STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **10**, March 2022, pp 439–456.
Published online in International Academic Press (www.IAPress.org)

Three-step Iterative Algorithm with Error Terms of Convergence and Stability Analysis for New NOMVIP in Ordered Banach Spaces

Iqbal Ahmad*

Department of Mechanical Engineering, College of Engineering, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia

Abstract This article undertakes to study a NOMVIP involving XNOR-operation and solved by employing a proposed three-step iterative algorithm in ordered Banach Space. Under suitable conditions, we obtain the strong convergence and existence results of NOMVIP involving XNOR-operation by applying the resolvent operator technique with XNOR and XOR operations and discuss the stability of the proposed algorithm. Finally, we provide a numerical example to confirm the convergence of the suggested algorithm in support of our considered problem which gives the grantee that all the proposed conditions of our main result have been formulated by using MATLAB programming.

Keywords Comparison, Convergence, Ordered Inclusion, XOR Operation, XNOR-operation, Stability.

AMS 2010 subject classifications 47H09, 49J40

DOI:10.19139/soic-2310-5070-990

1. Introduction

A wide class of inclusion problems has been investigated to find the zeros of the monotone operator G from $\mathbb{R}^n$ to itself that is find $\wp \in \mathbb{R}^n$ such that $0 \in G(\wp)$. Many problems in management sciences, economics, operations research, physics, and applied sciences can be formulated as an inclusion problem $0 \in G(\wp)$, for a given multi-valued mapping G in Hilbert spaces. In the 19th century, Martinet [19] and Rockafellar [24] proposed the proximal point iterative scheme for the monotone inclusion problem. The resolvent operator elegant methods introduced to prove the existence of a solution and some iterative procedures developed for several types of variational inclusions and their generalizations which provided us a powerful and novel framework for the study abroad class of nonlinear problems arising in optimization, convex programming problems, tomography, molecular biology, image restoring processing in applied and pure sciences (see, [1, 6, 7, 8, 9, 10, 11, 16, 21, 22, 23, 24, 25, 26, 27, 28, 29]).

After that, the problem of the inclusion involving multi-valued mappings which were introduced and studied by Moudafi et al. [20], Osilike [23], Bella [6], Jeong [12], Shelmas [26], Tan [28], Ahmad et al. [5], and Huang [11], is an important and useful extension of mathematical analysis and developed the various kinds of efficient and implementable iterative schemes to solve several types of variational inclusions problems in the literature, and among these authors used projection operator, the resolvent operator and proximal operator techniques to solve interesting and important variational inclusions problems. Recently, Glowinski et al. [10] and Noor [21, 22] investigated the convergence of the sequences generated by three-step predictor-corrector iterative schemes to solve several types of mixed variational inequalities and inclusion problem by applying the auxiliary principle and the Lagrangian multiplier techniques, and Glowinski et al. [10] proved that the three-step iterative schemes have more

*Correspondence to: Iqbal Ahmad (Email: iqbal@qec.edu.sa & i.ahmad@qu.edu.sa). Department of Mechanical, College of Engineering, Qassim University, Buraidah 51452, Al-Qassim, Saudi Arabia.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

reliable numerical results than the one-step and two-step iterative schemes to solve several problems, which aries in pure and applied sciences.

Very Recently, Li and his co-authors, focussed on the work done related to ordered variational inclusions involving multi-valued mappings with XOR-operation. The XOR-operation and XNOR-operations are the binary operations and behaves like ADD operations, which are insertable to each other, and have real-time applications in neural networks and digital communication systems. Several problems concerning in ordered variational inclusions have been solved by using the different kinds of multi-valued mappings to find the solutions of nonlinear ordered equations (inclusions) with or without XOR-operation and obtained their solutions in different settings (see [2, 3]).

Inspired and motivated by the above-described research, the aim of this work is proposed as follows. In section 2, contains certain basic results needed in this paper. In Section 3, we consider a new NOMVIP in real ordered Banach space and prove the existence of a unique solution. In section 4, we use the XOR and XNOR-operations technique and propose the three-step iterative schemes are better than the previously developed iterative schemes investigated by many authors (see, [2, 3, 13, 14, 15, 17, 18]). Moreover, we investigate the stability and convergence criteria of the proposed iterative schemes. In the last section, we demonstrate an example that ensures all the assumptions of our consider problem are fulfilled and show the convergence of the supposed iterative schemes by applying MATLAB programming.

2. Preliminaries

Throughout this article, $\mathcal{B}$ expresses a real ordered Banach space whose norm $\|.\|$ and inner product $\langle ., . \rangle$, respectively. Let K be a normal cone with normal constant δ_K, and $\leq$ is a partial ordering defined by for any $\wp, \widehat{\wp} \in E$, $\wp \leq \widehat{\wp}$ if and only if $\widehat{\wp} - \wp \in K$. For arbitrary elements $\wp, \widehat{\wp} \in \mathcal{B}$, $lub\{\wp, \widehat{\wp}\}$ is denoted by least upper bound of the set $\{\wp, \widehat{\wp}\}$ and $glb\{\wp, \widehat{\wp}\}$ is denoted by greatest lower bound of the set $\{\wp, \widehat{\wp}\}$, respectively. Let $glb\{\wp, \widehat{\wp}\}$ and $lub\{\wp, \widehat{\wp}\}$ exist, binary operations $\vee, \wedge$, $\oplus$ and $\odot$ which called as AND, OR, XNOR and XOR operations, respectively are defined as follows:

(i) $\wp \vee \widehat{\wp} = \sup\{\wp, \widehat{\wp}\}$;
(ii) $\wp \wedge \widehat{\wp} = \inf\{\wp, \widehat{\wp}\}$;
(iii) $\wp \oplus \widehat{\wp} = (\wp - \widehat{\wp}) \vee (\widehat{\wp} - \wp)$;
(iv) $\wp \odot \widehat{\wp} = (\wp - \widehat{\wp}) \wedge (\widehat{\wp} - \wp)$.

Definition 1 ([9, 14, 25])
A non-empty subset K of $\mathcal{B}$. Then

(i) K said to be a normal cone if and only if there exists a constant $\delta_K > 0$ such that for $0 \leq \wp \leq \widehat{\wp}$, we have $||\wp|| \leq \delta_K ||\widehat{\wp}||$, for any $\wp, \widehat{\wp} \in \mathcal{B}$;
(ii) For each $\wp, \widehat{\wp} \in \mathcal{B}$ if either $\wp \leq \widehat{\wp}$ or $\widehat{\wp} \leq \wp$ hold, then $\wp$ and $\widehat{\wp}$ are said to be comparable to each other (denoted by $\wp \propto \widehat{\wp}$).

Definition 2 ([9, 14])
A comparison mapping $S : \mathcal{B} \to \mathcal{B}$ is called

(i) a strongly comparison mapping, $\wp \propto \widehat{\wp}$ if and only if $S(\wp) \propto S(\widehat{\wp})$, for all $\wp, \widehat{\wp} \in \mathcal{B}$;
(ii) a δ_S-ordered compression mapping, if S is a comparison mapping and there exists $0 < \lambda_S < 1$ such that

$$S(\wp) \oplus S(\widehat{\wp}) \leq \lambda_S(\wp \oplus \widehat{\wp}), \text{ for all } \wp, \widehat{\wp} \in \mathcal{B}.$$

(iii) a ν-ordered non-extended mapping, if there exists $\nu > 0$ such that

$$S(\wp) \oplus S(\widehat{\wp}) \geq \nu(\wp \oplus \widehat{\wp}), \text{ for all } \wp, \widehat{\wp} \in \mathcal{B}.$$

Definition 3 ([14])
A single-valued bi-mapping $F : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is said to be (κ, μ)-ordered Lipschitz continuous, if $\wp \propto \widehat{\wp}$ and $q \propto \widehat{q}$, then $F(\wp, q) \propto F(\widehat{\wp}, \widehat{q})$ and there exist constants $\kappa, \mu > 0$ such that

$$F(\wp, q) \oplus F(\widehat{\wp}, \widehat{q}) \leq \kappa(\wp \oplus \widehat{\wp}) + \mu(q \oplus \widehat{q}),\ for\ all\ \wp, \widehat{\wp}, q, \widehat{q} \in \mathcal{B}.$$

Definition 4 ([14])
Let $G : \mathcal{B} \to \mathcal{B}$ be a single-valued mapping and $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be a multi-valued mapping. Then M is said to be a

(i) weak-comparison mapping, if for any $\vartheta_\wp \in M(\wp)$, $\wp \propto \vartheta_\wp$, and if $\wp \propto \widehat{\wp}$, then there exist $\vartheta_\wp \in M(\wp)$ and $\vartheta_{\widehat{\wp}} \in M(\widehat{\wp})$, $\vartheta_\wp \propto \vartheta_{\widehat{\wp}}$, for all $\wp, \widehat{\wp} \in \mathcal{B}$;

(ii) α_G-weak-non-ordinary difference mapping with respect to G, if for each $\wp, \widehat{\wp} \in \mathcal{B}$, there exist a constant α_G and $\vartheta_\wp \in M(G(\wp))$ and $\vartheta_{\widehat{\wp}} \in M(G(\widehat{\wp}))$ such that

$$(\vartheta_\wp \oplus \vartheta_{\widehat{\wp}}) \oplus \alpha_G(G(\wp) \oplus G(\widehat{\wp})) = 0;$$

(iii) λ-ordered different weak-comparison mapping with respect to G, if there exists a constant $\lambda > 0$ and $\vartheta_\wp \in M(G(\wp))$ and $\vartheta_{\widehat{\wp}} \in M(G(\widehat{\wp}))$ such that

$$\lambda(\vartheta_\wp - \vartheta_{\widehat{\wp}}) \propto \wp - \widehat{\wp}, \ \text{ for all } \wp, \widehat{\wp} \in \mathcal{B};$$

(iv) (α_G, λ)-weak-GNODD mapping with respect to G, if M is a α_G-weak-non-ordinary difference mapping with respect to G and a λ-ordered different weak-comparison mapping with respect to G and $[G + \lambda M](\mathcal{B}) = \mathcal{B}$ for $\alpha_G, \lambda > 0$.

Definition 5
Let $G : \mathcal{B} \to \mathcal{B}$ be a γ-ordered non-extended mapping. Let $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping with respect to G. The resolvent operator $\mathcal{R}^G_{\lambda,M} : \mathcal{B} \to \mathcal{B}$ associated with G and M is defined by

$$\mathcal{R}^G_{\lambda,M}(\wp) = [G + \lambda M]^{-1}(\wp), \ \text{ for all } \wp \in \mathcal{B}, \tag{1}$$

where $\lambda > 0$ is a constant.

Definition 6 ([27])
Let $R : \mathcal{B} \to \mathcal{B}$ be a single-valued mapping, $\wp_0 \in \mathcal{B}$ and let

$$\wp_{m+1} = S(R, \wp_m)$$

defines an iterative sequence which yields a sequence of points $\{\wp_m\}$ in $\mathcal{B}$. Suppose that $F(R) = \{\wp \in \mathcal{B} : R(\wp) = \wp\} \neq \emptyset$ and $\{\wp_m\}$ converges to a fixed point $\wp^*$ of R. Let $\{q_m\} \subset \mathcal{B}$ and

$$\varsigma_m = \|q_{m+1} - S(R, \wp_m)\|.$$

If $\lim_{m\to\infty} \varsigma_m = 0$, which implies that $q_m \to \wp^*$, then the iterative sequence $\{\wp_m\}$ is said to be stable with respect to R or R-stable.

Lemma 1 ([29])
Let $\{a_m\}$ and $\{b_m\}$ be the sequences such that $a_m \geq 0$ and $0 \leq b_m \leq 1$, for all $m \geq 0$, and $\sum_{m=0}^{\infty} b_m = \infty$. If there exists a number $p \geq 0$ such that

$$a_{m+1} \leq (1 - b_m)a_m + b_m c_m,\ \forall m \geq p,$$

where $c_m \geq 0$, for all $m \geq 0$ and $c_m \to 0$ $(m \to 0)$, then $\lim_{m\to\infty} a_m = 0$.

Lemma 2 ([9, 13, 14, 15])
Let $\oplus$ and $\odot$ be the XOR and XNOR-operations, respectively. Then the following properties satisfied:

(i) $\wp \odot \wp = 0$, $\wp \odot \widehat{\wp} = \widehat{\wp} \odot \wp = -(\wp \oplus \widehat{\wp}) = -(\widehat{\wp} \oplus \wp)$;
(ii) $(\ell\wp) \oplus (\ell\widehat{\wp}) = |\ell|(\wp \oplus \widehat{\wp})$;
(iii) if $\wp \propto \widehat{\wp}$, then $\wp \oplus \widehat{\wp} = 0$ if and only if $\wp = \widehat{\wp}$;
(iv) $(\wp + \widehat{\wp}) \odot (q + \widehat{q}) \geq (\wp \odot q) + (\widehat{\wp} \odot \widehat{q})$;
(v) if $\wp, \widehat{\wp}$ and s are comparative to each other, then $(\wp \oplus \widehat{\wp}) \leq \wp \oplus s + s \oplus \widehat{\wp}$;
(vi) if $\wp \propto \widehat{\wp}$, then $((\wp \oplus 0) \oplus (\widehat{\wp} \oplus 0)) \leq (\wp \oplus \widehat{\wp}) \oplus 0 = \wp \oplus \widehat{\wp}$;
(vii) $(\imath\wp) \oplus (\jmath\wp) = |\imath - \jmath|\wp = (\imath \oplus \jmath)\wp$;
$(viii)$ $\|\wp \oplus \widehat{\wp}\| \leq \|\wp - \widehat{\wp}\| \leq \delta_K \|\wp \oplus \widehat{\wp}\|$;
(ix) if $\wp \propto \widehat{\wp}$, then $\|\wp \oplus \widehat{\wp}\| = \|\wp - \widehat{\wp}\|$, for all $\wp, \widehat{\wp}, q, \widehat{q}, s \in \mathcal{B}$ and $\imath, \jmath, \ell \in \mathbb{R}$.

Lemma 3 ([14])
Let $G : \mathcal{B} \to \mathcal{B}$ be an γ-ordered non-extended mapping and $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping with respect to $\mathcal{R}^G_{\lambda,M}$. The resolvent operator $\mathcal{R}^G_{\lambda,M} : \mathcal{B} \to \mathcal{B}$ associated with G and M is well-defined, single valued, comparison and $\frac{1}{\gamma(\lambda\alpha_G - 1)}$-ordered Lipschitz continuous mapping with $\lambda\alpha_G > 1$, that is

$$\mathcal{R}^G_{\lambda,M}(\wp) \oplus \mathcal{R}^G_{\lambda,M}(\widehat{\wp}) \leq \frac{1}{\gamma(\lambda\alpha_G - 1)}(\wp \oplus \widehat{\wp}), \text{ for all } \wp, \widehat{\wp} \in \mathcal{B}. \tag{2}$$

3. Formulation of NOMVIP and existence result

Let $G, A, g, h : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued comparison mappings. Let $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping. We consider the following nonlinear ordered mixed variational inclusion problem involving XNOR-operation (in short, NOMVIP):

For any $\omega, \xi \in \mathbb{R}$ and some $\tau > 0$, find $\wp \in E$ such that

$$0 \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)). \tag{3}$$

Now, we first transfer our considered NOMVIP (3) into a fixed point formulation.

Lemma 4
Let $h, g, A, G : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued comparison mappings, and $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping with respect to g. Then the followings are equivalent:

(i) a NOMVIP (3) has a solution $\wp \in \mathcal{B}$;
(ii) a mapping $Q : \mathcal{B} \rightrightarrows \mathcal{B}$ defined by

$$Q(\wp) = \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)) + \wp,$$

has a the fixed point $\wp \in \mathcal{B}$;
(iii) the following equation

$$g(\wp) \odot \mathcal{R}^G_{\lambda,M}\big[(Gog)(\wp) + \frac{\lambda}{\tau}\big((\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)) \odot \omega\eta(\wp, h(\wp))\big)\big] = 0. \tag{4}$$

has a solution $\wp \in \mathcal{B}$.

Proof
$(i) \implies (ii)$ Adding $\wp$ to both sides of (3), we have

$$\begin{aligned} & 0 \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)) \\ \implies \quad & \wp \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)) + \wp = Q(\wp). \end{aligned}$$

Hence, $\wp$ is a fixed point of Q.

$(ii) \implies (iii)$ Let $\wp$ be a fixed point of Q, then

$$\begin{aligned}
&\wp \in Q(\wp) = \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)) + \wp \\
\implies\ & 0 \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)) \\
\implies\ & \omega\eta(\wp, h(\wp)) \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) \\
\implies\ & \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big) \in \lambda M(g(\wp)) \\
\implies\ & (Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big) \in (Gog)(\wp) + \lambda M(g(\wp)) \\
\implies\ & (Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big) \in G(g(\wp)) + \lambda M(g(\wp)) \\
\implies\ & (Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big) \in [G + \lambda M](g(\wp)),
\end{aligned}$$

that is

$$g(\wp) = \mathcal{R}^G_{\lambda,M}\Big[(Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big)\Big],$$

implies that $g(\wp) \odot \mathcal{R}^G_{\lambda,M}\Big[(Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big)\Big] = 0$.

Consequently, $\wp$ is a solution of the NOMVIP (3).

$(iii) \implies (i)$, from (4) we have

$$\begin{aligned}
g(\wp) \odot \mathcal{R}^G_{\lambda,M}\Big[(Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big)\Big] &= 0 \\
g(\wp) &= \mathcal{R}^G_{\lambda,M}\Big[(Gog)(\wp) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big)\Big] \\
g(\wp) &= [G + \lambda M]^{-1}\big[G(g(\wp)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\big)\big],
\end{aligned}$$

so

$$\begin{aligned}
G(g(\wp)) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big) &\in G(g(\wp)) + \lambda M(g(\wp))), \\
\omega\eta(\wp, h(\wp)) &\in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)),
\end{aligned}$$

which implies

$$0 \in \Big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\Big) \odot \tau M(g(\wp)) - \omega\eta(\wp, h(\wp)).$$

Therefore, $\wp \in \mathcal{B}$ is a solution of NOMVIP (3). □

Now, we are equipped to prove the following existence result for NOMVIP (3).

Theorem 1

Let $h, g, A, G : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued comparison mappings such that h is δ_h-ordered compression mapping, g is δ_g-ordered compression mapping, A is δ_A-ordered compression mapping, G is δ_G-ordered compression mapping with respect to g and γ_G-ordered non-extended mapping, and η is (κ, ν)-ordered Lipschitz continuous mapping with respect to g, respectively. Let $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping with respect to g.

In addition, if G, A, g, h, η, M and $\mathcal{R}^G_{\lambda,M}$ are compared to each other, the following conditions are satisfied:

$$\begin{cases} \lambda\Big(\Big(\frac{|\xi|}{\gamma(\lambda\alpha_G-1)}+\delta_A\Big)\oplus\Big(|\omega|(\kappa+\nu\delta_h)\Big)\Big)<\tau\delta_g(\gamma\lambda\alpha_G-(\gamma+\delta_G))\min\{\frac{1}{\delta_K},1\}, \\ \alpha_G>\frac{1}{\lambda}>0, \end{cases} \tag{5}$$

then, NOMVIP (3) admits a unique solution $\wp^*\in\mathcal{B}$, which is a fixed point of the mapping $F(\wp^*)=\wp^*-g(\wp^*)\oplus\mathcal{R}^G_{\lambda,M}\Big[G(g(\wp^*))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*)-A(\wp^*)\big)\odot\omega\eta(\wp^*,h(\wp^*))\Big)\Big]$.

Proof
Using Lemma 2 and Lemma 3, we have

$$\begin{aligned}
0 &\leq F(\wp)\oplus F(\widehat{\wp})\\
&\leq \Big[\wp-g(\wp)\oplus\mathcal{R}^G_{\lambda,M}\Big(G(g(\wp))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\odot\omega\eta(\wp,h(\wp))\Big)\Big)\Big]\\
&\quad\oplus\Big[\widehat{\wp}-g(\widehat{\wp})\oplus\mathcal{R}^G_{\lambda,M}\Big(G(g(\widehat{\wp}))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\odot\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})+\Big[g(\wp)\oplus\mathcal{R}^G_{\lambda,M}\Big(G(g(\wp))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\odot\omega\eta(\wp,h(\wp))\Big)\Big)\Big]\\
&\quad\oplus\Big[g(\widehat{\wp})+\mathcal{R}^G_{\lambda,M}\Big(G(g(\widehat{\wp}))\oplus\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\odot\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})+(g(\wp)\oplus g(\widehat{\wp}))\oplus\Big[\mathcal{R}^G_{\lambda,M}\Big(G(g(\wp))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\\
&\quad\odot\omega\eta(\wp,h(\wp))\Big)\Big)\oplus\mathcal{R}^G_{\lambda,M}\Big(G(g(\widehat{\wp}))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\odot\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})+\delta_g(\wp\oplus\widehat{\wp})\oplus\Upsilon\Big[\Big(G(g(\wp))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\odot\omega\eta(\wp,h(\wp))\Big)\Big)\\
&\quad\oplus\Big(G(g(\wp))+\frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\odot\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})+\delta_g(\wp\oplus\widehat{\wp})\oplus\Upsilon\Big[G(g(\wp))\oplus G(g(\widehat{\wp}))+\frac{\lambda}{\tau}\Big(\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\\
&\quad\odot\omega\eta(\wp,h(\wp))\Big)\oplus\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\odot\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})-\delta_g(\wp\oplus\widehat{\wp})\odot\Upsilon\Big[\delta_G\delta_g(\wp\oplus\widehat{\wp})+\frac{\lambda}{\tau}\Big(-\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\\
&\quad\oplus\omega\eta(\wp,h(\wp))\Big)\oplus\Big(-\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\oplus\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})-\delta_g(\wp\oplus\widehat{\wp})\odot\Upsilon\Big[\delta_G\delta_g(\wp\oplus\widehat{\wp})+\frac{\lambda}{\tau}\Big(\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\\
&\quad\oplus\omega\eta(\wp,h(\wp))\Big)\oplus\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\oplus\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})-\delta_g(\wp\oplus\widehat{\wp})\odot\Upsilon\Big[\delta_G\delta_g(\wp\oplus\widehat{\wp})+\frac{\lambda}{\tau}\Big(\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp)-A(\wp)\big)\\
&\quad\oplus\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp})-A(\widehat{\wp})\big)\Big)\oplus\Big(\omega\eta(\wp,h(\wp))\oplus\omega\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]\\
&\leq (\wp\oplus\widehat{\wp})-\delta_g(\wp\oplus\widehat{\wp})\odot\Upsilon\Big[\delta_G\delta_g(\wp\oplus\widehat{\wp})+\frac{\lambda}{\tau}\Big(|\xi|\Big(\mathcal{R}^G_{\lambda,M}(\wp)\oplus\mathcal{R}^G_{\lambda,M}(\widehat{\wp})\Big)\\
&\quad+A(\wp)\oplus A(\widehat{\wp})\Big)\oplus|\omega|\Big(\eta(\wp,h(\wp))\oplus\eta(\widehat{\wp},h(\widehat{\wp}))\Big)\Big)\Big]
\end{aligned}$$

$$\leq \quad (\wp \oplus \widehat{\wp}) - \delta_g(\wp \oplus \widehat{\wp}) \odot \Upsilon\Big[\delta_G\delta_g(\wp \oplus \widehat{\wp}) + \frac{\lambda}{\tau}\Big(\big(|\xi|\Upsilon + \delta_A\big)$$
$$\oplus\big(|\omega|(\kappa + \nu\delta_h)\big)\Big)(\wp \oplus \widehat{\wp})\Big]$$
$$\leq \quad \Big[1 - \delta_g \odot \Upsilon\Big(\delta_G\delta_g + \frac{\lambda}{\tau}\Big(\big(|\xi|\Upsilon + \delta_A\big) \oplus \big(|\omega|(\kappa + \nu\delta_h)\big)\Big)\Big)\Big](\wp \oplus \widehat{\wp}),$$

where

$$\Upsilon = \frac{1}{\gamma(\alpha_G\lambda - 1)}.$$

which implies that

$$0 \quad \leq \quad F(\wp) \oplus F(\widehat{\wp}) \leq \psi(\wp \oplus \widehat{\wp}),$$

where

$$\psi = \Big[1 - \delta_g \odot \Upsilon\Big(\delta_G\delta_g + \frac{\lambda}{\tau}\Big(\big(|\xi|\Upsilon + \delta_A\big) \oplus \big(|\omega|(\kappa + \nu\delta_h)\big)\Big)\Big)\Big].$$

By Definition 1 (i) and Lemma 2, we conclude that

$$\|F(\wp) - F(\widehat{\wp})\| \leq |\psi|\delta_K\|\wp - \widehat{\wp}\|. \tag{6}$$

Using the condition (5), we can see that $\delta_K|\psi| < 1$. It follows from (6) that the resolvent operator $F(.)$ is contraction mapping. Hence, there exists a unique $\wp^* \in \mathcal{B}$ such that

$$\wp^* = \wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\Big[G(g(\wp^*)) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\Big)\Big],$$

which implies that

$$g(\wp^*) \odot \mathcal{R}^G_{\lambda,M}\Big[G(g(\wp^*)) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\Big)\Big] = 0.$$

From Lemma 4, $\wp^*$ is a unique solution of NOMVIP (3). □

4. Stability and Convergence Analysis

In this section, we first describe the three-step iterative schemes based on Lemma 4 for approximating a solution of NOMVIP (3) and reveal the convergence and stability analysis of the suggested iterative schemes.

Algorithm 1
Let $h, g, A, G : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued comparison mappings. Let $M : \mathcal{B} \rightrightarrows \mathcal{B}$ be an ordered (α_G, λ)-weak-GNODD multi-valued mapping. Given any initial point $\wp_0 \in \mathcal{B}$, assume that $\wp_1 \propto \wp_0$. We define the sequence $\{\wp_n\}$ and let $\wp_{n+1} \propto \wp_n$ such that

$$\left.\begin{aligned}
\wp_{m+1} &= (1-\varrho_m)\wp_m + \varrho_m\Big(\widehat{\wp}_m - g(\widehat{\wp}_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\widehat{\wp}_m)) + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp}_m) \\
&\quad -A(\widehat{\wp}_m)\big) \odot \omega\eta(\widehat{\wp}_m, h(\widehat{\wp}_m))\big)\big)\Big) + \varrho_m d_m \\
\widehat{\wp}_m &= (1-\sigma_m)\wp_m + \sigma_m\Big(\widehat{q}_m - g(\widehat{q}_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\widehat{q}_m)) + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{q}_m) \\
&\quad -A(\widehat{q}_m)\big) \odot \omega\eta(\widehat{q}_m, h(\widehat{q}_m))\big)\big)\Big) + \sigma_m e_m \\
\widehat{q}_m &= (1-\omega_m)\wp_m + \omega_m\Big(\wp_m - g(\wp_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp_m)) + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp_m) \\
&\quad -A(\wp_m)\big) \odot \omega\eta(\wp_m, h(\wp_m))\big)\big)\Big) + \omega_m f_m \\
&.
\end{aligned}\right\} \tag{7}$$

Suppose $\{z_m\}$ is an arbitrary sequence in $\mathcal{B}$ and the sequence $\{\varphi_m\}$ is define by

$$\left.\begin{aligned}
\varphi_m &= \Big\| z_{m+1} - \Big[(1-\varrho_m)z_m + \varrho_m\Big(a_m - g(a_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(a_m)) \\
&\qquad + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(a_m) - A(a_m)\big) \odot \omega\eta(a_m, h(a_m))\big)\big)\Big) + \varrho_m d_m\Big]\Big\| \\
a_m &= (1-\sigma_m)z_m + \sigma_m\Big(b_m - g(b_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(b_m)) + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(b_m) \\
&\qquad - A(b_m)\big) \odot \omega\eta(b_m, h(b_m))\big)\big)\Big) + \sigma_m e_m \\
b_m &= (1-\omega_m)z_m + \omega_m\Big(z_m - g(z_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(z_m)) + \tfrac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(z_m) \\
&\qquad - A(z_m)\big) \odot \omega\eta(z_m, h(z_m))\big)\big)\Big) + \omega_m f_m
\end{aligned}\right\} \tag{8}$$

where $0 \le \varrho_m, \sigma_m, \omega_m \le 1, \forall\, m \ge 0$ and $\sum_{m=0}^{\infty} \varrho_m$ diverge. Let $\{d_m\}$, $\{e_m\}$ and $\{f_m\}$ be the sequences in $\mathcal{B}$ to take into account the possible inexact computation provided that $d_m \oplus 0 = d_m$, $e_m \oplus 0 = e_m$ and $f_m \oplus 0 = f_m$, $\forall m \ge 0$.

Theorem 2
Suppose all the mappings h, g, A, G, η and M are similar as in Theorem 1 such that all the hypotheses of Theorem 1 are satisfied. Besides, admit that the following assumptions hold:

$$\begin{cases} \lambda\Big(\Big(\frac{|\xi|}{\gamma(\lambda\alpha_G - 1)} + \delta_A\Big) \oplus \big(|\omega|(\kappa + \nu\delta_h)\big)\Big) < \tau\delta_g(\gamma\lambda\alpha_G - (\gamma + \delta_G))\min\{\frac{1}{\delta_K}, 1\}, \\ \alpha_G > \frac{1}{\lambda} > 0, \end{cases} \tag{9}$$

If $\lim_{m\to\infty}\|d_m \vee (-d_m)\| = \lim_{m\to\infty}\|e_m \vee (-e_m)\| = \lim_{m\to\infty}\|f_m \vee (-f_m)\| = 0$, then

(I) The sequence $\{\wp_m\}$ generated by the suggested Algorithm 1 converges strongly to the unique solution $\wp^*$ of NOMVIP (3).

(II) Furthermore, if $0 < \varepsilon < \varrho_m$, then $\lim_{m\to\infty} z_m = \wp^*$ if and only if $\lim_{m\to\infty}\varphi_m = 0$, where φ_m is given in (8) i.e., the sequence $\{\wp_m\}$ generated by (7) is $\mathcal{R}^G_{\lambda,M}$-stable.

Proof
(I). Suppose $\wp^*$ is a unique solution of NOMVIP (3). Then, we have

$$\begin{aligned}
\wp^* &= (1-\varrho_m)\wp^* + \varrho_m\Big(\wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) \\
&\quad - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big) \\
&= (1-\sigma_m)\wp^* + \sigma_m\Big(\wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) \\
&\quad - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big) \\
&= (1-\omega_m)\wp^* + \omega_m\Big(\wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) \\
&\quad - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big).
\end{aligned} \tag{10}$$

Using Algorithm 1, Lemma 2, Lemma 3, and (10) it follows that

$$\begin{aligned}
0 &\leq \wp_{m+1} \oplus \wp^* \\
&= \Big[(1-\varrho_m)\wp_m + \varrho_m\Big(\widehat{\wp}_m - g(\widehat{\wp}_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\widehat{\wp}_m)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp}_m) \\
&\quad -A(\widehat{\wp}_m)\big) \odot \omega\eta(\widehat{\wp}_m, h(\widehat{\wp}_m))\big)\big)\Big) + \varrho_m d_m\Big] \oplus \Big[(1-\varrho_m)\wp^* + \varrho_m\Big(\wp^* - g(\wp^*) \\
&\quad \oplus\mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big)\Big] \\
&\leq (1-\varrho_m)(\wp_m \oplus \wp^*) + \varrho_m\Big[\Big(\widehat{\wp}_m - g(\widehat{\wp}_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\widehat{\wp}_m)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{\wp}_m) \\
&\quad -A(\widehat{\wp}_m)\big) \odot \omega\eta(\widehat{\wp}_m, h(\widehat{\wp}_m))\big)\big)\Big) \oplus \Big(\wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) \\
&\quad +\frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big)\Big] + \varrho_m(d_m \oplus 0) \\
&\leq (1-\varrho_m)(\wp_m \oplus \wp^*) + \varrho_m\psi(\widehat{\wp}_m \oplus \wp^*) + \varrho_m(d_m \oplus 0), \qquad (11)
\end{aligned}$$

where

$$\psi = \Big[1 - \delta_g \odot \frac{1}{\gamma(\lambda\alpha_G - 1)}\Big(\delta_G\delta_g + \frac{\lambda}{\tau}\Big(\Big(\frac{|\xi|}{\gamma(\lambda\alpha_G - 1)} + \delta_A\Big) \oplus \big(|\omega|(\kappa + \nu\delta_h)\big)\Big)\Big)\Big].$$

Using the same argument as for (11), we calculate

$$\begin{aligned}
0 &\leq \widehat{\wp}_m \oplus \wp^* \\
&= \Big[(1-\sigma_m)\wp_m + \sigma_m\Big(\widehat{q}_m - g(\widehat{q}_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\widehat{q}_m)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\widehat{q}_m) \\
&\quad -A(\widehat{q}_m)\big) \odot \omega\eta(\widehat{q}_m, h(\widehat{q}_m))\big)\big)\Big) + \sigma_m e_m\Big] \oplus \Big[(1-\sigma_m)\wp^* + \sigma_m\Big(\wp^* - g(\wp^*) \\
&\quad \oplus\mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big)\Big] \\
&\leq (1-\sigma_m)(\wp_m \oplus \wp^*) + \sigma_m\psi(\widehat{q}_m \oplus \wp^*) + \sigma_m(e_m \oplus 0). \qquad (12)
\end{aligned}$$

Using the same argument as for (11), we calculate

$$\begin{aligned}
0 &\leq \widehat{q}_m \oplus \wp^* \\
&= \Big[(1-\omega_m)\wp_m + \omega_m\Big(\wp_m - g(\wp_m) \oplus \mathcal{R}^G_{\lambda,M}\big(G(g(\wp_m)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp_m) \\
&\quad -A(\wp_m)\big) \odot \omega\eta(\wp_m, h(\wp_m))\big)\big)\Big) + \sigma_m f_m\Big] \oplus \Big[(1-\omega_m)\wp^* + \omega_m\Big(\wp^* - g(\wp^*) \\
&\quad \oplus\mathcal{R}^G_{\lambda,M}\big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\big)\Big)\Big] \\
&\leq (1-\omega_m)(\wp_m \oplus \wp^*) + \omega_m\psi(\wp_m \oplus \wp^*) + \sigma_m(f_m \oplus 0) \\
&\leq (1-\omega_m(1-\psi))(\wp_m \oplus \wp^*) + \omega_m(f_m \oplus 0) \\
&\leq (\wp_m \oplus \wp^*) + \omega_m(f_m \oplus 0), \quad \text{since } (1-\omega_m(1-\psi)) \leq 1. \qquad (13)
\end{aligned}$$

Combining (12), (13) and (11) becomes

$$\begin{aligned}
0 &\leq \wp_{m+1} \oplus \wp^* \\
&\leq (1-\varrho_m)(\wp_m \oplus \wp^*) + \psi\varrho_m\big[(\wp_m \oplus \wp^*) + \psi\omega_m(f_m \oplus 0) \\
&\quad +\sigma_m(e_m \oplus 0)\big] + \varrho_m(d_m \oplus 0) \\
&\leq (1-\varrho_m(1-\psi))(\wp_m \oplus \wp^*) + \psi\varrho_m\big[\psi\omega_m(f_m \oplus 0) + \sigma_m(e_m \oplus 0)\big] \\
&\quad +\varrho_m(d_m \oplus 0) \\
&\leq (1-\varrho_m(1-\psi))(\wp_m \oplus \wp^*) + \big[\psi^2\varrho_m\omega_m(f_m \oplus 0) + \psi\varrho_m\sigma_m(e_m \oplus 0) \\
&\quad +\varrho_m(d_m \oplus 0)\big].
\end{aligned}$$

By Definition 1 (i) and Lemma 2, we have

$$\begin{aligned}
\|\wp_{m+1} - \wp^*\| &\leq (1-\delta_K\varrho_m(1-\psi))\|\wp_m - \wp^*\| + \delta_K\varrho_m(1-\psi) \\
&\quad \left(\frac{\psi^2\omega_m\|f_m \vee (-f_m)\| + \psi\sigma_m\|e_m \vee (-e_m)\| + \|d_m \vee (-d_m)\|}{(1-\psi)}\right)
\end{aligned} \tag{14}$$

On setting $c_m = \frac{\psi^2\omega_m\|f_m\vee(-f_m)\|+\psi\sigma_m\|e_m\vee(-e_m)\|+\|d_m\vee(-d_m)\|}{(1-\psi)}$, $a_m = \|\wp_m - \wp^*\|$, and $b_m = \delta_K\varrho_m(1-\psi)$, (14) can be rewrite as

$$a_{m+1} \leq (1-b_m)a_m + b_m c_m. \tag{15}$$

By Lemma 1 and the assumptions $\lim_{n\to\infty}\|d_m \vee (-d_m)\| = \lim_{n\to\infty}\|e_m \vee (-e_m)\| = \lim_{n\to\infty}\|f_m \vee (-f_m)\| = 0$, we can conclude that $a_m \to 0$, as $n \to \infty$, and so $\{\wp_m\}$ converges strongly to a unique solution $\wp^*$ of NOMVIP (3).

Proof of (II). Let $S(\wp^*) = \wp^* - g(\wp^*) \oplus \mathcal{R}^G_{\lambda,M}\Big(G(g(\wp^*)) + \frac{\lambda}{\tau}\big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp^*) - A(\wp^*)\big) \odot \omega\eta(\wp^*, h(\wp^*))\big)\Big)$. Using Algorithm 1, Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
0 &\leq z_{m+1} \oplus \wp^* \\
&\leq z_{m+1} \oplus ((1-\varrho_m)\wp^* + \varrho_m S(\wp^*)) \\
&\leq z_{m+1} \oplus ((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \\
&\quad + ((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \oplus ((1-\varrho_m)\wp^* + \varrho_m S(\wp^*)) \\
&\leq z_{m+1} \oplus ((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \\
&\quad +(1-\varrho_m)(z_m \oplus \wp^*) + \varrho_m\left(S(a_m) \oplus S(\wp^*)\right) + \varrho_m(d_m \oplus 0) \\
&\leq z_{m+1} \oplus ((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \\
&\quad +(1-\varrho_m)(z_m \oplus \wp^*) + \varrho_m\psi(a_m \oplus \wp^*) + \varrho_m(d_m \oplus 0),
\end{aligned} \tag{16}$$

From (16), we have

$$\begin{aligned}
0 &\leq a_m \oplus \wp^* \\
&= [(1-\sigma_m)z_m + \sigma_m S(b_m) + \sigma_m e_m] \oplus [(1-\sigma_m)\wp^* + \sigma_m S(\wp^*)] \\
&\leq (1-\sigma_m)(z_m \oplus \wp^*) + \sigma_m(S(b_m) \oplus S(\wp^*)) + \sigma_m(e_m \oplus 0) \\
&\leq (1-\sigma_m)(z_m \oplus \wp^*) + \sigma_m\psi(b_m \oplus \wp^*) + \sigma_m(e_m \oplus 0).
\end{aligned} \tag{17}$$

From (17), we have

$$
\begin{aligned}
0 &\leq b_m \oplus \wp^* \\
&= [(1-\omega_m)z_m + \omega_m S(z_m) + \omega_m f_m] \oplus [(1-\sigma_m)\wp^* + \sigma_m S(\wp^*)] \\
&\leq (1-\omega_m)(z_m \oplus \wp^*) + \omega_m (S(z_m) \oplus S(\wp^*)) + \omega_m (f_m \oplus 0) \\
&\leq (1-\omega_m)(z_m \oplus \wp^*) + \omega_m \psi (z_m \oplus \wp^*) + \omega_m (f_m \oplus 0) \\
&\leq (1-\omega_m(1-\psi))(z_m \oplus \wp^*) + \omega_m (f_m \oplus 0) \\
&\leq (z_m \oplus \wp^*) + \omega_m (f_m \oplus 0), \ \text{ since } \ (1-\omega_m(1-\psi)) \leq 1. \qquad (18)
\end{aligned}
$$

Combining (17) and (18), we have

$$
\begin{aligned}
0 &\leq a_m \oplus \wp^* \\
&\leq (1-\sigma_m)(z_m \oplus \wp^*) + \sigma_m \psi \big((z_m \oplus \wp^*) + \omega_m (f_m \oplus 0)\big) + \sigma_m (e_m \oplus 0) \\
&\leq (1-\sigma_m(1-\psi))(z_m \oplus \wp^*) + \sigma_m \omega_m (f_m \oplus 0) + \sigma_m (e_m \oplus 0) \\
&\leq (z_m \oplus \wp^*) + \sigma_m \omega_m (f_m \oplus 0) + \sigma_m (e_m \oplus 0), \ \text{since } (1-\sigma_m(1-\psi)) \leq 1. \qquad (19)
\end{aligned}
$$

Using (19), (16) becomes as

$$
\begin{aligned}
0 &\leq z_{m+1} \oplus \wp^* \\
&\leq [z_{m+1} \oplus ((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m)] \\
&\quad + (1-\varrho_m(1-\psi))(z_m \oplus \wp^*) + \varrho_m \big[\psi^2 \omega_m (f_m \oplus 0) + \psi \sigma_m (e_m \oplus 0) \\
&\quad + (d_m \oplus 0)\big].
\end{aligned}
$$

By Definition 1 (i) and Lemma 2, we have

$$
\begin{aligned}
\|z_{m+1} - \wp^*\| &\leq \delta_K \|z_{m+1} - [(1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m]\| \\
&\quad + \delta_K (1-\varrho_m(1-\psi))\|u_n - \wp^*\| + \varrho_m \delta_K (1-\psi) \\
&\quad \left(\frac{\psi^2 \omega_m \|f_m \vee (-f_m)\| + \psi \sigma_m \|e_m \vee (-e_m)\| + \|d_m \vee (-d_m)\|}{(1-\psi)}\right) \\
&\leq \delta_K \varphi_m + \delta_K (1-\varrho_m(1-\psi))\|u_n - \wp^*\| + \varrho_m \delta_K (1-\psi) \\
&\quad \left(\frac{\psi^2 \omega_m \|f_m \vee (-f_m)\| + \psi \sigma_m \|e_m \vee (-e_m)\| + \|d_m \vee (-d_m)\|}{(1-\psi)}\right). \qquad (20)
\end{aligned}
$$

Since $0 < \varepsilon \leq \varrho_m$, (20) becomes as

$$
\begin{aligned}
\|z_{m+1} - \wp^*\| &\leq (1-\delta_K \varrho_m (1-\psi))\|u_n - \wp^*\| + \varrho_m \delta_K (1-\psi) \Bigg[\frac{\varphi_m}{\varepsilon(1-\psi)} \\
&\quad + \left(\frac{\psi^2 \omega_m \|f_m \vee (-f_m)\| + \psi \sigma_m \|e_m \vee (-e_m)\| + \|d_m \vee (-d_m)\|}{(1-\psi)}\right)\Bigg]. \qquad (21)
\end{aligned}
$$

Assume that $\lim_{m\to\infty} \varphi_m = 0$, hence

$$\lim_{m\to\infty} z_m = \wp^*,$$

where

$$\lim_{m\to\infty} \|d_m \vee (-d_m)\| = \lim_{m\to\infty} \|e_m \vee (-e_m)\| = \lim_{m\to\infty} \|f_m \vee (-f_m)\| = 0.$$

Conversely, assume that $\lim_{m\to\infty} z_m = \wp^*$. From (10) and $\lim_{m\to\infty}\|d_m \vee (-d_m)\| = \lim_{m\to\infty}\|e_m \vee (-e_m)\| = \lim_{m\to\infty}\|f_m \vee (-f_m)\| = 0$, we have

$$\begin{aligned}
0 &\leq z_{m+1} \oplus [(1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m] \\
&\leq (z_{m+1} \oplus \wp^*) + [((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \oplus \wp^*] \\
&= (z_{m+1} \oplus \wp^*) + [((1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m d_m) \\
&\quad \oplus ((1-\varrho_m)\wp^* + \varrho_m S(\wp^*))] \\
&\leq z_{m+1} \oplus \wp^* + (1-\varrho_m)(z_m \oplus \wp^*) + \varrho_m (S(a_m) \oplus S(\wp^*)) + \varrho_m(d_m \oplus 0) \\
&\leq z_{m+1} \oplus \wp^* + (1-\varrho_m)(z_m \oplus \wp^*) + \wp_m \psi(a_m \oplus \wp^*) \\
&\quad + \varrho_m(d_m \oplus 0) \\
&\leq z_{m+1} \oplus \wp^* + (1-\varrho_m(1-\psi))(z_m \oplus \wp^*) + \varrho_m \big[\psi^2 \omega_m (f_m \oplus 0) \\
&\quad + \psi \widehat{\wp}_m (e_m \oplus 0) + (d_m \oplus 0)\big].
\end{aligned} \tag{22}$$

Applying again Definition 1 (i) and Lemma 2, it follows that

$$\begin{aligned}
\varphi_m &= \|z_{m+1} - [(1-\varrho_m)z_m + \varrho_m S(a_m) + \varrho_m e_m]\| \\
&\leq \delta_K \|z_{m+1} - \wp^*\| + \delta_K (1-\varrho_m(1-\psi)) \|z_m - \wp^*\| \\
&\quad + \varrho_m \delta_K \big[\psi^2 \omega_m \|f_m \vee (-f_m)\| + \psi \sigma_m \|e_m \vee (-e_m)\| + \|d_m \vee (-d_m)\|\big],
\end{aligned}$$

which implies that

$$\lim_{m\to\infty} \varphi_m = 0.$$

Hence, the generated sequence $\{z_m\}$ by (8) is $\mathcal{R}^G_{\lambda,M}$-stable. □

Remark 1

As a various selection of the mappings G, g, A, h, η and M, and the constants ω, ξ and τ, we can propose some another class of three-step iterative scheme to reveal the convergence and stability analysis of the various known problems which investigated by several authors (see [3, 13, 14, 15, 17, 18, 4]) as special cases of Theorem 1 and Theorem 2.

5. Numerical Example

In this segment, we utilize the following numerical example to demonstrate Algorithm 1 and defend our main problem.

Example 1

Suppose $\mathcal{B} = \mathbb{R}$ is the set of real numbers with standard inner product and norm, and let the normal cone $C = \{\wp \in \mathcal{B} : 0 \leq \wp \leq 3\}$ with normal constant $\delta_K = 3$. Let $G, g, A, h : \mathcal{B} \to \mathcal{B}$ and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the mappings defined by

$$G(\wp) = -\frac{\wp}{2} + 1, \quad g(\wp) = \frac{\wp}{3}, \quad A(\wp) = \frac{1-\wp}{4}, \quad h(\wp) = \frac{\wp}{24} \text{ and } \eta(\wp, h(\wp)) = (4h(\wp) - \wp).$$

For each $\wp, \widehat{\wp} \in \mathcal{B}$, $\wp \propto \widehat{\wp}$. Then, it is simple to verify that G is $\frac{1}{6}$-ordered non-extended mapping and $\frac{3}{4}$-ordered compression mapping, g is $\frac{1}{2}$-ordered compression mapping, A is $\frac{1}{3}$-ordered compression mapping and h is $\frac{1}{12}$-ordered compression mapping, respectively.

For $\wp, \widehat{\wp}, q, \widehat{q} \in \mathcal{B},\ \wp \propto q,\ \widehat{\wp} \propto \widehat{q}$, we calculate

$$\begin{aligned}
\eta(\wp, h(q)) \oplus \eta(\widehat{\wp}, h(\widehat{q})) &= (4h(\wp) - \wp) \oplus (4h(\widehat{q}) - \widehat{\wp}) \\
&\leq \big((4h(q)) \oplus (4h(\widehat{q}))\big) + \big((-\wp) \oplus (-\widehat{\wp})\big) \\
&= (\wp \oplus \widehat{\wp}) + 4\left(\left(\frac{q}{24} + \frac{1}{2}\right) \oplus \left(\frac{\widehat{q}}{24} + \frac{1}{2}\right)\right) \\
&= (\wp \oplus \widehat{\wp}) + \frac{1}{6}(q \oplus \widehat{q}),
\end{aligned}$$

i.e.,

$$\eta(\wp, g(q)) \oplus \eta(\widehat{\wp}, g(\widehat{q})) \leq (\wp \oplus \widehat{\wp}) + \frac{1}{6}(q \oplus \widehat{q}).$$

Hence, η is $(1, \frac{1}{6})$-ordered Lipschitz continuous mapping with respect to h.

Suppose the multi-valued mapping $M : \mathcal{B} \to \mathcal{B}$ is defined by

$$M(\wp) = \{9\wp\}, \quad \forall \wp \in \mathcal{B}.$$

It is simple to conclude that M is a comparison mapping, M is 18-weak non-ordinary difference mapping and $\frac{1}{3}$-XOR-ordered different comparison mapping. Moreover, it is obvious that for $\lambda = \frac{1}{3},\ [A + \lambda M](\mathcal{B}) = \mathcal{B}$. So, M is an $\left(\frac{1}{3}, 18\right)$-weak-GNODD multi-valued mapping. The resolvent operator defined by (1) associated with G and M is given by

$$\mathcal{R}^G_{\lambda,M}(\wp) = \frac{2(\wp - 1)}{5}, \ \ \forall \wp \in \mathcal{B}.$$

It is simple to check that $\mathcal{R}^G_{\lambda,M}$ is a comparison and single-valued mapping. In particular for $\mu = 1$, we obtain

$$\begin{aligned}
\mathcal{R}^G_{\lambda,M}(\wp) \oplus \mathcal{R}^G_{\lambda,M}(\widehat{\wp}) &= \left[\frac{2(\wp - 1)}{5}\right] \oplus \left[\frac{2(\widehat{\wp} - 1)}{5}\right] \\
&\leq \frac{2}{5}(\wp \oplus \widehat{\wp}) + \left(\frac{2}{5} \oplus \frac{2}{5}\right) \\
&= \frac{2}{5}(\wp \oplus \widehat{\wp}) \\
&\leq \frac{6}{5}(\wp \oplus \widehat{\wp}),
\end{aligned}$$

i.e.

$$\mathcal{R}^G_{\lambda,M}(\wp) \oplus \mathcal{R}^G_{\lambda,M}(\widehat{\wp}) \leq \frac{6}{5}(\wp \oplus \widehat{\wp}), \ \ \forall \wp, \widehat{\wp} \in \mathcal{B}.$$

Hence, $\mathcal{R}^G_{\lambda,M}$ is $\frac{6}{5}$-ordered Lipschitz type continuous.

On taking $\omega = 1, \tau = 9$ and $\xi = -1$, we calculate

$$\begin{aligned}
F(\wp) &= \left[\wp - g(\wp) \oplus \mathcal{R}^G_{\lambda,M}\Big(G(g(\wp)) + \frac{\lambda}{\tau}\Big(\big(\xi\mathcal{R}^G_{\lambda,M}(\wp) - A(\wp)\big) \odot \omega\eta(\wp, h(\wp))\Big)\Big)\right] \\
&= \left[\wp - g(\wp) \oplus \mathcal{R}^G_{\lambda,M}\Big(-\frac{\wp}{6} + 1 + 3\Big(\Big(-\frac{17\wp}{30}\Big) \odot \Big(-\frac{5\wp}{6}\Big)\Big)\Big)\right] \\
&= \left[\wp - g(\wp) \oplus \mathcal{R}^G_{\lambda,M}\Big(-\frac{29\wp}{30} + 1\Big)\right] \\
&= \left[\wp - \Big(\frac{\wp}{3}\Big) \oplus \Big(-\frac{58\wp}{150}\Big)\right] \\
&= \frac{42\wp}{150}.
\end{aligned}$$

Evidently, 0 is a fixed point of $F(\wp)$. It is confirmed that the condition (9) is fulfilled. Therefore, all the hypotheses of Theorem 1 are satisfied.

Let $\varrho_m = \frac{1}{3m+7}, \sigma_m = \frac{1}{2m^2+1}, \omega_m = \frac{m}{m^3+m^2+1}, d_m = -\frac{1}{m+m^2}, \ e_m = \frac{m^2-1}{m^3+2}$ and $f_m = \frac{m-2}{m^2-m+1}$. It is easy to show that the sequences $\{\varrho_m\}, \{\sigma_m\}, \{\omega_m\}, \{d_m\}, \{e_m\}$ and $\{f_m\}$ satisfying the conditions $0 \leq \varrho_m, \ \omega_m, \ \sigma_m \leq 1$, $\sum_{m=0}^{\infty} \varrho_m = \infty, \ \ d_m \oplus 0 = d_m, \ \ e_m \oplus 0 = e_m, \ \ f_m \oplus 0 = f_m.$

Now, we can determine the sequences $\{\wp_m\}, \ \{\widehat{\wp}_m\}$ and $\{\widehat{q}_m\}$ as:

$$\begin{aligned}
\wp_{m+1} &= \left(\frac{3m+6}{3m+7}\right)\wp_m + \left(\frac{3m+6}{3m+7}\right)\left(\frac{42\widehat{\wp}_m}{150}\right) - \left(\frac{1}{3m+7}\right)\left(\frac{1}{m^2+m}\right),\\
\widehat{\wp}_m &= \left(\frac{2m^2}{2m^2+1}\right)\wp_m + \left(\frac{1}{2m^2+1}\right)\left(\frac{42\widehat{q}_m}{150}\right) + \left(\frac{1}{2m^2+1}\right)\left(\frac{m^2-1}{m^3+2}\right),\\
\widehat{q}_m &= \left(\frac{m^3+m^2-m+1}{m^3+m^2+1}\right)\wp_m + \left(\frac{m}{m^3+m^2+1}\right)\left(\frac{42\wp_m}{150}\right)\\
&\quad + \left(\frac{m}{m^3+m^2+1}\right)\left(\frac{m-2}{m^2-m+1}\right),
\end{aligned}$$

It is also confirmed that assumption (5) is fulfilled. So, all the conditions of Theorem 2 are satisfied. Therefore, the sequence $\{\wp_m\}$ converges strongly to the unique solution $\wp^* = 0$ of the NOMVIP (3).

All codes are given in MATLAB version $R2019a$, for a different choice of initial values $\wp_0 = 5, \ 10$ and 15 which reveals that the sequence $\{\wp_m\}$ converge to $\wp^* = 0$.

Table 1. The values of $\wp_m$ with initial values $\wp_0 = 5$, $\wp_0 = 10$ and $\wp_0 = 15$

No. of Iteration	For $\wp_0 = 5$ $\wp_m$	For $\wp_0 = 10$ $\wp_m$	For $\wp_0 = 15$ $\wp_m$
m=1	5	10	15
m=2	-2.64919019157088	-5.36938881226054	-8.08958743295020
m=3	-0.681376233372378	-1.43077589209397	-2.14819637010015
m=4	2.69201299070253	5.44856667193379	8.18625764630015
m=5	-2.39868122239568	-4.79510289932239	-7.21309495322547
m=6	0.656547967211196	1.22213909691756	1.84079312502577
m=7	1.02614222324223	2.14978955539579	3.22810035447828
m=8	-1.57443680084547	-3.17726620027428	-4.77732623330388
m=9	0.961963976920110	1.85848845505862	2.79575386165318
m=10	0.0992418599415578	0.299210765650301	0.447069810112438
m=15	0.490354997519306	0.996984027201863	1.49861331397862
m=20	0.108331086025402	0.178530404331916	0.268739941015830
m=25	-0.0663999735322326	-0.149652163453438	-0.224933199569455
m=28	0	0	-0.000194296441690925
m=30	0	0	0
m=35	0	0	0

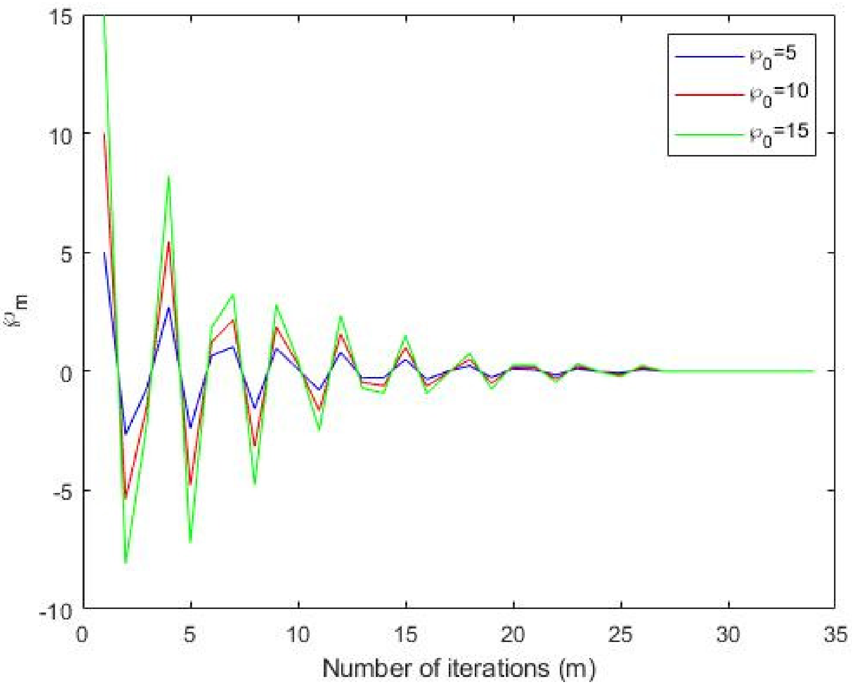

Figure 1. The convergence of $\wp_m$ with initial values $\wp_0 = 5$, $\wp_0 = 10$ and $\wp_0 = 15$

Remark 2

We adopt similar mappings as in Example 1 and compare our suggested Algorithm 1 with the Ishikawa-type Algorithm and Manntype Algorithm.

On taking $\omega_m = 0, \forall\, m \geq 0$, then our proposed Algorithm 1 becomes Ishikawa-type Algorithm, we can determine the sequences $\{\wp_m\}$ and $\{\widehat{\wp}_m\}$ by the following Ishikawa-type iterative schemes:

$$\begin{aligned} \wp_{m+1} &= \left(\frac{3m+6}{3m+7}\right)\wp_m + \left(\frac{3m+6}{3m+7}\right)\left(\frac{42\widehat{\wp}_m}{150}\right) - \left(\frac{1}{3m+7}\right)\left(\frac{1}{m^2+m}\right), \\ \widehat{\wp}_m &= \left(\frac{2m^2}{2m^2+1}\right)\wp_m + \left(\frac{1}{2m^2+1}\right)\left(\frac{42\widehat{q}_m}{150}\right) + \left(\frac{1}{2m^2+1}\right)\left(\frac{m^2-1}{m^3+2}\right). \end{aligned}$$

Also, on taking $\omega_m = \rho_m = 0, \forall\, m \geq 0$, then our proposed Algorithm 1 becomes Mann-type Algorithm, we can determine the sequence $\{\wp_n\}$ by the following Mann-type iterative scheme:

$$\wp_{m+1} = \left(\frac{3m+6}{3m+7}\right)\wp_m + \left(\frac{3m+6}{3m+7}\right)\left(\frac{42\widehat{\wp}_m}{150}\right) - \left(\frac{1}{3m+7}\right)\left(\frac{1}{m^2+m}\right).$$

The iterative schemes will be suspended when the stopping criteria $\|\wp_{m+1} - \wp_m\| \leq 10^{-7}$ is fulfilled. Figure 2 and Table 2 show the comparisons of our recommended Algorithm, Ishikawa-type Algorithm, and Mann-type Algorithm on taking initial value $\wp_0 = 10$.

Table 2. The values of $\wp_m$ with initial value $\wp_0 = 10$.

No. of Iteration	Proposed Algorithm $\wp_m$	Ishikawa-type Algorithm $\wp_m$	Mann-type Algorithm $\wp_m$
m=1	10	10	10
m=2	-5.36938881226054	-5.96866039707419	-6.16277483424535
m=3	-1.43077589209397	-1.11212811675287	-0.882788772565949
m=4	5.44856667193379	6.27190168501902	6.44489035131048
m=5	-4.79510289932239	-6.57446088093008	-7.24447314676890
m=11	0.299210765650301	0.741538833008721	1.622613995187101
m=20	0.178530404331916	1.43269358463089	2.40552101052545
m=28	0	0.384685286039858	1.12054858914816
m=35	0	-0.289596577331598	-0.798022642435171
m=45	0	0.149332546280771	0.245659597107873
m=55	0	-0.0274723743225643	0.0673564657329054
m=65	0	0	-0.0877701132102605
m=70	0	0	-0.00231912786917975
m=74	0	0	0.000832378092252251
m=80	0	0	0

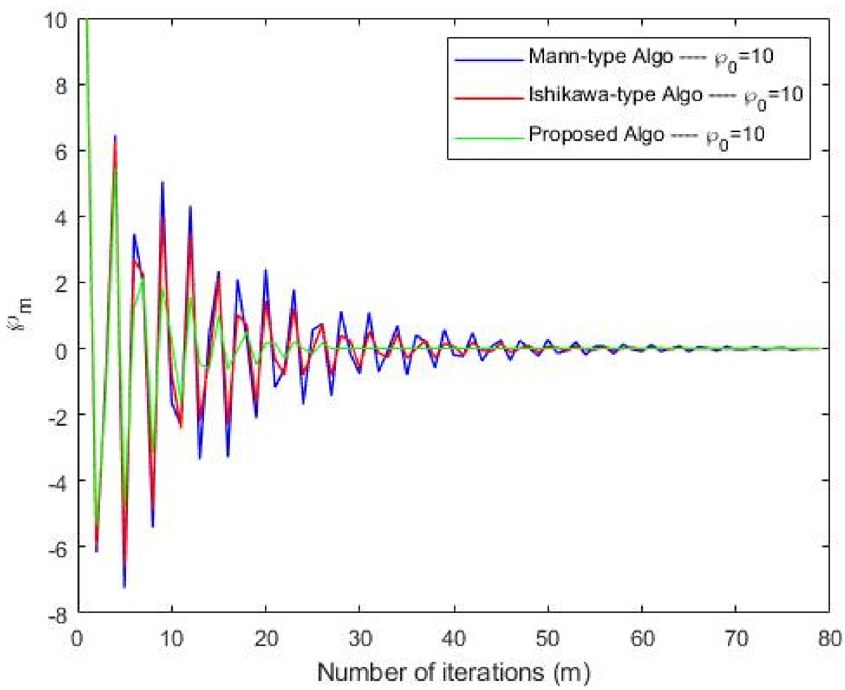

Figure 2. The convergence of $\wp_m$ with initial value $\wp_0 = 10$.

The numerical result in Table 2 and Graph 2 indicates that our suggested three-step iterative scheme has a better performance and shows to have an ambitious advantage. We can decide that our algorithm fast, efficient and stable, and it takes an average of iterations to converge.

6. Conclusion

In this work, we introduced and analyzed a NOMVIP involving XNOR operation and proved the existence of the solution to our main problem. We constructed three-step iterative schemes based on the fixed point formulation with XNOR operation and discussed the convergence of the iterative sequences generated by the proposed algorithms which suggested that algorithms converge to a solution to our proposed problem. Also, we discussed the stability of the convergence. Finally, we created a numerical example to verify that convergence of the suggested algorithm in support of our considered problem has better convergence as compare to Mann-type and Ishikawa-type iterative algorithms. The achieved results in this article are an important and significant generalization to recent known results in nonlinear analysis. Note that it needs further research on the forward-backward splitting method based on the inertial technique for solving ordered inclusion problems and also it needs to develop the algorithms for solving the image deblurring and image recovery problems by using the Tseng method and viscosity method in real ordered Hilbert spaces with XOR and XNOR operations.

Acknowledgments

The authors would like to express their gratitude to Deanship of Scientific Research, Qassim University, Saudi Arabia for technical support. The author thanks to the anonymous referees for their constructive suggestions and helpful comments which led to significant improvement of the original manuscript of this paper.

REFERENCES

1. H. Attouch, *Variational convergence for functions and operators*, Boston, MA, Pitman, 1984.
2. I. Ahmad, C.T. Pang, R. Ahmad, and M. Ishtyak, *System of Yosida inclusions involving XOR operator*, J. Nonlinear Convex Anal., vol 18, no. 5, pp. 831–845, 2017.
3. I. Ahmad, R. Ahmad, and J. Iqbal, *A Resolvent approach for solving a set-valued variational inclusion problem using weak-RRD set-valued mapping*, Korean J. Math., vol. 24, no. 2, pp. 199–213, 2016..
4. R. Ahmad, I. Ahmad, I. Ali, S. Al-Homidan, and Y.H. Wang, *$H(.,.)$-ordered-compression mapping for solving XOR-variational inclusion problem*, J. Nonlinear Convex Anal., vol. 19, no. 12, pp. 2189–2201, 2018.
5. I. Ahmad, M. Rahaman, and R. Ahmad, *Relaxed resolvent operator for solving a variational inclusion problem*, Stat., Optim. Inf. Comput., vol. 4, pp. 183–193, 2016.
6. B.D. Bella, *An existence theorem for a class of inclusions*, Appl. Math. Lett., vol. 13, no. 3, pp. 15–19, 2000.
7. F.E. Browder, *Nonlinear variational inequalities and maximal monotone mapinggs in Banach spaces*, Math. Annln., vol. 183, pp. 213-231, 1969.
8. J. Crank, *Free and moving boundary problems*, Clarendon Press, Oxford, U.K., 1984.
9. Y.H. Du, *Fixed points of increasing operators in ordered Banach spaces and applications*, Appl. Anal., vol. 38, pp. 1–20, 1990.
10. G. Glowinski, and P.L. Tallec, *Augmented Lagrangian and operator spliting methods in nonlinear mechanics*, SIAM, Philadelphia, 1989.
11. N.J. Huang, *A new class of generalized set-valued implicit variational inclusions in Banach spaces with an application*, Comput. Math. Appl., vol. 41, pp. 937–943, 2001.
12. J.U. Jeong, *Generalized set-valued variational inclusions and resolvent equations in Banach spaces*, Comput. Math. Appl., vol. 47, pp. 1241-1247, 2004.
13. H.G. Li, L.P. Li, and M.M. Jin, *A class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ)-ANODM set-valued mappings with strong comparison mapping A*, Fixed Point Theory Appl., 2014: 79, 2014.
14. H.G. Li, D. Qiu, and Y. Zou, *Characterizations of weak-ANODD set-valued mappings with applications to approximate solution of GNMOQV inclusions involving $\oplus$ operator in ordered Banach spaces*, Fixed Point Theory Appl., doi: 10.1186/1687-1812-2013-241, 2013.
15. H.G. Li, *A nonlinear inclusion problem involving (α, λ)-NODM set-valued mappings in ordered Hilbert space*, Appl. Math. Lett., vol. 25, pp. 1384-1388, 2012.
16. D. T. Luc, and N.X. Tan, *Existence conditions in variational inclusions with constraints*, Optimization, vol. 53, pp. 505-515, 2004.
17. H.G. Li, Y. Yang, M.M. Jin, and Q. Zhang, *Stability for a new class of GNOVI with (γ_G, λ)-weak-GRD mapping in positive Hilbert spaces*, Math. Probl. Engg., vol. 2016, Article ID 9217091, 7 pages, 2016.
18. H.G. Li, X.B. Pan, Z.Y. Deng, and C.Y. Wang, *Solving GNOVI frameworks involving (γ_G, λ)-weak-GRD set-valued mappings in positive Hilbert spaces* Fixed Point Theory Appl., 2014:146, 2014.
19. B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Fr. D'Informatique Rech. OpéRationnelle, vol. 3, pp. 154-158, 1970.
20. A. Moudafi, and M. Thera, *Finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., vol. 94, pp. 424-448, 1997.

21. M.A. Noor, *A predictor-corrector algorithm for general variational inequalities*, Appl. Math. Lett., vol. 14, pp. 53-58, 2001.
22. M.A. Noor, *Three-step iterative algorithms for multivaled quasi-variational inclusions*, J. Math. Anal. Appl., vol. 255, pp. 589-604, 2001.
23. M.O. Osilike, *Stability results for the Ishikawa fixed point iteration procedure*, Ind. J. Pure Appl. Math., vol. 26, no. 10, pp. 937-945, 1995.
24. R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control. Optim., vol. 14, pp. 877-898, 1976.
25. H.H. Schaefer, *Banach lattices and positive operators*, Springer Berlin Heidelberg, New York, 1974.
26. E.A. Shelmas, *Resolvent operator method for general variational inclusions*, J. Math. Ineqal., vol. 3, pp. 455-462, 2009.
27. S. Simson, *From Hahn-Banach to monotonicity*, Second Edition, Lecture Notes in Math., Springer, New York, 2008.
28. N.X. Tan, *On the existence of solutions of quasivariational inclusion problem*, J. Optim. Theory Appl., vol. 123, pp. 619-638, 2004.
29. X.L. Weng, *Fixed point iteration for local srictly pseudocontractive mapping*, Proc. A.M.S., vol. 113, no. 3, pp. 727-731, 1991.