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On Mean Field Games with Common Noise Based on Stable-Like Processes

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Abstract In this paper, we study Mean Field Games with common noise based on nonlinear stable-like processes. The MFG limit is specified by a single quasi-linear deterministic infinite-dimensional partial differential second order backward equation. The main result is that any its solution provides an $1/N$-Nash equilibrium for the initial game of N agents. Our approach is based on interpreting the common noise as a kind of binary interaction of agents and our previous results on regularity and sensitivity with respect to the initial conditions of the solution to the nonlinear stochastic differential equations of McKean-Vlasov type generated by stable-like processes.

Keywords Mean-field games with Common Noise, Stable-like processes, McKean-Vlasov SPDE, Regularity, Sensitivity, Nash equilibrium.

AMS 2010 subject classifications 60H15, 60J99, 91A15, 91A06, 82C22

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1. Introduction

Mean field games with common noise present a quickly developing part of the mean field game theory. The theory of mean field games was initiated by Lasry-Lions [28] and Huang-Malhame-Caines [19, 17, 18], see [5, 6, 14, 15, 7, 8, 3, 4] for recent surveys, as well as [9, 10, 16, 2] and references therein. Notice that there is quite an extensive literature on the mean field games with common noise (e.g. [11, 12, 27, 23, 24, 13, 1] and references therein).

Seemingly first serious contributions to the general theory of mean field games with common noise are the works [12] and [27]. In [12], existence of weak solutions for mean field games with common noise is shown to hold under very general assumptions, existence and uniqueness of a strong solution are proved under additional assumptions. However, [12] and [27] work mostly with controlled SDEs. Another approach which based on the sensitivity analysis for McKean-Vlasov SPDEs was developed in the papers [23, 24]. Some simple concrete models of mean field game types with common noise applied to modeling interbank loans are analyzed in detail in [13]. A model of common noise with constant coefficients is discussed in [1].

Another new trend concerns the theory of mean field games with a major player, see [29, 30, 31], and references therein. Common noise can be considered as a kind of neutral major player, but the usual setting for the latter [29] introduces the corresponding noise into the coefficients of the SDEs of the minor players, rather than adding additional common stochastic differential. One of the ideas (and results) of our contribution is to use the method of stochastic characteristics to link these two models.

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Let us consider N agents, whose positions are governed by the system of SDEs

$$dX_t^i = b(X_t^i, \mu_t^N, u_t^i)\, dt + \sigma_{com} dW_t + a^{1/\alpha}(X_t^i) dY_t^i, \tag{1}$$

where all X_t^i belong to $\mathbf{R}^d$, W_t is d'-dimensional standard Brownian motion referred to as the common noise and Y_t^i are independent symmetric Lévy processes with the index α, σ_{com} is a constant $d \times d'$ matrix. The parameters $u_t^i \in U$ are controls available to the players, trying to minimize their payoffs

$$V_t^i(x) = \mathbf{E}\left[\int_t^T J(s, X_s^i, \mu_s^N, u_s^i)\, ds + V_T(X_T^i, \mu_T^N)\right], \tag{2}$$

depending on the action of other players, with the given functions J and V_T. The coefficient $b(x, \mu, u)$ is a function of $x \in \mathbf{R}^d, u \in U$ and a measure $\mu \in \mathcal{M}(\mathbf{R}^d)$, and μ_t^N in (1) is

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

Here U is a closed interval in $\mathbf{R}$ and $\mathcal{M}(\mathbf{R}^d)$ is the set of bounded positive Borel measures on $\mathbf{R}^d$.

In general, the function b needs to be defined only for μ from the set of probability measures $\mathcal{P}(R^d)$. However, to use smoothness with respect to μ it is convenient (though not necessary) to have this function defined on a larger space. In the usual examples, b depend on μ via a finite set of moments of type

$$F_j(\mu) = \int \tilde{F}_j(x_1, \cdots, x_k)\mu(dx_1)\cdots\mu(dx_k), \tag{3}$$

with some bounded measurable symmetric functions $\tilde{F}_j$.

As shown below, under appropriate regularity assumptions on the coefficients b, σ_{com} and a in (1), the corresponding Markov evolution of the empirical measures μ_t^N converges, as $N \to \infty$ to the unique solution μ_t of the following nonlinear stochastic partial differential equation of the McKean-Vlasov type generated by the stable-like process,

$$d(f, \mu_t) = (L(\mu_t, u_t)f + \frac{1}{2}(\sigma_{com}\sigma_{com}^T \nabla, \nabla)f, \mu_t)\, dt + (\sigma_{com}\nabla f, \mu_t)\, dW_t. \tag{4}$$

This equation is written in the weak form meaning that it should hold for all $f \in C^2(\mathbf{R}^d)$. Here

$$L(\mu, u)f(x) = (b(x, \mu, u(x, \mu)), \nabla)f(x) - a(x)|\Delta|^{\alpha/2} f(x), \tag{5}$$

is the generator for the stable-like processes in $\mathbf{R}^d$ with stability index $\alpha \in (1, 2)$, $\mu \in \mathcal{M}(\mathbf{R}^d)$.

By the usual rule $Y \circ dX = Y dX + \frac{1}{2} dY dX$, equation (4) rewrites in a more transparent Stratonovich form as

$$d(f, \mu_t) = (L(\mu_t, u_t)f, \mu_t)\, dt + (\sigma_{com}\nabla f, \mu_t) \circ dW_t. \tag{6}$$

Recall that the fractional Laplacian can be expressed via the integral operator

$$|\Delta|^{\alpha/2} f(x) = C_\alpha \int_{\mathbf{R}^d} \left(f(x+y) - f(x) - \frac{(\nabla f(x), y)}{1 + |y|^2}\right) \frac{dy}{|y|^{d+\alpha}}, \tag{7}$$

with a certain constant C_α.

Let us introduce the following conditions:

(C0) The function b is linear in u, that is,

$$b(x, \mu, u) = b_1(x, \mu) + b_2(x, \mu)u, \tag{8}$$

and the function $J(t,x,\mu,u)$ is convex and smooth in u.

(C1) The function $a(x) \in C^2_\infty\left(\mathbf{R}^d\right)$ and satisfies the inequalities

$$M^{-1} \le a(x) \le M$$

for all $x \in \mathbf{R}^d$ and a constant $M > 1$;

(C2) Functions $b_1(x,\mu), b_2(x,\mu)$ are continuous and bounded on $\mathbf{R}^d \times \mathcal{M}(\mathbf{R}^d)$, function $u(t,x,\mu)$ is continuous and bounded on $[0,T] \times \mathbf{R}^d \times \mathcal{M}(\mathbf{R}^d)$, $b_1(.,\mu), b_2(.,\mu), u(t,.,\mu) \in C^2(\mathbf{R}^d)$, and b_1, b_2, u are Lipshitz continuous as a functions of x, uniformly in other variables;

(C3) The first and second order variational derivatives of $b_1(x,\mu)$, $b_2(x,\mu)$, $u(t,x,\mu)$ and $J(t,x,\mu,u)$ with respect to μ are well defined, bounded and

$$b_1(x,.), b_2(x,.), u(t,x,.), J(t,x,.,u) \in (C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$$

for any $\lambda > 0$, the classes of functions $C^{2,1\times1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ and $C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ are introduced below.

Recall now that the optimal control problem facing each player, say X_t^1, is to minimize cost (2). Now the crucial difference with the games without common noise starts to reveal itself. For games without noise, one expects to get a deterministic curve μ_t in the limit of large N, so that in the limit, each player should solve a usual optimization problem for a stable-like process in $\mathbf{R}^d$. Here the limit is stochastic, and thus even in the limit the optimization problem faced by each player is an optimization with respect to an infinite-dimensional, in fact measure-valued, process.

In fact, for fixed N, if all players, apart from the first one, are using the same control $u_{com}(t,x,\mu)$, the optimal payoff for the first player is found from the HJB equation for the stable-like process governed by (1), that is, the HJB equation (where we denote X^1 by x),

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t,x,\mu) + \inf_u\left[(b(x,\mu,u),\nabla)V + J(t,x,\mu,u)\right] + \frac{1}{2}(\sigma_{com}\sigma_{com}^T\nabla,\nabla)V - a(x)|\Delta|^{\alpha/2}V \\
&+ \sum_{j\neq1}\left[(b(x_j,\mu,u_{com}(t,x_j,\mu),\nabla)V + \frac{1}{2}((\sigma_{com}\sigma_{com}^T\nabla,\nabla)V - a(x_j)|\Delta|^{\alpha/2}V\right] \\
&+ \sum_{j\neq1}\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial x},\frac{\partial}{\partial x_j}\right)V + \sum_{\{i,j\}\subset\{1,\dots,N\}}\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)V = 0. \qquad (9)
\end{aligned}$$

As will be shown, in the limit when $(\delta_{x_1}+\dots+\delta_{x_N})/N$ converge to the process μ_t, this equation turns to the limiting HJB equation

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \inf_u\left[(b(x,\mu,u),\nabla)V + J(t,x,\mu,u)\right] + \frac{1}{2}(\sigma_{com}\sigma_{com}^T\nabla,\nabla)V - a(x)|\Delta|^{\alpha/2}V \\
&+ \Lambda_{lim}V(t,x,\mu) + \int\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\delta V(t,x,\mu)}{\delta\mu(y)}\mu(dy) = 0, \qquad (10)
\end{aligned}$$

where the operator Λ_{lim} acting on the variable μ is calculated in (48) with u_{com} as the control.

If J is convex, the infimum here is achieved on the single point

$$\hat{u}_{ind}(t,x,\mu) = argmin_u\left[(b(x,\mu,u),\nabla)V + J(t,x,\mu_t,u)\right].$$

Moreover, if $\hat{u}_{ind}(t,x,\mu)$ belongs to the internal part of U then

$$\hat{u}_{ind}(t,x,\mu) = -\left(\frac{\partial J}{\partial u}\right)^{-1}(b_2(x,\mu),\nabla)V. \qquad (11)$$

Instead of a pair of coupled forward-backward equations in the usual MFG we have now one single infinite-dimensional equation (10). Namely, for any curve $u_{com}(t,x,\mu)$ (defining Λ_{lim} in (48) and thus in (10)), we should solve equation (10) with a given terminal condition leading to the optimal control (11). The key *MFG consistency requirement* is now given by the equation

$$\hat{u}_{ind}(t, x, \mu) = u_{com}(t, x, \mu). \tag{12}$$

This can be interpreted as having a limiting game of two players, a tagged player and a measure-valued player, for which we are looking for a symmetric Nash equilibrium (see e.g. [32]).

Equivalently, the MFG consistency (12) can be encoded into a single quasi-linear deterministic infinite-dimensional partial differential second order backward equation on the function $V(t, x, \mu)$, which we present now in full substituting Λ_{lim} from (48) and (12) in (10):

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t,x,\mu) + [(b(x,\mu,\hat{u}_{ind}(t,x,\mu)),\nabla)V + J(t,x,\mu,\hat{u}_{ind}(t,x,\mu))] \\
&+\frac{1}{2}((\sigma_{com}\sigma_{com}^T\nabla,\nabla)V - a(x)|\Delta|^{\alpha/2}V + \frac{1}{2}\int\int\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\frac{\delta^2 V(t,x,\mu)}{\delta\mu(y)\delta\mu(z)}\mu(dy)\mu(dz) \\
&+\int\left(\left[(b(.,\mu,\hat{u}_{ind}(t,.,\mu)),\nabla) + \frac{1}{2}(\sigma_{com}\sigma_{com}^T\nabla,\nabla) - a(.)|\Delta|^{\alpha/2}\right]\frac{\delta V(t,x,\mu)}{\delta\mu(.)}\right)(y)\mu(dy) \\
&+\int\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{\delta V(t,x,\mu)}{\delta\mu(y)}\mu(dy) = 0,
\end{aligned} \tag{13}$$

with a given terminal condition

$$V(t, x, \mu)|_{t=T} = V_T(x, \mu), \quad \mu_t|_{t=0} = \mu_0. \tag{14}$$

The MFG methodology suggests that for large N the optimal behavior of players arises from the control $\hat{u}$ given by (11) with V solving (13), or equivalently, satisfying the consistency condition (12).

In this paper, we are going to concentrate exclusively on proving that the solutions to MFG provide the $\epsilon(N)$-Nash-equilibria error-order $\epsilon(N) \sim 1/N$. Our approach will be based on interpreting (by means of Ito's formula) the common noise as a kind of binary interaction of agents (in addition to the usual mean field interaction of the standard situation without common noise) and then reducing the problem to the sensitivity analysis for McKean-Vlasov type SPDE.

Our paper is organized as follows. Section 2 formulates our main results and indicates the strategy of their proof. Sections 3-4 are devoted to the well-posedness and sensitivity analysis of the McKean-Vlasov type SPDEs and the related properties of the corresponding measure-valued Markov processes. The last two Sections prove the Theorems formulated in Section 2.

The following basic notations will be used:

$C^n(\mathbf{R}^d)$ is the Banach space of n times continuously differentiable and bounded functions f on $\mathbf{R}^d$ such that each derivative up to and including order n is bounded, equipped with norm $\|f\|_{C^n}$ which is the sum of the suprema of the magnitudes of all mixed derivatives up to and including order n.

$C_\infty(\mathbf{R}^d)$ is the Banach space of bounded continuous functions $f : \mathbf{R}^d \to \mathbf{R}$ with $\lim_{x\to\infty} f(x) = 0$, equipped with sup-norm.

$C^n_\infty(\mathbf{R}^d)$ is a closed subspace of $C^n(\mathbf{R}^d)$ with f and all its derivatives up to and including order n belonging to $C_\infty(\mathbf{R}^d)$.

If $\mathcal{M}$ is a closed subset of a Banach space $\mathbf{B}$, then

$C([0, T], \mathcal{M})$ is a metric space of continuous functions $t \to \mu_t \in \mathcal{M}$ with distance $\|\eta - \xi\|_{C([0,T],\mathcal{M})} = \sup_{t\in[0,T]} \|\eta_t - \xi_t\|_{\mathbf{B}}$. An element from $C([0, T], \mathcal{M})$ is written as $\{\mu_.\} = \{\mu_t, t \in [0, T]\}$.

$\mathcal{M}(\mathbf{R}^d)$ is the Banach space of finite signed Borel measures on $\mathbf{R}^d$.

$\mathcal{M}^+(\mathbf{R}^d)$ and $\mathcal{P}(\mathbf{R}^d)$ are the subsets of $\mathcal{M}(\mathbf{R}^d)$ of positive and positive normalized (probability) measures, respectively.

Let $\mathcal{M}_{<\lambda}(\mathbf{R}^d)$ (resp. $\mathcal{M}_{\le\lambda}(\mathbf{R}^d)$ or $\mathcal{M}_\lambda(\mathbf{R}^d)$) and $\mathcal{M}^+_{<\lambda}(\mathbf{R}^d)$ (resp. $\mathcal{M}^+_{\le\lambda}(\mathbf{R}^d)$ or $\mathcal{M}^+_\lambda(\mathbf{R}^d)$) denote the parts of these sets containing measures of the norm less than λ (resp. not exceeding λ or equal λ).

Let $C^{k\times k}(\mathbf{R}^{2d})$ denote the subspace of $C(\mathbf{R}^{2d})$ consisting of functions f such that the partial derivatives $\partial^{\alpha+\beta}f/\partial x^\alpha \partial y^\beta$ with multi-index α,β, $|\alpha|\le k, |\beta|\le k$, are well defined and belong to $C(\mathbf{R}^{2d})$. Supremum of the norms of these derivatives provide the natural norm for this space.

For a function F on $\mathcal{M}^+(\mathbf{R}^d)$ or $\mathcal{M}(\mathbf{R}^d)$ the variational derivative is defined as the directional derivative of $F(\mu)$ in the direction δ_x:

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \frac{d}{dh}|_{h=0}F(\mu + h\delta_x).$$

The same formula defines the variational derivative in the case when F is a Banach space-valued mapping.

For instance, if $F = F_j$ is given by (3), then

$$\frac{\delta F}{\delta \mu(x)} = k\int \tilde{F}_j(x, x_2, \cdots, x_k)\mu(dx_2)\cdots\mu(dx_k),$$

$$\frac{\delta^2 F}{\delta \mu(x)\delta\mu(y)} = k(k-1)\int \tilde{F}_j(x, y, x_3, \cdots, x_k)\mu(dx_3)\cdots\mu(dx_k).$$

The higher derivatives $\delta^l F(\mu)/\delta\mu(x_1)...\delta\mu(x_l)$ are defined inductively.

Let $C^k(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ denote the space of functionals such that the kth order variational derivatives are well defined and represent continuous functions of all variables with measures considered in their weak topology.

Let $C^{k,l}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ denote the subspace of functionals F from $C^k(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ such that $\delta^m F(\mu)/\delta\mu(.)...\delta\mu(.) \in C^l(\mathbf{R}^{dm})$ for all $m \le k$ uniformly in $\mu \in \mathcal{M}_{\le\lambda}(\mathbf{R}^d)$.

Let $C^{2,k\times k}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ be the space of functionals such that their second order variational derivatives are continuous as functions of all variables and belong to $C^{k\times k}(\mathbf{R}^{2d})$ as functions of the spatial variable; the norm of this space is

$$\|F\|_{C^{2,k\times k}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} = \sup_{\mu\in\mathcal{M}_{\le\lambda}(\mathbf{R}^d)} \left\|\frac{\delta^2 F}{\delta\mu(.)\delta\mu(.)}\right\|_{C^{k\times k}(\mathbf{R}^{2d})}.$$

$(f,\mu) = \int f(x)\mu(dx)$ denotes the usual pairing of functions and measures on $\mathbf{R}^d$.

$\mathbf{E}$ denotes the expectation.

2. Our strategy and results

Our main result is the following.

Theorem 1

Let functions b, a, u, J satisfy the Conditions (C0)-(C3), σ_{com} be a constant $d\times d'$ matrix and let $V(t,x,\mu)$ be a classical solution to problem (13), (14). Assume $V(t,x,\mu)$, as the function of μ belongs to the space $(C^{2,1\times 1}\cap C^{1,2})(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$, as the function of x belongs to the space $C^2(\mathbf{R}^d)$ and its derivatives with respect to x belong to the space $C^{1,1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$. Then the profile of symmetric strategies $\hat{u}_t(x,\mu)$ given by (11) is an ϵ-Nash equilibrium of the N-player game given by (1), (2), with $\epsilon(N)\sim 1/N$ as $N\to\infty$.

As an important ingredient in our proof we use our previous results on the regularity and sensitivity of the nonlinear stochastic differential equations of McKean-Vlasov type generated by stable-like processes [25]. As a by-product of our analysis, we obtain a result of independent interest, not linked with any optimization problem, namely the $1/N$-rates of convergence for interacting stable-like processes to the limiting measure-valued stable-like process, Theorem 2 (often interpreted as the 'propagation of chaos' property).

Let us explain our strategy for proving Theorem 1.

For any N and a fixed common strategy $u_t(x,\mu)$, solutions to the system of SDEs (1) on $t\in[0,T]$ define a backward propagator (also referred in the literature as a flow or as a two-parameter semigroup) $U_N^{s,t} = U_N^{s,t}[u(.)]$, $0\le s\le t\le T$, of linear contractions on the space $C_{sym}(\mathbf{R}^{dN})$ of symmetric functions via the formula

$$(U_N^{s,t}f)(x_1,\cdots,x_N) = \mathbf{E}f(X_1,\cdots,X_N)_{s,t}(x_1,\cdots,x_N), \tag{15}$$

where $(X_1, \cdots, X_N)_{s,t}(x_1, \cdots, x_N)$ is the solution to (1) at time t with the initial condition

$$(X_1, \cdots, X_N)_{s,s}(x_1, \cdots, x_N) = (x_1, \cdots, x_N)$$

at time s. The corresponding dual forward propagator $V_N^{t,s} = (U_N^{s,t})'$ is defined by the equation

$$(f, V_N^{t,s}\mu) = (U_N^{s,t} f, \mu). \tag{16}$$

It acts on the probability measures on $\mathbf{R}^{dN}$, so that if μ is the initial distribution of $(X_1, \cdots, X_N)$ at time s, then $V_N^{t,s}\mu$ is the distribution of $(X_1, \cdots, X_N)$ at time t.

By the standard inclusion

$$(x_1, \cdots, x_N) \to \frac{1}{N}(\delta_{x_1} + \cdots + \delta_{X_N}) \tag{17}$$

the set $\mathbf{R}^{dN}$ is mapped to the set $\mathcal{P}_N(\mathbf{R}^d)$ of normalized sums of N Dirac's measures, so that $U_N^{s,t}$, $V_N^{t,s}$ can be considered as propagators in $C(\mathcal{P}_N(\mathbf{R}^d))$ and $\mathcal{P}(\mathcal{P}_N(\mathbf{R}^d))$ respectively.

On the other hand, for a fixed function $u_t(x, \mu)$, the solution of SPDE (4) specifies a stochastic process, a stable-like process, on the space of probability measures $\mathcal{P}(\mathbf{R}^d)$ defining the backward propagator $U^{s,t} = U^{s,t}[u(.)]$ on $C(\mathcal{P}(\mathbf{R}^d))$:

$$(U^{s,t} f)(\mu) = \mathbf{E} f(\mu_{s,t}(\mu)), \tag{18}$$

where $\mu_{s,t}(\mu)$ is the solution to (4) at time t with a given initial condition μ at time $s \leq t$.

From the convergence of the empirical measures μ_t^N, mentioned above, it follows that $U_N^{s,t}$ tend $U^{s,t}$, as $N \to \infty$. The following result provides the rates for the weak convergence.

Theorem 2
Let functions b, a, u satisfy the Conditions (C0)-(C3), σ_{com} be a constant $d \times d'$ matrix. Then for any $\mu \in \mathcal{P}_N(\mathbf{R}^d)$ and $F \in (C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_{\leq\lambda}(\mathbf{R}^d))$

$$\|(U^{s,t} - U_N^{s,t})F(\mu)\|_{C(\mathcal{M}_{\leq\lambda}(\mathbf{R}^d))} \leq \frac{C(T)}{N} \left(\|F\|_{C^{2,1\times 1}(\mathcal{M}_{\leq\lambda}(\mathbf{R}^d))} + \|F\|_{C^{1,2}(\mathcal{M}_{\leq\lambda}(\mathbf{R}^d))}\right) \tag{19}$$

for $0 \leq s \leq t \leq T$.

This result belongs to the statistical mechanics of interacting stable-like processes, so that its significance goes beyond any links with games or control theory.

This result is not sufficient for us, as we have to allow one of the agent to behave differently from the others. To tackle this case we shall consider the corresponding problem with a tagged agent. Namely, consider the Markov process on pairs $(X_t^{1,N}, \mu_t^N)[u^{ind}(.), u^{com}(.)]$, where u^{ind} and u^{com} are some U-valued functions $u_t^{ind}(x, \mu)$, $u_t^{com}(x, \mu)$, $(X_t^{1,N}, \cdots, X_t^{N,N})$ solves (1) under the assumptions that the first agent uses the control $u_t^{ind}(X_t^{1,N}, \mu_t^N)$ and all other agents $i \neq 1$ use the control $u_t^{com}(X_t^{i,N}, \mu_t^N)$, and $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$.

Remark 1
The coordinates $(X_t^{1,N}, \mu_t^N)$ of our pair process are not independent. Quite opposite, $X_t^{1,N}$ is the position of the first δ-function in μ_t^N. However, we are aiming at the limit $N \to \infty$ where the influence of $X_t^{1,N}$ on μ_t^N becomes negligible, and we do not want it to be lost in the limit. Alternatively, to avoid this dependence, one can consider (as some authors do), instead of our μ_t^N, the measures that do not take $X_t^{1,N}$ into account, that is $\tilde{\mu}_t^N = \frac{1}{N}\sum_{i=2}^N \delta_{X_t^{i,N}}$, but this would neither change the results nor simplifies the notations.

Let us now define the corresponding tagged propagators $U_{N,tag}^{s,t} = U_{N,tag}^{s,t}[u^{ind}(.), u^{com}(.)]$ and $U_{tag}^{s,t} = U_{tag}^{s,t}[u^{ind}(.), u^{com}(.)]$:

$$(U_{N,tag}^{s,t} F)(x, \mu) = \mathbf{E} F(X_t^{1,N}, \mu_t^N)[u^{ind}(.), u^{com}(.)](x, \mu), \tag{20}$$

where $\mu = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ is the position of the process at time s and where $x = x_1$;

$$(U_{tag}^{s,t} F)(x, \mu) = \mathbf{E} F(X_t^1, \mu_t)[u^{ind}(.), u^{com}(.)](x, \mu), \tag{21}$$

where the process $(X_t^1, \mu_t)[u^{ind}(.), u^{com}(.)](x, \mu)$ with the initial data x, μ at time s is the solution to the system of stochastic equations

$$dX_t^1 = b(X_t^1, \mu_t, u_t^{ind}(X_t^1, \mu_t)) + \sigma_{com}(X_t^1) dW_t + a^{1/\alpha}(X_t^i) dY_t^i, \tag{22}$$

$$d(f, \mu_t) = (L(\mu_t, u_t^{com}(., \mu_t)) f + \frac{1}{2}(\sigma_{com}\sigma_{com}^T \nabla, \nabla) f, \mu_t)\, dt + (\sigma_{com} \nabla f, \mu_t)\, dW_t \tag{23}$$

(the second equation is actually independent of the first one).

The following is the basic convergence result for the tagged processes.

Theorem 3
Under the assumptions of Theorem 2 (with both u_t^{com}, u_t^{ind} satisfying these assumptions), let $F(x, \mu)$, $x \in \mathbf{R}^d$, belongs to the space $(C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ as a function of μ, $F \in C^2(\mathbf{R}^d)$ as a function of x and $\frac{\partial F}{\partial x}(x, .) \in C^{1,1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$. Then, for any $\mu \in \mathcal{P}_N(\mathbf{R}^d)$

$$\begin{aligned} &\|(U_{tag}^{s,t} - U_{N,tag}^{s,t}) F\|_{C(\mathbf{R}^d \times \mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \\ \le \frac{C(T)}{N} \Bigg(&\sup_x \|F(x, .)\|_{C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} + \sup_x \|F(x, .)\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \\ &+ \sup_\mu \|F(., \mu)\|_{C^2(\mathbf{R}^d)} + \sup_x \|\frac{\partial F}{\partial x}(x, .)\|_{C^{1,1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \Bigg). \end{aligned} \tag{24}$$

Theorem 2 is a particular case of Theorem 3 obtained from the latter by ignoring the first coordinate. However, by methodological reason, we first prove simpler Theorem 2, and then its extension Theorem 3.

Now Theorem 1 can be derived.

Proof of Theorem 1

Let u_1 be any adaptive control of the first player and V_1 the corresponding payoff in the game of N players, where all other players are using $u_{com}(t, x, \mu)$ arising from a solution to (13),(14). Then $V_1 \ge V_2$, where V_2 is obtained by playing optimally, that is using control u_2 arising from the solution to (9). By Theorem 3,

$$|V_2 - V_{2,lim}| \le C/N,$$

where $V_{2,lim}$ is obtained by playing u_2 in the limiting game specified by equations (22), (23). But $V_{2,lim} \ge V$, where V is the optimal payoff for the first player in the limiting game of two players, where the second, measure-valued, player uses u_{com}. Consequently,

$$V_1 \ge V_2 \ge V_{2,lim} - \frac{C}{N} \ge V - \frac{C}{N},$$

completing the proof. □

3. Sensitivity for stochastic McKean-Vlasov type equations

In this section, we formulate the well-posedness and the sensitivity results of equations (4) or (6), proved in our previous paper [25].

For any measure μ and a vector y the measure $\mu(. + y)$ denotes the measure μ shifted by y.

By using the method of stochastic characteristics, equations (4) and (6) are transferred to the non-stochastic equations with random coefficients.

Namely, according to Lemma 1 of [25], equation (6) rewrites in terms of the measures $\zeta_t = T^*(-W_t)\mu_t = \mu_t(. + \sigma_{com} W_t)$ as the following non-stochastic equation with random coefficients

$$\frac{d}{dt}(f, \zeta_t) = (\tilde{L}_t(\zeta_t) f, \zeta_t) = (L_t^{dress}(\zeta_t(. - \sigma_{com} W_t)) f, \zeta_t), \tag{25}$$

with

$$L_t^{dress}(\mu)f(x) = (b(x+\sigma_{com}W_t,\mu,u),\nabla)f(x) - a(x+\sigma_{com}W_t)|\Delta|^{\alpha/2}f(x) \tag{26}$$

and

$$\begin{aligned}\tilde{L}_t(\zeta_t)f(x) &= L_t^{dress}(\zeta_t(x-\sigma_{com}W_t))f(x)\\ &= (\tilde{b}(W_t,x,\zeta_t),\nabla)f(x) - \tilde{a}(W_t,x)|\Delta|^{\alpha/2}f(x),\end{aligned} \tag{27}$$

where

$$\tilde{b}(W_t,x,\zeta_t) = b(x+\sigma_{com}W_t,\zeta_t(x-\sigma_{com}W_t),u),$$
$$\tilde{a}(W_t,x) = a(x+\sigma_{com}W_t).$$

Theorem 4
Let functions b,a,u satisfy the Conditions (C0)-(C3), σ_{com} be a constant $d\times d'$ matrix. Then for any given $T>0$ the following holds.

(i) The Cauchy problem for equation (6) is well posed almost surely, that is, for any initial condition $Y\in\mathcal{M}^+(\mathbf{R}^d)$ it has the unique bounded nonnegative solution $\mu_t(Y)$ such that $\zeta_t=\mu_t(x+\sigma_{com}W_t)$ solves (25).

(ii) For all $t>0$, $\|\zeta_t\|\le\|Y\|$ and ζ_t have densities with respect to Lebesgue measure. With some abuse notation, we shall denote these densities again by ζ_t. They satisfy the mild equation

$$\zeta_t(x) = \int G_t(x,y)Y(dy) - \int_0^t ds\int\frac{\partial}{\partial y}\left(G_{(t-s)}(x,y),\tilde{b}(W_s,y,\zeta_s)\zeta_s(y)\right)dy. \tag{28}$$

Consequently, $\|\mu_t\|\le\|Y\|$ and μ_t also have densities, v_t and $\mu_t(dy)=v_t(y)dy\to Y$ weakly, as $t\to0$. If the initial condition Y has a density, v_0, then $v_t\to v_0$ in $L^1(\mathbf{R}^d)$.

(iii) For any two solutions μ_t^1 and μ_t^2 of (6) with the initial conditions Y^1,Y^2 the estimate

$$\|\mu_t^1-\mu_t^2\|_{L_1(R^d)} \le \|Y^1-Y^2\|_{\mathcal{M}(R^d)}C(T,\|Y^1\|), \tag{29}$$

with C depending on the bounds of the derivatives in conditions (C1)-(C3).

Denote by

$$\xi_t(x;.) = \xi_t(x;.)[\mu_0] = \frac{\delta\mu_t}{\delta\mu_0(x)} = \frac{d}{dh}|_{h=0}\mu_t[\mu_0+h\delta_x], \tag{30}$$

and

$$\eta_t(x,z;.) = \eta_t(x,z;.)[\mu_0] = \delta^2\mu_t/\delta\mu_0(x)\delta\mu_0(z)$$

the first and second variational derivatives of the solutions to (6) with respect to the initial data respectively.

Lemma 1
Let equation (25) be well posed and its solutions ζ_t depend smoothly on the initial condition in the sense that the variational derivatives $\delta\zeta_t/\delta\zeta_0(x)$ and $\delta^2\zeta/\delta\zeta_0(x)\delta\zeta_0(y)$ exist as signed measures and are continuous bounded functions of x and y. Then the solutions $\mu_t=\zeta_t(x-\sigma_{com}W_t)$ to equation (6) also depend smoothly on the initial condition $\mu_0=\zeta_0$ and the variational derivatives are given by the formulas

$$\left(f,\frac{\delta\mu_t}{\delta\mu_0(x)}\right) = \frac{\delta}{\delta\mu_0(x)}(f,\mu_t) = \frac{\delta}{\delta\zeta_0(x)}(f(x-\sigma_{com}W_t),\zeta_t), \tag{31}$$

$$\left(f,\frac{\delta^2\mu_t}{\delta\mu_0(x)\delta\mu_0(y)}\right) = \frac{\delta^2}{\delta\zeta_0(x)\delta\zeta_0(y)}(f(x-\sigma_{com}W_t),\zeta_t). \tag{32}$$

Theorem 5
Let functions b,a,u satisfy the Conditions (C0)-(C3), σ_{com} be a constant $d\times d'$ matrix. Then the mapping $Y=\mu_0\mapsto\mu_t$ from Theorem 4 is continuously differentiable in Y, so that the derivative (30) is well defined as

a continuous function of two variables such that

$$\sup_x \sup_{\mu_0 \in \mathcal{M}^+_{\le\lambda}(\mathbf{R}^d)} \|\xi_t(x;.)[\mu_0]\| \le C(T, \lambda, \|Y\|) \tag{33}$$

for $t \in [0, T]$ uniformly for all values of W_t. Moreover, the variational derivatives $\xi_t(x;.)$ are twice differentiable in x weakly, as the functionals on the spaces of smooth functions, that is,

$$\partial\xi_t(x;.)/\partial x \in (C^1_\infty(\mathbf{R}^d))^*, \quad \partial^2\xi_t(x;.)/\partial x^2 \in (C^2_\infty(\mathbf{R}^d))^*,$$

and

$$\|\partial\xi_t(x;.)/\partial x\|_{(C^1_\infty(\mathbf{R}^d))^*} \le C(T, \lambda, \|Y\|),$$
$$\|\partial^2\xi_t(x;.)/\partial x^2\|_{(C^2_\infty(\mathbf{R}^d))^*} \le C(T, \lambda, \|Y\|), \tag{34}$$

again uniformly for all values of the noise W_t.

Theorem 6
(i) Let functions b, a, u satisfy the Conditions (C0)-(C3), σ_{com} be a constant $d \times d'$ matrix. Then the mapping $Y = \mu_0 \mapsto \mu_t$ from Theorem 4 is twice continuously differentiable in Y, so that the derivative $\eta_t(x, z;.)$ is well defined as a continuous function of three variables such that

$$\sup_x \sup_z \sup_{\mu_0 \in \mathcal{M}^+_{\le\lambda}(\mathbf{R}^d)} \|\eta_t(x, z;.)[\mu_0]\| \le C(T, \lambda, \|Y\|) \tag{35}$$

for $t \in [0, T]$ uniformly for all values of W_t.

Moreover, the derivatives of $\eta_t(x, z;.)$ with respect to x and z of order at most one are well-defined as elements of $(C^2(\mathbf{R}^d))^*$ and

$$\|\frac{\partial^\gamma}{\partial x^\gamma}\frac{\partial^\beta}{\partial z^\beta}\eta_t(x, z;.)\|_{(C^2(\mathbf{R}^d))^*} \le C(T, \lambda, \|Y\|) \tag{36}$$

for $\gamma, \beta = 0, 1$.

4. On the domain of the Markov semigroups generated by the McKean-Vlasov type SPDEs

The solutions of equation (6) define a Markov process, in fact a measure-valued stable-like process. The corresponding Markov propagator is given on the continuous functionals of measures in the usual way:

$$U^{s,t}F(\mu) = \mathbf{E}F(\mu_t(\mu, [W])), \tag{37}$$

where μ_t is the solution to (6) for $t > s$ with given $\mu = \mu_s$ at time s.

The main conclusion we need from the sensitivity analysis developed above is the invariance of the set of smooth functionals under this propagator, that is the following fact:

Theorem 7
Under assumption of Theorem 6 the spaces of functionals $C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ and its intersection with $C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ are invariant under the action of the operators (37), so that

$$\|U^{s,t}F\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \le C(T)\|F\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))}, \tag{38}$$

$$\|U^{s,t}F\|_{C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \le C(T)\left(\|F\|_{C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} + \|F\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))}\right) \tag{39}$$

with a constant $C(T)$.

Proof
Since

$$\frac{(\delta U^{s,t}(F))(v)}{\delta v(x)} = \mathbf{E}\frac{\delta F(v_t)}{\delta v(x)} = \mathbf{E}\int_{\mathbf{R}^d} \frac{\delta F(v_t)}{\delta v_t(z)}\xi_t(z;x)[v]\,dz,$$

it follows that

$$\|U^{s,t}F\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \le \mathbf{E}\left\|\int_{\mathbf{R}^d} \frac{\delta F(v_t)}{\delta v_t(z)}\xi_t(z;.)[v]\,dz\right\|_{C^2(\mathbf{R}^d)}$$

$$\le \mathbf{E}\left\|\frac{\delta F(v_t)}{\delta v_t(.)}\right\|_{C^2(\mathbf{R}^d)} \left\|\frac{\partial^2}{\partial x^2}\xi_t(.;x)[v]\right\|_{(C^2(\mathbf{R}^d))^*} \le C(T).$$

It follows from Theorem 6 and the formula

$$\frac{(\delta^2 U^{s,t}(F))(v)}{\delta v(x)\delta v(y)} = \mathbf{E}\frac{\delta^2 F(v_t)}{\delta v(x)\delta v(y)}$$

$$= \mathbf{E}\int_{\mathbf{R}^d} \frac{\delta F(v_t)}{\delta v_t(z)}\eta_t(z;x,y)[v_0]dz + \mathbf{E}\int_{\mathbf{R}^{2d}} \frac{\delta^2 F(v_t)}{\delta v_t(z)\delta v_t(w)}\xi_t(z;x)[v_0]\xi_t(w;y)[v_0]\,dzdw,$$

that

$$\|U^{s,t}F\|_{C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))} \le \mathbf{E}\left\|\frac{\delta^2 F(v_t)}{\delta v(.)\delta v(.)}\right\|_{C^{1\times 1}(\mathbf{R}^{2d})}$$

$$\le \left\|\frac{\partial^\alpha}{\partial x_1^\alpha}\frac{\partial^\beta}{\partial x_2^\beta}\eta_t(.;x_1,x_2)\right\|_{[C^2(\mathbf{R}^d)]^*} \|F\|_{C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))}$$

$$+\left\|\frac{\partial^\alpha}{\partial x^\alpha}\xi_t(.;x)[v]\right\|_{(C^1(\mathbf{R}^d))^*} \left\|\frac{\partial^\beta}{\partial y^\beta}\xi_t(.;y)[v]\right\|_{(C^1(\mathbf{R}^d))^*} \left\|\frac{\delta^2 F(v_t)}{\delta v_t(.)\delta v_t(.)}\right\|_{C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))},$$

leading to (39). □

5. Proof of Theorem 2

Let us return to our initial equation (1). Under the conditions (C0)-(C3) and the constant correlations σ_{com}, equation (1) is well-posed in $\mathbf{R}^{dN}$ and specifies a Feller stable-like process and the corresponding backward and forward propagators U_N, V_N given by (15), (16). We are interested in the limit of this stable-like process as $N \to \infty$.

Applying Ito's formula we obtain the generator of the stable-like process specified by (1):

$$A_N f(x_1,\cdots,x_N) = \sum_{j=1}^N (B_\mu^1)_i f(x_1,\cdots,x_N) + \sum_{\{i,j\}\subset\{1,\dots,N\}} (B_\mu^2)_{ij} f(x_1,\cdots,x_N), \tag{40}$$

where $\mu = (\delta_{x_1}+\cdots+\delta_{x_N})/N$, $(B_\mu^1)_i$ and $(B_\mu^2)_{ij}$ denote the actions of the operators B_μ^1 and B_μ^2 on the variables x_i and x_i, x_j respectively. Here B_μ^1 and B_μ^2 are the operators, depending on a measure μ as on a parameter and allowing for an additional mean field interaction:

$$B_\mu^1 g(x) = (b(x,\mu,u(t,x,\mu)),\nabla)g(x) + \frac{1}{2}(\sigma_{com}\sigma_{com}^T\nabla,\nabla)g(x) - a(x)|\Delta|^{\alpha/2}g(x), \tag{41}$$

$$B_\mu^2 g(x,y) = \left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)g(x,y). \tag{42}$$

Here and everywhere by a time-dependent generator, say A_N above, of a non-homogeneous Markov process we mean a time-dependent family of operators such that for f from some invariant dense subspace of bounded continuous functions the equation

$$\frac{d}{ds}U_N^{s,t}f = -A_N U_N^{s,t} f$$

holds for $s \le t$. In the case of the N-particle stable-like process, the invariant subspace can be usually taken to be the space of twice differentiable functions (which are invariant if σ and b are twice and once differentiable respectively). In the case of the limiting measure-valued process the invariant domains will be given by the subspaces $C^{2,k\times k}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$.

The first term in (40) can be considered as describing a stable-like process arising from the system of particles with a mean field interaction and the second term as giving an additional binary interaction (though not of a standard potential type that can be easily included in the mean field interaction).

By inclusion (17), the process specified by (40) can be equivalently considered as a measure-valued process defined on the set of linear combinations $\mathcal{P}_N(\mathbf{R}^d)$ of the Dirac atomic measures. On the level of propagators this correspondence arises from the identification of symmetric functions f on R^{dN} with the functionals $F = F_f$ on $\mathcal{P}_N(\mathbf{R}^d)$ via the equation

$$f(x_1, \cdots, x_N) = F_f[(\delta_{x_1} + \cdots + \delta_{x_N})/N].$$

To recalculate the generator (40) in terms of functionals F on measures we use the following simple formulas for differentiation of functionals on measures (proofs can be found e.g. in [21]): for $\mu = h(\delta_{x_1} + \cdots + \delta_{x_N})$ with $h = 1/N$

$$\frac{\partial}{\partial x_j}F(\mu) = h\frac{\partial}{\partial x_j}\frac{\delta F(\mu)}{\delta\mu(x_j)}, \tag{43}$$

$$\left(G\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right)F(\mu) = h\left(G\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right)\frac{\delta F(\mu)}{\delta\mu(x_j)} + h^2\left(G\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\delta^2 F(\mu)}{\delta\mu(y)\delta\mu(z)}\bigg|_{y=z=x_j}, \tag{44}$$

$$\left(\gamma\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)F(\mu) = h^2\left(\gamma\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\delta^2 F(\mu)}{\delta\mu(x_i)\delta\mu(x_j)}, \quad i \ne j. \tag{45}$$

Applying these formulas in conjunction with the obvious identity

$$h^2\sum_{i<j:i,j\in\{1,\dots,N\}}\phi(x_i, x_j) = \frac{1}{2}\int\int\phi(z_1, z_2)\mu(dz_1)\mu(dz_2) - \frac{h}{2}\int\phi(z, z)\mu(dz), \tag{46}$$

leads to the following expression of A_N in terms of $F(\mu)$:

$$A_N F(\mu) = \Lambda_{lim}F(\mu) + \frac{1}{N}\Lambda_{corr}F(\mu), \tag{47}$$

with

$$\Lambda_{lim}F(\mu) = \int_{\mathbf{R}^d}\left(B^1_\mu\frac{\delta F}{\delta\mu(.)}\right)(x)\mu(dx)$$
$$+\int_{\mathbf{R}^{2d}}\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\delta^2 F(\mu)}{\delta\mu(y)\delta\mu(z)}\mu(dy)\mu(dz), \tag{48}$$

and

$$\Lambda_{corr}F(\mu) = \frac{1}{2}\int_{\mathbf{R}^d}\left(\sigma_{com}\sigma_{com}^T\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\delta^2 F(\mu)}{\delta\mu(y)\delta\mu(z)}\bigg|_{y=z=x}\mu(dx)$$
$$+a(x)C_\alpha\int_0^1(1-s)ds\int_{\mathbf{R}^{2d}}\left(\frac{\delta^2 F}{\delta\mu(.)\delta\mu(.)}(\mu + \frac{s}{N}(\delta_{x+y} - \delta_x)), (\delta_{x+y} - \delta_x)^{\otimes 2}\right)$$
$$+a(x)C_\alpha\int_0^1(1-s)ds\int_{\mathbf{R}^{2d}}\left(\frac{\delta^2 F}{\delta\mu(.)\delta\mu(.)}(\mu + \frac{s}{N}(\delta_{x+y} - \delta_x)), (\delta_{x+y} - \delta_x)^{\otimes 2}\right)\frac{dy}{|y|^{d+\alpha}}\mu(dx). \tag{49}$$

Thus we have an explicit expression for the limit of A_N as $N \to \infty$ and for the correction term, which are well defined for functional F from the spaces $C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d)) \cap C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$.

It is straightforward to check by Ito's formula that the operator Λ_{lim} generates the measure-valued process defined by the solution of equation (1). Hence we have the convergence of the generators of N-particle approximations to the generator of the process given by (1) on the space $C^{1,2}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d)) \cap C^{2,1\times 1}(\mathcal{M}_{\le\lambda}(\mathbf{R}^d))$ with the uniform rate of convergence of order $1/N$.

But according to Theorem 7, the propagator of the process generated by (1) acts by bounded operators on this subspace. Hence Theorem 2 follows from the standard representation of the difference of two propagators in terms of the difference of their generators:

$$U_N^{t,r} - U^{t,r} = \int_t^r U_N^{t,s}(A_N - \Lambda_{lim})_s U^{s,r} ds. \tag{50}$$

6. Proof of Theorem 3

The well-posedness of the process on pairs (x, μ) solving equations (22) and (23) is straightforward once the well-posedness of the process solving (23) is proved, because equation (23) does not depend on x, and once it is solved, equation (22) is just a usual Ito's equation. Straightforward extension of the above calculations for the generator of the process solving (23) shows that the process solving (22)-(23) is generated by the operator

$$\Lambda_{lim} F(x, \mu) + \tilde{\Lambda}_{lim} F(x, \mu),$$

where Λ_{lim} is given by (48) and acts on the variable μ,

$$\tilde{\Lambda}_{lim} F(x, \mu) = (b(x, \mu, u_t^{ind}(x, \mu)), \nabla) F(x, \mu)$$
$$+ \frac{1}{2} \left(\sigma_{com}\sigma_{com}^T \nabla, \nabla\right) F(x, \mu) - a(x)|\Delta|^{\alpha/2} F(x, \mu) + \int \left(\sigma_{com}\sigma_{com}^T \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \frac{\delta F}{\delta \mu(y)} \mu(dy), \tag{51}$$

and with the same correction term (49). Thus the proof of Theorem 3 is the same as for Theorem 2.

7. Conclusion

Mean field games with common noise based on nonlinear stable-like processes are studied in this paper. We consider the common noise as a multidimensional noise with constant correlations. The MFG limit is specified by a single quasi-linear deterministic infinite-dimensional partial differential second order backward equation. The main result is that any solution of this equation provides an $1/N$-Nash equilibrium for the initial game of N agents. Our approach is based on interpreting the common noise as a kind of binary interaction of agents. We use our previous results on regularity and sensitivity with respect to the initial conditions of the solution to the nonlinear stochastic differential equations of McKean-Vlasov type generated by stable-like processes.

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