate solution. These facts motivated us to use resolvent operator and altering points techniques to solve them efficiently. By using resolvent operators one may develop powerful and efficient iterative algorithms for several classes of variational inequalities. In fact, proximal algorithms are well suited to the problems of substantial recent interests involving large or higher dimensional data sets.

In this paper, we consider a generalized system of mixed variational-like inclusion problems involving $\alpha\beta$-symmetric η-monotone mapping and a fixed point problem of nonlinear Lipschitz mapping in Hilbert spaces. We use resolvent operator technique to calculate approximate common solution of generalized system of mixed variational-like inclusion problems involving $\alpha\beta$-symmetric η-monotone mapping and a fixed point problem for nonlinear Lipchitz mappings. We discuss the strong convergence analysis of the sequences generated by proposed Mann type iterative algorithm under some mild conditions. Moreover, we consider an altering points problem associated to generalized system of variational-like inclusion problems and propose a parallel S-iterative algorithm. We establish the existence of solution for our considered system of variational inclusions by using the technique of altering points problem. Finally, we prove convergence result for the proposed parallel S-iterative algorithm. The results presented in this paper improve and extend the results existing in the literature, see; for example, [1, 6, 18, 19, 27, 31, 33, 36].

Let $\mathcal{H}$ be a real Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$ which induces norm $\|\cdot\|$ and d be the metric induced by the norm $\|\cdot\|$. Let K be a nonempty closed and convex subset of $\mathcal{H}$, $\mathcal{CB}(\mathcal{H})$ be the family of all nonempty closed and bounded subsets of $\mathcal{H}$ and let $\mathcal{D}(\cdot,\cdot)$ be the Hausdorff metric on $\mathcal{CB}(\mathcal{H})$ defined by

$$\mathcal{D}(X,Y) = \max\{\sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(X,y)\},\ \forall X,Y \in \mathcal{CB}(\mathcal{H})$$

where $d(x,Y) = \inf_{y\in Y} d(x,y)$ and $d(X,y) = \inf_{x\in X} d(x,y)$.

Before presenting our main results, we collect relevant definitions and results which will be needed in our subsequent discussion.

Definition 1.1. A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be Lipchitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \le k\|x - y\|,\ \forall x, y \in \mathcal{H}.$$

If $k = 1$, the mapping T is called nonexpansive. We denote the set of all fixed points of T by $F(T)$, that is, $F(T) = \{x \in \mathcal{H} : Tx = x\}$.

Definition 1.2. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$; $A, B : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings and let $M : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping. Then

(i) $M(A,\cdot)$ is said to be α-strongly η-monotone with respect to A, if there exists a constant $\alpha > 0$ such that

$$\langle M(Ax,u) - M(Ay,u), \eta(y,x)\rangle \ge \alpha\|x-y\|^2,\ \forall x,y,u \in \mathcal{H};$$

(ii) $M(\cdot,B)$ is said to be β-relaxed η-monotone with respect to B, if there exists a constant $\beta > 0$ such that

$$\langle M(u,Bx) - M(u,By), \eta(y,x)\rangle \ge (-\beta)\|x-y\|^2,\ \forall x,y,u \in \mathcal{H};$$

(iii) $M(A,B)$ is said to be $\alpha\beta$-symmetric η-monotone with respect to A and B, if $M(A,\cdot)$ is α-strongly η-monotone with respect to A and $M(\cdot,B)$ is β-relaxed η-monotone with respect to B;

(iv) $M(\cdot,\cdot)$ is said to be ξ-mixed Lipschitz continuous with respect to A and B, if there exists a constant $\xi > 0$ satisfying

$$\|M(Ax, Bx) - M(Ay, By)\| \leq \xi\|x - y\|, \ \forall x, y \in \mathcal{H};$$

(v) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \ \forall x, y \in \mathcal{H}.$$

Definition 1.3. Let $A, B : \mathcal{H} \to \mathcal{H}; M, N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mappings and $T, P : \mathcal{H} \to \mathcal{CB}(\mathcal{H})$ be the set-valued mappings. Then

(i) T is said to be $\mathcal{D}$-Lipschitz continuous, if there exists a constant $\delta_T > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \delta_T\|x - y\|, \ \forall x, y \in \mathcal{H};$$

(ii) M is said to be mixed strongly monotone with respect to A and B, if there exists a constant $k > 0$ such that

$$\langle M(Ax, Bx) - M(Ay, By), x - y\rangle \geq k\|x - y\|^2, \ \forall x, y \in \mathcal{H};$$

(iii) P is said to be strongly monotone with respect to N in the first argument, if there exists a constant $c > 0$ such that

$$\langle N(u, \cdot) - N(v, \cdot), x - y\rangle \geq c\|x - y\|^2, \ \forall x, y \in \mathcal{H} \text{ and for some } u \in P(x), v \in P(y);$$

(iv) N is said to be γ-Lipschitz continuous in the first argument, if there exists a constant $\gamma > 0$ such that

$$\|N(u, \cdot) - N(v, \cdot)\| \leq \gamma\|u - v\|, \ \forall x \in \mathcal{H}, u \in P(x), v \in P(y).$$

Similarly, one can define strong monotonicity of P in the second argument with respect to N and Lipschitz continuity of N in the second argument.

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$. Define the norm $\|\cdot\|_*$ on $\mathcal{H} \times \mathcal{H}$ by

$$\|(x, y)\|_* = \|x\| + \|y\|, \ \forall x, y \in \mathcal{H}. \tag{1}$$

Note that $(\mathcal{H} \times \mathcal{H}, \|\cdot\|_*)$ is a Banach space.

Definition 1.4. Let A and B be two nonempty subsets of a metric space X and let $P : A \to B$ and $Q : B \to A$ be mappings. Then $x \in A$ and $y \in B$ are called altering points of the mappings P and Q, if

$$\begin{cases} P(x) = y, \\ Q(y) = x. \end{cases}$$

We denote the set of altering points of the mappings $P : A \to B$ and $Q : B \to A$ by

$$\text{Alt}(P, Q) = \{(x, y) \in A \times B : P(x) = y \text{ and } Q(y) = x\}.$$

Example 1.1. Let $\mathcal{H} = \mathbb{R}$ and $C = D = [0, k], k > 0$. Define $P, Q : \mathcal{H} \to \mathcal{H}$ by $P(x) = Q(x) = k - x$. Then $QP : C \to C$ is defined by $QP(x) = Q(k - x) = x$ and $PQ : D \to D$ is defined by $PQ(x) = P(k - x) = x$. It is easy to see that each point of C is a fixed point of QP and each point of D is a fixed point of PQ. Thus, altering points $x \in C$ and $y \in D$ are given by

$$\text{Alt } (P, Q) = \{(x, y) \in C \times D : x + y = k\}.$$

Example 1.2. Let $\mathcal{H} = \mathbb{R}, C = [0, 2]$, and $D = [2, 4]$. Define $P : C \to D$ by $P(x) = x + 2, \forall x \in C$ and $Q : D \to C$ by $Q(x) = \frac{x^2}{8}, \forall x \in D$. Note that, $PQ(x) = \frac{x^2}{8} + 2, \forall x \in D$ and $QP(x) = \frac{(x+2)^2}{8}, \forall x \in C$. Then $(2, 4)$ are altering points of P and Q, i.e., $(2, 4) \in \text{Alt}(P, Q)$.

Definition 1.5. [38]. A functional $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-*DQCV*) in x_i, if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{H}$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0.$$

Definition 1.6. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a single-valued mapping. A proper functional $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $x \in \mathcal{H}$, if there exists a point $f^* \in \mathcal{H}$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \ \forall y \in \mathcal{H};$$

where f^* is called η-subdgradient of ϕ at $x \in dom\phi$. The set of all η-subgradients of ϕ at x is denoted by $\partial_\eta \phi(x)$. The mapping $\partial_\eta \phi : \mathcal{H} \to 2^{\mathcal{H}}$ defined by

$$\partial_\eta \phi(x) = \begin{cases} f^* \in \mathcal{H} : \phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \ \forall y \in \mathcal{H}, & x \in dom\phi \\ \emptyset, & x \notin dom\phi \end{cases}$$

is said to be η-subdifferential of ϕ at x.

Definition 1.7. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$; $A, B : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional and let $M : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a nonlinear mapping. If for any given point $z \in \mathcal{H}$ and $\rho > 0$, there exists a unique point $x \in \mathcal{H}$ satisfying

$$\langle M(Ax, Bx) - z, \eta(y, x) \rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \ \forall y \in \mathcal{H},$$

then the mapping $z \mapsto x$, denoted by $R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}(z)$ is called resolvent operator of ϕ. Then, we have $z - M(Ax, Bx) \in \rho \partial_\eta \phi(x)$ and it follows that $R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}(z) = [M(A, B) + \rho \partial_\eta \phi]^{-1}(z)$.

Lemma 1.1. *[20]. Let $\mathcal{H}$ be a real Hilbert space. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a continuous mapping such that $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in \mathcal{H}$; let $M : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an $\alpha\beta$-symmetric η-monotone mapping with respect to A and B; let for any $z \in \mathcal{H}$, the function $h(y, x) = \langle z - M(Ax, Bx), \eta(y, x) \rangle$ be 0-DQCV in y and let $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable (may not be convex) functional. Then for any given constant $\rho > 0$ and $z \in \mathcal{H}$, there exists a unique $x \in \mathcal{H}$ such that*

$$\langle M(Ax, Bx) - z, \eta(y, x) \rangle \geq \rho\phi(x) - \rho\phi(y), \ \forall y \in \mathcal{H}, \tag{2}$$

that is, $x = R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}(z)$.

Lemma 1.2. *[20]. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be τ-Lipschitz continuous such that $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in \mathcal{H}$; let $M : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an $\alpha\beta$-symmetric η-monotone mapping with respect to A and B; let for any $z \in \mathcal{H}$, the function $h(y, x) = \langle z - M(Ax, Bx), \eta(y, x) \rangle$ be 0-DQCV in y and let $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and η-subdifferentiable functional and let $\rho > 0$ be any given constant. Then the resolvent operator $R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}$ of ϕ is $\frac{\tau}{\alpha - \beta}$-Lipschitz continuous, that is,*

$$\|R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}(z_1) - R^{\partial_\eta \phi}_{\rho, M(\cdot,\cdot)}(z_2)\| \leq \frac{\tau}{\alpha - \beta} \|z_1 - z_2\|, \ \forall z_1, z_2 \in \mathcal{H}.$$

Example 1.3. Let $\mathcal{H} = \mathbb{R}$, $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a mapping defined by $\eta(y, x) = \dfrac{x - y}{c}$, $\forall x, y \in \mathcal{H}$, $c > 0$ be any real number. Then, it is easy to verify that

(i) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in \mathbb{R}$;
(ii) $|\eta(x, y)| \leq \frac{1}{c}|x - y|, \forall x, y \in \mathbb{R}$; i.e., η is $\frac{1}{c}$–Lipschitz continuous.

Let $M : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $A, B : \mathcal{H} \to \mathcal{H}$ be the mappings defined by $M(x, y) = x - y, A(x) = 3kx, B(x) = 2kx$ for any $x \in \mathbb{R}$ and $k > 0$. Then

$$\begin{aligned}\langle M(Ax, u) - M(Ay, u), \eta(y, x)\rangle &= \langle A(x) - A(y), \eta(y, x)\rangle \\ &= \langle 3k(x - y), \frac{x - y}{c}\rangle \\ &\geq \frac{3k}{c}|x - y|^2, \forall x, y \in \mathbb{R},\end{aligned}$$

i.e., M is $\frac{3k}{c}$–strongly η-monotone with respect to A.

$$\begin{aligned}\langle M(u, Bx) - M(u, By), \eta(y, x)\rangle &= \langle -(B(x) - B(y)), \eta(y, x)\rangle \\ &= \langle -2k(x - y), \frac{x - y}{c}\rangle \\ &\geq \frac{-2k}{c}|x - y|^2, \forall x, y \in \mathbb{R},\end{aligned}$$

i.e., M is $\frac{2k}{c}$–relaxed η-monotone with respect to B. Hence, $M(A, B)$ is $\alpha\beta$-symmetric η-monotone mapping with respect to A and B.

Next, for any $x, z \in \mathbb{R}$, the mapping

$$\begin{aligned}h(y, x) &= \langle z - M(Ax, Bx), \eta(y, x)\rangle \\ &= (z - kx)(\tfrac{x-y}{c})\end{aligned}$$

is 0-$DQCV$ in y. If, it is false, then there exists a finite set $\{y_1, y_2, \cdots, y_m\}$ and $x_0 = \sum_{i=1}^{m} t_i y_i$ with $t_i \geq 0$ and $\sum_{i=1}^{m} t_i = 1$ such that for each $i = 1, 2, \cdots, m$,

$$\begin{aligned}0 < h(y_i, x_0) &= \langle z - M(Ax_0, Bx_0), \eta(y_i, x_0)\rangle \\ &= (z - kx_0)(\tfrac{x_0-y_i}{c}) \\ &= (z - kx_0)\Big(\tfrac{\sum_{i=1}^{m} t_i y_i - y_i}{c}\Big) = 0,\end{aligned}$$

which is not possible. Thus, for any $x, z \in \mathbb{R}$, the mapping $h(y, x)$ is is 0-$DQCV$ in y.

Lemma 1.3. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \ \ \forall n \geq n_0$$

for some $n_0 \in \mathbb{N}$, $\{\lambda_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty, b_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq ka_n + b_n, \ \ \forall n \in \mathbb{N},$$

$k \in (0, 1)$ with $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

2. Formulation of the Problem and Existence Results

Let $\mathcal{H}$ be a real Hilbert space and assume that $I = \{1, 2\}$ is an index set; for each $i \in I$, let $M_i, N_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$; $A_i, B_i, g_i : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings and $P, Q : \mathcal{H} \to \mathcal{CB}(\mathcal{H})$ be the multi-valued mappings;

$\eta_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a single-valued nonlinear mapping and $\phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be mapping such that for fixed $x, y \in \mathcal{H}$, $\phi_1(g_1(\cdot), x)$ and $\phi_2(g_2(\cdot), y)$ be lower semi-continuous, η-subdifferential, proper functionals on $\mathcal{H}$ satisfying $g_1(\mathcal{H}) \cap \text{dom}(\partial_\eta \phi(g_1(\cdot), x)) \neq \emptyset$ and $g_2(\mathcal{H}) \cap \text{dom}(\partial_\eta \phi(g_2(\cdot), y)) \neq \emptyset$. We consider the following generalized system of multi-valued mixed variational-like inclusion problems $(GSMVLIP)$:

Find (x, y, u, v) such that $x, y \in \mathcal{H}, v \in P(x), u \in Q(y)$ and

$$GSMVLIP : \begin{cases} \langle N_1(u, v) + M_1(A_1(x), B_1(x)) - M_1(A_1(y), B_1(y)), \eta_1(z_1, g_1(x)) \rangle \\ \qquad \geq \rho_1[\phi_1(g_1(x), x) - \phi_1(z_1, x)], \ \forall z_1 \in \mathcal{H}, \\ \langle N_2(v, u) + M_2(A_2(y), B_2(y)) - M_2(A_2(x), B_2(x)), \eta_2(z_2, g_2(y)) \rangle \\ \qquad \geq \rho_2[\phi_2(g_2(y), y) - \phi_2(z_2, y)], \ \forall z_2 \in \mathcal{H}. \end{cases} \tag{3}$$

Some special cases of $GSMVLIP$ (3) are listed below.

(i) If $M_1 = M_2 \equiv 0, g_1, g_2, P, Q \equiv I$, the identity mappings, then $GSMVLIP$ (3) is equivalent to the problem of finding $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{cases} \langle M_1(x_1, x_2), \eta_1(y_1, x_1) \rangle + \phi_1(x_1, y_1) - \phi_1(x_1, x_1) \geq 0, \ \forall y_1 \in \mathcal{H}_1, \\ \langle M_2(x_1, x_2), \eta_2(y_2, x_2) \rangle + \phi_2(x_2, y_2) - \phi_2(x_2, x_2) \geq 0, \ \forall y_2 \in \mathcal{H}_2. \end{cases} \tag{4}$$

System (4) was studied by Kazmi and Khan [19].

(ii) If $M_1 = M_2 \equiv 0, g_1 = g_2 \equiv I$, the identity mappings, $\eta_i(z_i, a_i) = z_i - a_i, N_1(u, v) = S(x, v), N_2(v, u) = T(y, u)$ and $\phi_i(x, y) = f(x)$, then $GSMVLIP$ (3) coincides with the problem of finding $x, y \in K, u \in E(x)$ and $v \in F(y)$ such that

$$\begin{cases} \langle S(x, v)), z - x \rangle + f(z) - f(x) \geq 0, \ \forall z \in K, \\ \langle T(y, u)), z - y \rangle + f(z) - f(y) \geq 0, \ \forall z \in K. \end{cases} \tag{5}$$

Problem (5) was considered and studied by Li and Li [22].

(iii) If $E = F \equiv I$, the identity mapping, then Problem (5) reduces to the problem of finding $x, y \in K$ such that

$$\begin{cases} \langle S(x, y)), z - x \rangle + f(z) - f(x) \geq 0, \ \forall z \in K, \\ \langle T(y, x)), z - y \rangle + f(z) - f(y) \geq 0, \ \forall z \in K. \end{cases} \tag{6}$$

(iv) If $S(x, y) \equiv sA(y, x) + x - y$ and $T(y, x) \equiv rA(x, y) + y - x$, then System (6) is equivalent to the problem of finding $x, y \in \mathcal{H}$ such that

$$\begin{cases} \langle sA(y, x)) + x - y, z - x \rangle + \phi(z) - \phi(x) \geq 0, \ \forall z \in \mathcal{H}, s > 0 \\ \langle rA(x, y)) + y - x, z - y \rangle + \phi(z) - \phi(y) \geq 0, \ \forall z \in \mathcal{H}, r > 0. \end{cases} \tag{7}$$

System (7) was studied by Petrot [27].

(v) If K is a closed convex subsets of $\mathcal{H}$ and $\phi(x) = \delta_K(x), \forall x \in K$, where the indicator function δ_K is defined by

$$\delta_K = \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the system (7) reduces to the following problem of finding $x, y \in K$ such that

$$\begin{cases} \langle sA(y, x)) + x - y, z - x \rangle \geq 0, \ \forall z \in \mathcal{H}, s > 0 \\ \langle rA(x, y)) + y - x, z - y \rangle \geq 0, \ \forall z \in \mathcal{H}, r > 0. \end{cases} \tag{8}$$

A System of type (8) was studied by Chang et al. [6].

In the following theorem, we establish the fixed point formulation of $GSMVLIP$ (3) then we prove the existence of common solution of $GSMVLIP$ (3) for $\alpha\beta$-symmetric η-monotone mapping and a fixed point problem of nonlinear Lipschitz mappings.

Theorem 2.1. *Let $\mathcal{H}$ be a real Hilbert space. For each $i \in \{1,2\}$; let $A_i, B_i, g_i : \mathcal{H} \to \mathcal{H}$; $N_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mappings and $P, Q : \mathcal{H} \to CB(\mathcal{H})$ be the set-valued mappings. Let $\eta_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a τ-Lipschitz continuous mapping such that $\eta_i(x_i, y_i) + \eta_i(y_i, x_i) = 0$, for all $x_i, y_i \in \mathcal{H}$. Let $M_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an $\alpha_i\beta_i$-symmetric η_i-monotone mapping with respect to A_i and B_i; let for any $z_i \in \mathcal{H}$, the function $\langle z_i - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$ be 0-DQCV in y_i and let $\phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be such that for fixed fixed $x, y \in \mathcal{H}$, $\phi_1(g_1(\cdot), x)$ and $\phi_2(g_2(\cdot), y)$ be lower semi-continuous, η-subdifferential, proper functionals on $\mathcal{H}$ satisfying $g_1(\mathcal{H}) \cap dom(\partial_\eta \phi(g_1(\cdot), x)) \neq \emptyset$ and $g_2(\mathcal{H}) \cap dom(\partial_\eta \phi(g_2(\cdot), y)) \neq \emptyset$. Then (x, y, u, v), where $x, y \in \mathcal{H}, v \in P(x), u \in Q(y)$ is a solution of $GSMVLIP$ (3), if and only if (x, y, u, v) satisfies the following relation:*

$$\begin{aligned} x &= R^{\partial_\eta \phi_1(g_1(\cdot),x)}_{\rho_1, M_1(\cdot,\cdot)} \left[M_1(A_1(y), B_1(y)) - N_1(u, v)\right], \\ y &= R^{\partial_\eta \phi_2(g_2(\cdot),y)}_{\rho_2, M_2(\cdot,\cdot)} \left[M_2(A_2(x), B_2(x)) - N_2(v, u)\right], \end{aligned} \tag{9}$$

where $\rho_1, \rho_2 > 0$ are constants and $R^{\partial_\eta \phi_1(g_1(\cdot),x)}_{\rho_1, M_1(\cdot,\cdot)} = [M_1(A_1, B_1) + \rho \partial_\eta \phi(g_1(\cdot), x)]^{-1}, R^{\partial_\eta \phi_2(g_2(\cdot),y)}_{\rho_2, M_2(\cdot,\cdot)} = [M_2(A_2, B_2) + \rho \partial_\eta \phi(g_2(\cdot), y)]^{-1}$.

Proof
The conclusion follows directly from the definition of resolvent operators $R^{\partial_\eta \phi_1(g_1(\cdot),x)}_{\rho_1, M_1(\cdot,\cdot)}$ and $R^{\partial_\eta \phi_2(g_2(\cdot),y)}_{\rho_2, M_2(\cdot,\cdot)}$ of the functionals $\phi_1(g_1(\cdot), x)$ and $\phi_2(g_2(\cdot), y)$, respectively. □

Lemma 2.1. *If all the assumptions of the Theorem 2.1 are same and $T : \mathcal{H} \to \mathcal{H}$ be a nonlinear Lipschitz mapping such that $\{x, y\} \in (GSMVLIP) \cap F(T)$, then*

$$\begin{aligned} x = T(x) &= T[R^{\partial_\eta \phi_1(g_1(\cdot),x)}_{\rho_1, M_1(\cdot,\cdot)} \left[M_1(A_1(y), B_1(y)) - N_1(u, v)\right]], \\ y = T(y) &= T[R^{\partial_\eta \phi_2(g_2(\cdot),y)}_{\rho_2, M_2(\cdot,\cdot)} \left[M_2(A_2(x), B_2(x)) - N_2(v, u)\right]]. \end{aligned} \tag{10}$$

Based on Lemma 2.1, we suggest the following Mann-type iterative algorithm for finding a common element of the solution set of $GSMVLIP$ (3) involving $\alpha\beta$-symmetric η-monotone mapping and the set of fixed points $F(T)$ of a nonlinear Lipschitz mapping T.

Algorithm 2.1. *For any $x_0, y_0 \in \mathcal{H}, v_0 \in P(x_0), u_0 \in Q(y_0)$; compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ by the following iterative scheme:*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[R^{\partial_\eta \phi_1(g_1(\cdot),x_n)}_{\rho_1, M_1(\cdot,\cdot)} \left[M_1(A_1(y_n), B_1(y_n)) - N_1(u_n, v_n)\right]], \\ y_n = (1 - \beta_n)x_n + \beta_n T[R^{\partial_\eta \phi_2(g_2(\cdot),y_n)}_{\rho_2, M_2(\cdot,\cdot)} \left[M_2(A_2(x_n), B_2(x_n)) - N_2(v_n, u_n)\right]], \end{cases} \tag{11}$$

and

$$\begin{aligned} v_n \in P(x_n), \ \ \|v_{n+1} - v_n\| &\leq \mathcal{D}(P(x_{n+1}), P(x_n)); \\ u_n \in Q(y_n), \ \ \|u_{n+1} - u_n\| &\leq \mathcal{D}(Q(x_{n+1}), Q(x_n)); \end{aligned}$$

where $n = 1, 2, 3, \cdots, \rho_i > 0$ are constants and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

Theorem 2.2. *Let $\mathcal{H}$ be a real Hilbert space. For each $i \in \{1,2\}$; let $A_i, B_i, g_i : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Let $M_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an $\alpha_i\beta_i$-symmetric η_i-monotone mapping such that M_i be mixed Lipchitz continuous with constant t_i, mixed strongly monotone with constant k_i with respect to A_i and B_i. Let $N_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mapping such that $N_i(\cdot, \cdot)$ be γ_i and $\gamma_i^{'}$-Lipschitz continuous in first and second argument, respectively; $P, Q : \mathcal{H} \to CB(\mathcal{H})$ be the set-valued mappings such that Q is $\mathcal{D}$-Lipchitz continuous with constant ξ_1, strongly monotone with respect to N_1 with constant c_1 and P is $\mathcal{D}$-Lipchitz continuous with*

constant ξ_2, strongly monotone with respect to N_2 with constant c_2. Let $\eta_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous mapping with constant τ_i such that $\eta_i(x_i, y_i) = -\eta_i(y_i, x_i),\ \forall x_i, y_i \in \mathcal{H}$, and for any $z \in \mathcal{H}$, the mapping $h_i(y_i, x_i) = \langle z_i - M_i(A_i x_i, B_i x_i), \eta_i(z_i, g_i(\cdot))\rangle$ be 0-DQCV in z_i. Let $\phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $x, y \in \mathcal{H}$, $\phi_1(g_1(\cdot), x)$ and $\phi_2(g_2(\cdot), y)$ be lower semicontinuous, η_i-subdifferentiable, proper functionals on $\mathcal{H}$ satisfying $g_1(\mathcal{H}) \cap$ dom$(\partial_\eta \phi(g_1(\cdot), x)) \neq \emptyset$ and $g_2(\mathcal{H}) \cap$ dom$(\partial_\eta \phi(g_2(\cdot), y)) \neq \emptyset$. Suppose that there exist constants $\rho_i > 0, \mu_i > 0$ such that for each $z \in \mathcal{H}$

$$\left\| R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}(z) - R^{\partial_\eta \phi_1(g_1(\cdot), x)}_{\rho_1, M_1(\cdot,\cdot)}(z) \right\| \leq \mu_1 \|x_n - x\|; \tag{12}$$

$$\left\| R^{\partial_\eta \phi_2(g_2(\cdot), y_n)}_{\rho_2, M_2(\cdot,\cdot)}(z) - R^{\partial_\eta \phi_2(g_2(\cdot), y)}_{\rho_2, M_2(\cdot,\cdot)}(z) \right\| \leq \mu_2 \|y_n - y\|. \tag{13}$$

In addition, $GSMVLIP$ (3) $\cap F(T) \neq \phi$ and the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \beta_n = 1$;
(iii) $k\Omega_1\Omega_3 < \Omega_3 - k\Omega_2$; *where,*

$\Omega_1 = \mu_1 + L_1\gamma_1'\xi_2, \Omega_2 = L_1(\theta_1 + \theta_2)[(1 - \beta_n) + \beta_n k L_2(\theta_3 + \theta_4)], \Omega_3 = (1 - \beta_n k L_2 \gamma_2' \xi_1 - \beta_n k \mu_2)$,
$L_i = \frac{\tau_i}{\alpha_i - \beta_i}, \theta_1 = \sqrt{t_1^2 - 2k_1 + 1}, \theta_2 = \sqrt{\gamma_1^2\xi_1^2 - 2c_1 + 1}, \theta_3 = \sqrt{t_2^2 - 2k_2 + 1}, \sqrt{\gamma_2^2\xi_2^2 - 2c_2 + 1}$. *Then the iterative sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 2.1 converges strongly to x, y, u and v, respectively and (x, y, u, v) is the common solution of $GSMVLIP$ (3) and $F(T)$.*

Proof
By applying Lemma 1.2, Algorithm 2.1, Lipschitz continuity of T and condition (12), we have

$$\begin{aligned}
\|x_{n+1} - x\| &= \|(1 - \alpha_n)x_n + \alpha_n T[R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y_n), B_1(y_n)) - N_1(u_n, v_n)]] - x\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n \Big\| T\left[R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y_n), B_1(y_n)) - N_1(u_n, v_n)]\right] \\
&\quad - T\left[R^{\partial_\eta \phi_1(g_1(\cdot), x)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y), B_1(y)) - N_1(u, v)]\right] \Big\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n k \Big\| R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y_n), B_1(y_n)) - N_1(u_n, v_n)] \\
&\quad - R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y), B_1(y)) - N_1(u, v)] \Big\| \\
&\quad + \alpha_n k \Big\| R^{\partial_\eta \phi_1(g_1(\cdot), x_n)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y), B_1(y)) - N_1(u, v)] \\
&\quad - R^{\partial_\eta \phi_1(g_1(\cdot), x)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y), B_1(y)) - N_1(u, v)] \Big\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n k L_1 \| [M_1(A_1(y_n), B_1(y_n)) - N_1(u_n, v_n)] \\
&\quad - [M_1(A_1(y), B_1(y)) - N_1(u, v)] \| + \alpha_n k \mu_1 \|x_n - x\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n k L_1 \|M_1(A_1(y_n), B_1(y_n)) - M_1(A_1(y), B_1(y)) \\
&\quad - (y_n - y)\| + \alpha_n k L_1 \|N_1(u_n, v_n) - N_1(u, v) - (y_n - y)\| + \alpha_n k \mu_1 \|x_n - x\| \\
&\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n k L_1 \|M_1(A_1(y_n), B_1(y_n)) - M_1(A_1(y), B_1(y)) \\
&\quad - (y_n - y)\| + \alpha_n k L_1 \|N_1(u_n, v_n) - N_1(u, v_n) - (y_n - y)\| \\
&\quad + \alpha_n k L_1 \|N_1(u, v_n) - N_1(u, v)\| + \alpha_n k \mu_1 \|x_n - x\|.
\end{aligned} \tag{14}$$

Now,

$$\begin{aligned}
\|M_1(A_1(y_n), B_1(y_n)) &- M_1(A_1(y), B_1(y)) - (y_n - y)\|^2 \\
&= \|M_1(A_1(y_n), B_1(y_n)) - M_1(A_1(y), B_1(y))\|^2 \\
&\quad - 2\langle M_1(A_1(y_n), B_1(y_n)) - M_1(A_1(y), B_1(y)), y_n - y\rangle + \|y_n - y\|^2.
\end{aligned}$$

Since M_1 is mixed Lipschitz continuous with constant t_1 and mixed strongly monotone with constant k_1 with respect to A_1 and B_1, therefore

$$\begin{aligned}
\|M_1(A_1(y_n), B_1(y_n)) - M_1(A_1(y), B_1(y)) - (y_n - y)\|^2 &\leq (t_1^2 - 2k_1 + 1)\|y_n - y\|^2 \\
&= \theta_1^2 \|y_n - y\|^2.
\end{aligned} \tag{15}$$

Since Q is strongly monotone with respect to N_1 with constant c_1 and N_1 is γ_1-Lipschitz continuous in the first argument and Q is $\mathcal{D}$-Lipschitz continuous with constant ξ_1, we get

$$\begin{aligned}\|N_1(u_n,v_n)-N_1(u,v_n)-(y_n-y)\|^2 &= \|N_1(u_n,v_n)-N_1(u,v_n)\|^2\\ &\quad -2\langle N_1(u_n,v_n)-N_1(u,v_n),y_n-y\rangle+\|y_n-y\|^2\\ &\leq (\gamma_1^2\xi_1^2-2c_1+1)\|y_n-y\|^2\\ &= \theta_2^2\|y_n-y\|^2.\end{aligned} \tag{16}$$

Since N_1 is $\gamma_1^{'}$-Lipschitz continuous in the second argument and P is $\mathcal{D}$-Lipschitz continuous with constant ξ_2, we have

$$\|N_1(u,v_n)-N_1(u,v)\leq\gamma_1^{'}\|v_n-v\|\leq\gamma_1^{'}\xi_2\|x_n-x\|. \tag{17}$$

It follows from (14), (15), (16) and (17) that

$$\|x_{n+1}-x\|\leq\left[(1-\alpha_n)+\alpha_nk\mu_1+\alpha_nkL_1\gamma_1^{'}\xi_2\right]\|x_n-x\|+\alpha_nkL_1(\theta_1+\theta_2)\|y_n-y\|. \tag{18}$$

Again employing Lemma 1.2, Algorithm 2.1, Lipschitz continuity of T and condition (13), we have

$$\begin{aligned}\|y_n-y\| &= \|(1-\beta_n)x_n+\beta_nT[R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]]-y\|\\ &\leq (1-\beta_n)\|x_n-y\|+\beta_n\Big\|T\left[R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]\right]\\ &\quad -T\left[R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\right]\Big\|\\ &\leq (1-\beta_n)\|x_n-y\|+\beta_nk\Big\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]\\ &\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\Big\|\\ &\quad +\beta_nk\Big\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\\ &\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\Big\|\\ &\leq (1-\beta_n)\|x_n-x\|+(1-\beta_n)\|x-y\|\\ &\quad +\beta_nkL_2\|[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]\\ &\quad -[M_2(A_2(x),B_2(x))-N_2(v,u)]\|+\beta_nk\mu_2\|y_n-y\|\\ &\leq (1-\beta_n)\|x_n-x\|+(1-\beta_n)\|x-y\|+\beta_nkL_2\|M_2(A_2(x_n),B_2(x_n))\\ &\quad -M_2(A_2(x),B_2(x))-(x_n-x)\|\\ &\quad +\beta_nkL_2\|N_2(v_n,u_n)-N_2(v,u_n)-(x_n-x)\|\\ &\quad +\beta_nkL_2\|N_2(v,u_n)-N_2(v,u)\|+\beta_nk\mu_2\|y_n-y\|.\end{aligned} \tag{19}$$

Now,

$$\begin{aligned}\|M_2(A_2(x_n),B_2(x_n)) &- M_2(A_2(x),B_2(x))-(x_n-x)\|^2\\ &= \|M_2(A_2(x_n),B_2(x_n))-M_2(A_2(x),B_2(x))\|^2\\ &\quad -2\langle M_2(A_2(x_n),B_2(x_n))-M_2(A_2(x),B_2(x)),x_n-x\rangle\\ &\quad +\|x_n-x\|^2.\end{aligned}$$

Since M_2 is mixed Lipschitz continuous with constant t_2 and mixed strongly monotone with constant k_2 with respect to A_2 and B_2, we have

$$\begin{aligned}\|M_2(A_2(x_n),B_2(x_n))-M_2(A_2(x),B_2(x))-(x_n-x)\|^2 &\leq (t_2^2-2k_2+1)\|x_n-x\|^2\\ &= \theta_3^2\|x_n-x\|^2.\end{aligned} \tag{20}$$

Since P is strongly monotone with respect to N_2 with constant c_2 and N_2 is γ_2-Lipschitz continuous in the first argument and P is $\mathcal{D}$-Lipschitz continuous with constant ξ_2, we get

$$\begin{aligned}\|N_2(v_n,u_n)-N_2(v,u_n)-(x_n-x)\|^2 &= \|N_2(v_n,u_n)-N_2(v,u_n)\|^2-2\langle N_2(v_n,u_n)\\ &\quad -N_2(v,u_n),x_n-x\rangle+\|x_n-x\|^2\\ &\leq (\gamma_2^2\xi_2^2-2c_2+1)\|x_n-x\|^2\\ &= \theta_4^2\|x_n-x\|^2.\end{aligned} \tag{21}$$

Since N_2 is $\gamma_2^{'}$-Lipschitz continuous in the second argument and Q is $\mathcal{D}$-Lipschitz continuous with constant ξ_1, we have

$$\|N_2(v,u_n)-N_2(v,u)\leq\gamma_2^{'}\|u_n-u\|\leq\gamma_2^{'}\xi_1\|y_n-y\|. \tag{22}$$

It follows from (19), (20), (21) and (22) that

$$\|y_n-y\|\leq[(1-\beta_n)+\beta_nkL_2(\theta_3+\theta_4)]\|x_n-x\|+\left[\beta_nkL_2\gamma_2^{'}\xi_1+\beta_nk\mu_2\right]\|y_n-y\|+(1-\beta_n)\|x-y\|,$$

which implies that

$$\|y_n-y\|\leq\frac{[(1-\beta_n)+\beta_nkL_2(\theta_3+\theta_4)]\|x_n-x\|+(1-\beta_n)\|x-y\|}{\left[1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2\right]}. \tag{23}$$

Thus, from (18) and (23), we have

$$\begin{aligned}
\|x_{n+1}-x\| &\leq \left[(1-\alpha_n)+\alpha_nk\mu_1+\alpha_nkL_1\gamma_1^{'}\xi_2\right]\|x_n-x\| \\
&\quad+\alpha_nkL_1(\theta_1+\theta_2)\left[\frac{(1-\beta_n)+\beta_nkL_2(\theta_3+\theta_4)\|x_n-x\|+(1-\beta_n)\|x-y\|}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\right] \\
&= \left[(1-\alpha_n)+\alpha_nk\mu_1+\alpha_nkL_1\gamma_1^{'}\xi_2\right]\|x_n-x\| \\
&\quad+\alpha_nkL_1(\theta_1+\theta_2)\left[\frac{(1-\beta_n)+\beta_nkL_2(\theta_3+\theta_4)}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\right]\|x_n-x\| \\
&\quad+\frac{\alpha_nkL_1(\theta_1+\theta_2)(1-\beta_n)}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\|x-y\| \\
&= 1-\alpha_n\left[1-k\left[\left(\mu_1+L_1\gamma_1^{'}\xi_2\right)+L_1(\theta_1+\theta_2)\frac{(1-\beta_n)+\beta_nkL_2(\theta_3+\theta_4)}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\right]\right]\|x_n-x\| \\
&\quad+\frac{\alpha_nkL_1(\theta_1+\theta_2)(1-\beta_n)}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\|x-y\| \\
&= 1-\alpha_n\left[1-k\left(\Omega_1+\frac{\Omega_2}{\Omega_3}\right)\right]+b_n.
\end{aligned} \tag{24}$$

Setting;
$a_n=\|x_n-x\|, \lambda_n=\alpha_n\left[1-k\left(\Omega_1+\frac{\Omega_2}{\Omega_3}\right)\right]$ and $b_n=\frac{\alpha_nkL_1(\theta_1+\theta_2)(1-\beta_n)}{(1-\beta_nkL_2\gamma_2^{'}\xi_1-\beta_nk\mu_2)}\|x-y\|$. It follows from condition (iii) that $\lambda_n\in(0,1),\ \forall n\in\mathbb{N}$ and condition (ii) implies that $b_n=o(\lambda_n)$. By using condition (iii), we have $\lambda_n>\alpha_n(1-\Omega_1),\ \forall n\in\mathbb{N}$. Hence, condition (i) implies that $\sum_{n=0}^{\infty}\lambda_n=\infty$. Thus all the conditions of Lemma 1.3 are satisfied and so $\|x_n-x\|\to 0$ as $n\to\infty$. Consequently, condition (ii) and (23) implies that $y_n\to y$ as $n\to\infty$. Since the mappings P and Q are $\mathcal{D}$-Lipschitz continuous and using Algorithm 2.1, it follows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $\mathcal{H}$ such that $u_n\to u$ and $v_n\to v$ as $n\to\infty$.

Now, we show that $u\in P(x)$ and $v\in Q(y)$;

$$\begin{aligned}
d(u,P(x)) &= \inf\{\|u-y\|:y\in P(x)\} \\
&\leq \|u-u_n\|+d(u_n,P(x)) \\
&\leq \|u-u_n\|+\mathcal{D}(P(x_n),P(x)) \\
&\leq \|u-u_n\|+\xi_1\|x_n-x\|\to 0,\ \text{as } n\to\infty,
\end{aligned} \tag{25}$$

which implies that $d(u,P(x))=0$. Since $P(x)\in\mathcal{CB}(\mathcal{H})$, it follows that $u\in P(x)$. Similarly, we can verify that $v\in Q(y)$. Therefore, in view of Theorem 2.1 and Algorithm 2.1, we conclude that (x,y,u,v) such that $x,y\in\mathcal{H}, u\in P(x)$ and $v\in Q(y)$ is a common solution of $GSMVLIP$ (3) and $F(T)$. □

3. Convergence Theorem via Altering Points Problem

In this section, we approximate the solution of $GSMVLIP$ (3) by using the altering points problem. First, we propose a parallel S-iterative algorithm for altering points problem associated to generalized system of mixed variational-like inclusion problems involving $\alpha\beta$-symmetric η-monotone mappings and then we study strong convergence analysis of $GSMVLIP$ (3) by using proposed iterative algorithm.

Let C and D be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. Let $S_1 : C \to D$ and $S_2 : D \to C$ be contraction mappings with Lipschitz constants τ_1 and τ_2, respectively. Since $S_2S_1 : C \to C$ is a contraction, there exists a unique element $(x, y) \in C \times D$ of the following altering points problem for operators S_1 and S_2.

Find $(x, y) \in C \times D$ such that

$$\begin{cases} S_1(x) = y, \\ S_2(y) = x. \end{cases} \tag{26}$$

It is well known that the Picard iterative algorithm converges faster than the Mann iterative algorithm [23] for contraction mappings, see; [2]. Sahu [28] has introduced Normal S-iteration Process, whose rate of convergence similar to the Picard iteration process and faster than other fixed point iteration processes (see; [28], Theorem 3.6) which is defined as follows:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T(x_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{27}$$

where T is a self mapping on a convex subset of a normed space X and $\alpha_n \subseteq [0, 1]$ is a real sequence. Normal S-iterative algorithm is independent of Mann algorithm and it has attracted the attention of many researchers, see; for example, [34, 15, 30] due to fast convergence rate and its simplicity.

Very recently, Gursoy et al. [16] studied the following normal S-iterative algorithm:

$$\begin{cases} p_0 \in H, \\ p_{n+1} = S[q_n - g(q_n) + P_H[g(q_n) - \sigma T q_n]], \\ q_n = (1 - \xi_n)p_n + \xi_n S[p_n - g(p_n) + P_H[g(p_n) - \sigma T p_n]], \end{cases} \tag{28}$$

where $\{\xi_n\}_{n=0}^{\infty} \subset [0, 1]$. They have approximated the solution of a generalized nonlinear variational inequalities and studied convergence analysis.

For approximate calculation of altering points of contraction mappings $S_1 : C \to D$ and $S_2 : D \to C$, motivated by normal S-iteration process, Sahu [29] has introduced the following parallel S-iteration process:

$$\begin{cases} x_{n+1} = T_2[(1 - \alpha)y_n + \alpha T_1 x_n], \\ y_{n+1} = T_1[(1 - \alpha)x_n + \alpha T_2 y_n], \end{cases} \tag{29}$$

where $\alpha \in (0, 1)$. Further, Zhao et al. [37] generalized the parallel S-iteration process (29) and studied the following parallel S-iteration process:

$$\begin{cases} x_{n+1} = T_2[(1 - \alpha_n)y_n + \alpha_n T_1(x_n)], \\ y_{n+1} = T_1[(1 - \beta_n)x_n + \beta_n T_2 y_n], \end{cases} \tag{30}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. The parallel S-iteration process (30) is a natural generalization of the parallel S-iteration process (29).

Based on Theorem 2.1 and Definition 1.4, we pose the following altering points problem associated to $GSMVLIP$ (3).

$$\begin{cases} S_1[R^{\partial_\eta \phi_1(g_1(\cdot),x)}_{\rho_1, M_1(\cdot,\cdot)}[M_1(A_1(y), B_1(y)) - N_1(u, v)]] = y, \\ S_2[R^{\partial_\eta \phi_2(g_2(\cdot),y)}_{\rho_2, M_2(\cdot,\cdot)}[M_2(A_2(x), B_2(x)) - N_2(v, u)]] = x, \end{cases} \tag{31}$$

where $S_1 : C \to D$ and $S_2 : D \to C$ be contraction mappings with Lipschitz constants κ_1 and κ_2, respectively.

Now, we propose following parallel S-iterative algorithm to compute the approximate solution of $GSMVLIP$ (3).

Algorithm 3.1. *For any $(x_0, y_0) \in C \times D$, compute the sequences $\{x_n, y_n\} \in C \times D$ generated by the following parallel S-iteration scheme.*

$$\begin{cases} x_{n+1} = S_2\Big[(1-\alpha_n)R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}\left[M_2(A_2(x_n),B_2(x_n)) - N_2(v_n,u_n)\right] \\ \qquad + \alpha_n S_1\left(R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}\left[M_1(A_1(y_n),B_1(y_n)) - N_1(u_n,v_n)\right]\right)\Big], \\ y_{n+1} = S_1\Big[(1-\beta_n)R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}\left[M_1(A_1(y_n),B_1(y_n)) - N_1(u_n,v_n)\right] \\ \qquad + \beta_n S_2\left(R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}\left[M_2(A_2(x_n),B_2(x_n)) - N_2(v_n,u_n)\right]\right)\Big], \end{cases} \tag{32}$$

and

$$v_n \in P(x_n), \ \ \|v_{n+1} - v_n\| \le \mathcal{D}(P(x_{n+1}), P(x_n));$$
$$u_n \in Q(y_n), \ \ \|u_{n+1} - u_n\| \le \mathcal{D}(Q(x_{n+1}), Q(x_n));$$

where $\{\alpha_n, \beta_n\}$ is a sequence in $(0,1) \times (0,1)$.

Now we establish strong convergence of the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by parallel S-iterative scheme.

Theorem 3.1. *Let C and D be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. For each $i \in \{1,2\}$; let $A_i, B_i, g_i : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Let $M_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be $\alpha_i\beta_i$-symmetric η_i-monotone mappings such that M_i be mixed Lipchitz continuous with constant t_i, mixed strongly monotone with constant k_i with respect to A_i and B_i. Let $N_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mapping such that $N_i(\cdot,\cdot)$ be γ_i and $\gamma_i^{'}$-Lipschitz continuous in first and second argument, respectively; $P, Q : \mathcal{H} \to CB(\mathcal{H})$ be the set-valued mappings such that $\overline{Dom(P)} \subseteq D$ and $\overline{Dom(Q)} \subseteq C$ and Q is $\mathcal{D}$-Lipchitz continuous with constant ξ_1, strongly monotone with respect to N_1 with constant c_1 and P is $\mathcal{D}$-Lipchitz continuous with constant ξ_2, strongly monotone with respect to N_2 with constant c_2. Let $S_1 : C \to \mathcal{H}$ and $S_2 : D \to \mathcal{H}$ be contraction mappings with Lipschitz constant κ_1 and κ_2, respectively. Let $\eta_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous mapping with constant τ_i such that $\eta_i(x_i, y_i) = -\eta_i(y_i, x_i), \ \forall x_i, y_i \in \mathcal{H}$, and for any $z \in \mathcal{H}$, the mapping $h_i(y_i, x_i) = \langle z_i - M_i(A_i x_i, B_i x_i), \eta_i(z_i, g_i(\cdot))\rangle$ be 0-$DQCV$ in z_i. Let $\phi_i : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be such that for each $x_i \in \mathcal{H}$, $\phi_1(g_1(\cdot), x)$ and $\phi_2(g_2(\cdot), y)$ be lower semicontinuous, η_i-subdifferentiable, proper functionals. suppose that there exist constants $\rho_i > 0, \mu_i > 0$ such that for each $z \in \mathcal{H}$*

$$\left\| R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}(z) - R^{\partial_\eta\phi_1(g_1(\cdot),x)}_{\rho_1,M_1(\cdot,\cdot)}(z) \right\| \le \mu_1\|x_n - x\|. \tag{33}$$

$$\left\| R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}(z) - R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}(z) \right\| \le \mu_2\|y_n - y\|. \tag{34}$$

In addition, the following condition is satisfied:

$$\begin{aligned} where, \quad & \Theta = \max\{\Theta_1, \Theta_2\} \in [0,1), \\ & \Theta_1 = L_2(\theta_3 + \theta_4)(\kappa_2 + \kappa_1\kappa_2) + (L_1\gamma_1^{'}\xi_2 + \mu_1)(\kappa_1 + \kappa_1\kappa_2), \\ & \Theta_2 = L_1(\theta_1 + \theta_2)(\kappa_1 + \kappa_1\kappa_2) + (L_2\gamma_2^{'}\zeta_2 + \mu_2)(\kappa_2 + \kappa_1\kappa_2), \\ & \theta_1 = \sqrt{t_1^2 - 2k_1 + 1}, \ \theta_2 = \sqrt{\gamma_1^2\xi_1^2 - 2c_1 + 1}, \\ & \theta_3 = \sqrt{t_2^2 - 2k_2 + 1}, \ \theta_4 = \sqrt{\gamma_2^2\zeta_1^2 - 2c_2 + 1}. \end{aligned} \tag{35}$$

Then the iterative sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ generated by parallel S-iterative algorithm 3.1 converges strongly to x, y, u and v, respectively.

Proof
By Algorithm 3.1 and using the Lipschitz continuity of contraction mappings S_1 and S_2, we have

$$\begin{aligned}
\|x_{n+1}-x\| &= \|S_2\Big[(1-\alpha_n)R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)] \\
&\quad +\alpha_n S_1\left(R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)]\right)\Big]-S_2(y)\| \\
&\leq \kappa_2\|(1-\alpha_n)R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)] \\
&\quad +\alpha_n S_1(R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)])-y\| \\
&\leq \kappa_2\Big[(1-\alpha_n)\|R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]-y\| \\
&\quad +\alpha_n\|S_1(R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)])-S_1(x)\|\Big] \\
&\leq \kappa_2\Big[(1-\alpha_n)\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]\| \\
&\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\| \\
&\quad +\alpha_n\kappa_1\|R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)] \\
&\quad -R^{\partial_\eta\phi_1(g_1(\cdot),x)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y),B_1(y))-N_1(u,v)]\|\Big] \\
&\leq \kappa_2(1-\alpha_n)\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)]\| \\
&\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\| \\
&\quad +\kappa_2\alpha_n\kappa_1\|R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)] \\
&\quad -R^{\partial_\eta\phi_1(g_1(\cdot),x)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y),B_1(y))-N_1(u,v)]\|.
\end{aligned} \tag{36}$$

Following the same arguments as in (14)–(22), we have

$$\begin{aligned}
\|x_{n+1}-x\| &\leq \kappa_2(1-\alpha_n)\left[L_2\left[(\theta_3+\theta_4)\|x_n-x\|+\gamma_2'\xi_1\|y_n-y\|\right]+\mu_2\|y_n-y\|\right] \\
&\quad +\kappa_2\alpha_n\kappa_1\left[L_1\left[(\theta_1+\theta_2)\|y_n-y\|+\gamma_1'\xi_2\|x_n-x\|\right]+\mu_1\|x_n-x\|\right].
\end{aligned} \tag{37}$$

Again using the Lipschitz continuity of contraction mappings S_1 and S_2 and by Algorithm 3.1, we have

$$\begin{aligned}
\|y_{n+1}-y\| &= \|S_1\Big[(1-\beta_n)R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)] \\
&\quad +\beta_n S_2(R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)])\Big]-S_1(x)\| \\
&\leq \kappa_1\|(1-\beta_n)R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)] \\
&\quad +\beta_n S_2(R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)])-x\| \\
&\leq \kappa_1\Big[(1-\beta_n)\|R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_{12},M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)]-x\| \\
&\quad +\beta_n\|S_2(R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)])-S_2(y)\|\Big] \\
&\leq \kappa_1\Big[(1-\beta_n)\|R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)]\| \\
&\quad -R^{\partial_\eta\phi_1(g_1(\cdot),y)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y),B_1(y))-N_1(u,v)]\| \\
&\quad +\beta_n\kappa_2\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)] \\
&\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\|\Big] \\
&\leq \kappa_1(1-\beta_n)\|R^{\partial_\eta\phi_1(g_1(\cdot),x_n)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y_n),B_1(y_n))-N_1(u_n,v_n)]\| \\
&\quad -R^{\partial_\eta\phi_1(g_1(\cdot),y)}_{\rho_1,M_1(\cdot,\cdot)}[M_1(A_1(y),B_1(y))-N_1(u,v)]\| \\
&\quad +\kappa_1\beta_n\kappa_2\|R^{\partial_\eta\phi_2(g_2(\cdot),y_n)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x_n),B_2(x_n))-N_2(v_n,u_n)] \\
&\quad -R^{\partial_\eta\phi_2(g_2(\cdot),y)}_{\rho_2,M_2(\cdot,\cdot)}[M_2(A_2(x),B_2(x))-N_2(v,u)]\|\Big].
\end{aligned} \tag{38}$$

Again following the same arguments as in (14)–(22), we have

$$\begin{aligned}\|y_{n+1}-y\| \leq\ & \kappa_1(1-\beta_n)\left[L_1\left[(\theta_1+\theta_2)\|y_n-y\|+\gamma_1^{'}\xi_2\|x_n-x\|\right]+\mu_1\|x_n-x\|\right]\\ & +\kappa_2\beta_n\kappa_1\left[L_2\left[(\theta_3+\theta_4)\|x_n-x\|+\gamma_2^{'}\xi_1\|y_n-y\|\right]+\mu_2\|y_n-y\|\right].\end{aligned} \tag{39}$$

Adding the inequalities (37) and (39), we have

$$\begin{aligned}\|x_{n+1}-x\|+\|y_{n+1}-y\| \leq\ & \kappa_2(1-\alpha_n)[L_2[(\theta_3+\theta_4)\|x_n-x\|\\ & +\gamma_2^{'}\xi_1\|y_n-y\|]+\mu_2\|y_n-y\|]+\kappa_2\alpha_n\kappa_1[L_1[(\theta_1+\theta_2)\|y_n-y\|]\\ & +\mu_1\|x_n-x\|+\kappa_1(1-\beta_n)[L_1[(\theta_1+\theta_2)\|y_n-y\|\\ & +\gamma_1^{'}\xi_2\|x_n-x\|]]+\mu_1\|x_n-x\|+\kappa_2\beta_n\kappa_1[L_2[(\theta_3+\theta_4)\|x_n-x\|\\ & +\gamma_2^{'}\xi_1\|y_n-y\|]+\mu_2\|y_n-y\|]\\ \leq\ & [L_2(\theta_3+\theta_4)(\kappa_2+\kappa_1\kappa_2)+(\gamma_1^{'}\xi_2L_1+\mu_1)(\kappa_1+\kappa_1\kappa_2)]\|x_n-x\|\\ & +[L_1(\theta_1+\theta_2)(\kappa_1+\kappa_1\kappa_2)+(\gamma_2^{'}\xi_1L_2+\mu_2)(\kappa_2+\kappa_1\kappa_2)]\|y_n-y\|\\ =\ & \Theta_1\|x_n-x\|+\Theta_2\|y_n-y\|\\ \leq\ & \Theta[\|x_n-x\|+\|y_n-y\|],\end{aligned} \tag{40}$$

where

$$\Theta=\max\{\Theta_1,\Theta_2\}$$

and

$$\begin{cases}\Theta_1=[L_2(\theta_3+\theta_4)(\kappa_2+\kappa_1\kappa_2)+(\gamma_1^{'}\xi_2L_1+\mu_1)(\kappa_1+\kappa_1\kappa_2)]\\ \Theta_2=[L_1(\theta_1+\theta_2)(\kappa_1+\kappa_1\kappa_2)+(\gamma_2^{'}\xi_1L_2+\mu_2)(\kappa_2+\kappa_1\kappa_2)].\end{cases} \tag{41}$$

It follows from (1) and (40) that

$$\|(x_{n+1},y_{n+1})-(x,y)\|_*\leq\Theta\|(x_n,y_n)-(x,y)\|_*.$$

Since $\Theta\in(0,1)$ by condition (35), then from Lemma 1.4, we have

$$\lim_{n\to\infty}\|(x_n,y_n)-(x,y)\|_*=0.$$

Thus, we have

$$\lim_{n\to\infty}\|x_n-x\|=\lim_{n\to\infty}\|y_n-y\|=0.$$

Therefore, $\{x_n\}$ and $\{y_n\}$ converges to x and y, respectively. Since the mappings P and Q are $\mathcal{D}$-Lipschitz continuous and using Algorithm 3.1, it follows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $\mathcal{H}$ such that $u_n\to u$ and $v_n\to v$ as $n\to\infty$. It can be proved by using the same techniques as in Theorem 2.2 that $u\in P(x)$ and $v\in Q(y)$. Note that $S_2S_1:C\to C$ is a contraction mapping with Lipschitz constant $\kappa_1\kappa_2$. In view of Theorem 2.1 and (31), we conclude that (x,y,u,v) such that $x,y\in\mathcal{H},u\in P(x)$ and $v\in Q(y)$ is a unique solution of $GSMVLIP$ (3), where x and y are altering points of mappings S_1 and S_2. This completes the proof. □

4. Concluding Remarks

In this paper, a generalized system of variational-like inclusion problem for $\alpha\beta$-symmetric η-monotone mapping is considered. We apply Mann type iterative algorithm to analyze the common solution of a generalized system of variational-like inclusion problem for $\alpha\beta$-symmetric η-monotone mapping and a fixed point problem of nonlinear Lipschitz mapping in Hilbert spaces. Further, we considered altering point problem and proposed a parallel S-iterative algorithm. We prove the existence of solution of our system by using altering point problem. The result presented in this paper may be viewed as generalizations and refinements of the results existing in the literature.

Competing Interests

The authors declare that there is no competing interests regarding the publication of this article.

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