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Optimality of Reinsurance Treaties under a Mean-Ruin Probability Criterion

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Abstract The minimization of the probability of ruin is a crucial criterion for determining the effect of the form of reinsurance on the wealth of the cedant and is a very important factor in choosing optimal reinsurance. However, this optimization criterion alone does not generally lead to a rational decision for an optimal choice of reinsurance. This criterion acts only on the risk (minimizing it via the probability of ruin), but it does not affect the technical benefit. That is to say, the insurer should not choose the optimal reinsurance treaty if it is not beneficial. We propose a new reinsurance optimization strategy that maximizes the technical benefit of an insurance company while maintaining a minimal level for the probability of ruin. The objective is to optimize reinsurance with efficiency and ease of computation, using Genetic algorithms.

Keywords Adjustment coefficient; Genetic algorithms; Probability of ruin; Optimization; Technical benefit; Reinsurance.

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1. Introduction

The objective of reinsurance differs from one insurer to another insofar as an insurer can choose reinsurance to minimize the variance of its technical profit (De Finetti [21]), or minimize risk measures, such as Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) (Cai & Tan [17]), while another insurer chooses reinsurance which allows it to minimize its probability of ruin (Schmidli [47]).

Indeed, the criterion of minimizing the probability of ruin is a very important factor in the choice of optimal reinsurance, it has a significant impact on the stabilization and terminal wealth of the insurance company M.Kaluszka & A.Okolewski [35], S.Luoa & al [46].

In this context, several studies have been developed to study the effect of the probability of ruin on the optimal choice of reinsurance. Centeno [19] used the Panjer algorithm to calculate the probability of ruin in order to optimize reinsurance. Dickson & Waters [24], Aase [1], Krvavych [32], Schmidli [47], Deelstra & Plantin [22] chose to seek optimality by the criterion of minimizing the probability of ruin based on the Cramer-Lundberg model.

In addition, Schmidli [47] and several other authors have considered the reinsurance treaties as optimal if and only if they make it possible to maximize the Lundberg adjustment coefficient from the Cramer-Lundberg approximation G.Willmot & X.Lin [26].

In these approaches, the authors only act on the probability of ruin via maximizing the Lundberg adjustment coefficient to determine the optimal reinsurance treaty parameters. However, several authors, such as Ben Dbabis [15], have pointed out that maximizing the adjustment coefficient alone does not generally lead to a rational

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decision for an optimal choice of reinsurance, because this criterion acts only on the risk, but not on the technical benefit, ie the insurer should not choose the optimal reinsurance treaty if it is not profitable.

So it must also refer to a new method that acts on minimizing the probability of ruin and maximizing the technical benefit, both at the same time, by seeking both optimal reinsurance treaty parameters and adjustment factors that maximize the technical benefit and minimize the risk of an insurance company.

Therefore, it must also refer to a new method that acts on the minimization of the probability of ruin (by maximizing the adjustment coefficient) and on maximizing the technical benefit both at the same time and by seeking the parameters of the reinsurance treaties and the optimal adjustment coefficient; which maximize the technical benefit and minimize the risk of an insurance company at the same time.

In this paper we propose a new reinsurance optimization strategy that maximizes the technical benefit of an insurance company while maintaining a minimal level for the probability of ruin, using Genetic algorithms Davis.L [20]. the latter, contrary to the deterministic methods (classical Lagrangian and Kuhn-Tucker theorem), use only the value of the studied function, not its derivative nor any other auxiliary knowledge, which makes it easier to implement their effectiveness for more complex optimization problems.

The organization of this paper is as follows: in the second section, we start with a preliminary, then in the third section, we formulate our optimization problem for the different cases of reinsurance treaties, then we propose the optimization procedure by the Augmented Lagrangian method and the Genetic Algorithms. Finally, in the last section, we illustrate our model of optimization by a sample application.

2. Preliminary

Let a portfolio of claims expenses be represented by continuous and positive random variables $X_1, \ldots, X_N$, with distribution functions $F_1, \ldots, F_N$ and density functions $f_1, \ldots, f_N$ and corresponding to the premiums $P_1, \ldots, P_N$, with $E(X_i) = \mu$.

The risks are considered independent and identically distributed, and independent of N .

With the sum of$(X_i)_{i=1,\ldots,N}$being zero, if $N = 0$.

For a reinsurance contract, a risk X_i is defined by:

$$X_i = X_i^A + X_i^R \quad , \quad \forall i = [1, ..., N] \tag{1}$$

The componentX_i^A is the insurer's claims burden andX_i^R s the claims burden transferred to the reinsurer.

The reinsurer's charge shall in no way exceed the total claims burden, as it should not be negative, i.e. $0 < X_i^R \leq X_i \;\; , \quad \forall i = [1, ..., N]$.

The ruin probability is the probability that the total cost of claims exceeds the corresponding collection over time. Ruin occurs when reserves fall below 0.

The reserve of the insurance company at the moment $t$is given by the following random function:

Cramer-Lundberg has modeled the reserve of the insurance company by the following risk process**:**

$$R(u,t) = u + Ct - \sum_{i=1}^{N} X_i^A \tag{2}$$

or

- $\left(X_i^A\right)_{i \in [1,\ldots,N]}$ are random variables (i.i.d) non-negative, represent expenses claims of the insurer;
- N the number of claims in $(0, t]$ is a Poisson process independent of X_i with rate λ;
- C is the recipe for the premium collected at the moment t.

The probability of ruin $\psi(u,t)$for finite horizon can be written as follows:

$$\psi(u,t) = P\left(\exists t \in [0,T], R(u,t) = u + Ct - \sum_{i=1}^{N(t)} X_i < 0\right) \quad , \quad 0 < T \;\; < \infty \tag{3}$$

And for an infinite horizon, the probability of ruin can be written as follows:

$$\psi(u,t) = P\left(\exists t \geq 0, R(u,t) = u + Ct - \sum_{i=1}^{N(t)} X_i < 0\right) \tag{4}$$

Note. The surplus process can be naturally continuous, and can also be considered in discrete time;

We focus here on the static approach where the reserve of the insurance company is defined by the following surplus process:

$$R(u) = u + NC - S_N^A \tag{5}$$

With

- N is the number of claims, independent of X_i;
- $S_N^A = \sum_{i=1}^{N} X_i^A$ is the sum of the claims expenses of the insurer;
- C is the rate of return premiums (the recipe for the premium collected at the moment) t;

The recipe for the premium collected C is obtained by subtracting premiums charged from insurance, reinsurance premiums for a period.

In this case the probability of ruin is given as follows:

$$\psi(u) = P\left(\exists N > 0, R_N(u,\lambda) = u + C(\lambda)N - \sum_{i=1}^{N} X_i^A \leq 0\right) \tag{6}$$

The probability of ruin is evaluated by several approaches, numerical or by simulations, namely, exact solutions (Erlang model, $\cdots$), numerical methods (inverse Laplace transform, differential and integral equations, etc), and the approximations (composite Poisson model, Lundberg inequality, etc).

Recently, several works have been developed to find approximations of the probability of ruin by improving the previous approximations. To know: M.Longué & Y.Darmaillac [36], which were able to give an approximation of the Ruin probability using the ARMA model. P.Goffard [38] have proposed polynomial approximations of probability densities to apply them in insurance.

In this work we consider the Lundberg inequality treated in Schmidli [47] which is the most used in practice and adequately adequate with our problem:

Schmidli [47] has demonstrated that the probability of ruin $\psi(u)$which may be increased by an upper boundary known as the Lundberg boundary, such as :

$$\psi(u) = P\left(\exists N > 0, R_N(u,\lambda) = u + C(\lambda)N - \sum_{i=1}^{N} X_i^A \leq 0\right) \leq e^{-\rho u} \tag{7}$$

With $\rho > 0$ the Lundberg fitting coefficient and being the unique root for the following equation:

$$e^{-C(\lambda)\rho} E\left[e^{\rho X_i^A}\right] = e^{-C(\lambda)\rho} M_{X_i^A}(\rho) = 1, \ \ \forall i \in [1, ..., N] \tag{8}$$

Or $M_{X_i^A}$ is the generating function of the moments for the random variable X_i^A.

According to the Lundberg inequality, the probability of ruin is minimal if the adjustment coefficient is maximal.

Schmidli [47] determined an optimal choice of reinsurance based on the maximization criterion of the Lundberg adjustment coefficient. However, the insurance company is still seeking to realize higher profits (Ben Dbabis [15]). So to make a good decision, it must be melted on the criteria of profitability (maximizing the expected technical benefit) while keeping the condition of the adjustment factor which must be as maximal.

In this work, we look at maximizing the technical benefit of an insurance company while maintaining a minimal level for the probability of ruin (by maximizing the Lundberg adjustment coefficient). The objective is to optimize the efficiency and ease of calculation, using genetic algorithms.

3. Formulation of optimization problem

We then construct the following optimization program which maximizes both the expected technical benefit of the cedant and minimizes its probability of ruin (via the maximum setting of the adjustment coefficient):

$$\begin{cases} \max_{(\lambda,\rho)} \langle Z(X,\lambda) = E(B(X,\lambda))\rangle \\ s.t \quad \left\langle \begin{array}{ll} e^{-C(\lambda)\rho} E\left[e^{\rho X_i^A}\right] = 1 & , \quad \forall i \in [1,...,N] \\ \lambda \in [\lambda^-, \lambda^+] & \\ \rho > 0 & \end{array} \right. \end{cases} \tag{9}$$

For instances $E(B(X,\lambda))$ represents the expected technical benefit of the ceding company.
Let $\eta^r > 0$ and $\eta > 0$ respectively, the reinsurer's and the insurer's security charges, such as $\eta^r > \eta$.
Assume that the reinsurance premium uses the principle of premium based on mathematical expectation with a safety load η^r, i.e.

$$\Pi\left(X_i^R\right) = (1+\eta^r) E\left(X_i^R\right). \forall i = [1,...,N] \tag{10}$$

The technical benefit of the ceding company is defined by:

$$E(B(X,\lambda)) = \sum_{i=1}^{N}\left[P_i - \prod\left(X_i^R\right) - E\left(X_i^A\right)\right] = \sum_{i=1}^{N}\left[P_i - (1+\eta^r) E\left(X_i^R\right) - E\left(X_i^A\right)\right] \tag{11}$$

The parameter λ is the reinsurance treaty parameter applied to cover the risk of loss. This parameter represents the transfer rate in the case of proportional reinsurance, and the retention limit in the case of non-proportional reinsurance.
λ^- and λ^+ are respectively the lower bound and the upper bound of the reinsurance treaties parameter.
We will address the above optimization problem in different forms of reinsurance cases.

3.1. Case of treaties in "quota share"

In the case of proportional reinsurance of the "quote part" type with a proportionality factor $\alpha \in \ ]0,1]$ constant, the insurer supports the portion $X_i^A = (1-\alpha) X_i$ and the portion $X_i^R = \alpha X_i$ transferred to the reinsurer.
The premium charged by the reinsurer for a period i is given by:

$$C_r(\alpha) = \Pi\left(X_i^R\right) = (1+\eta^r)\alpha E(X_i) = (1+\eta^r)\alpha\mu \ , \quad \forall i \in [1,...,N] \tag{12}$$

Then the premium charged to the insurer after the reinsurance for a period i is given by:

$$C(\alpha) = (1+\eta) E(X_i) - C_r(\alpha) = \mu((1+\eta) - \alpha(1+\eta^r)) \ , \quad \forall i \in [1,...,N] \tag{13}$$

The static surplus process is given by the following formula:

$$\begin{aligned} R_N(u,\alpha) &= u + C(\alpha) N - \textstyle\sum_{i=1}^N X_i^A \\ &= u + \mu N((1+\eta) - \alpha(1+\eta^r)) - \textstyle\sum_{i=1}^N X_i^A \end{aligned}$$

Then the expected reserve is given by:

$$\begin{aligned} E(R_N(u,\alpha)) &= u + \mu N((1+\eta) - \alpha(1+\eta^r)) - E\left(\textstyle\sum_{i=1}^N X_i^A\right) \\ &= u + \mu N((1+\eta) - \alpha(1+\eta^r)) - N(1-\alpha)\mu \\ &= u + \mu N(\eta - \alpha\eta^r) \end{aligned} \tag{14}$$

The proportionality factor α must check the following safety condition:

$$E\left(R_N\left(u,\alpha\right)-u\right)>0 \Leftrightarrow \alpha<\frac{\eta}{\eta^r} \tag{15}$$

The Lundberg adjustment coefficient ρ is always the positive solution of the following equation:

$$e^{-C(\lambda)\rho}M_{X_i^A}\left(\rho\right)=1 \quad \Leftrightarrow \quad \left(M_{X_i}\left(\rho\right)-1\right)-\mu\left(\left(1+\eta\right)-\alpha\left(1+\eta^r\right)\right)\tfrac{\rho}{1-\alpha}=0 \ \ ,\forall i\in\left[1,...,N\right] \tag{16}$$

Or $M_{X_i}\left(\rho\right)$ is the generating function of the moments of the random variable X_i evaluated at ρ.
(See Hald & Schmidli [30]).
Let us now calculate the mathematical expectation of the technical benefit:

$$\begin{aligned} E\left(B\left(X,\alpha\right)\right) &= \textstyle\sum_{i=1}^{N}\left[P_i-\Pi\left(X_i^R\right)-E\left(X_i^A\right)\right] \\ &= \textstyle\sum_{i=1}^{N}\left[P_i-\left(1+\eta^r\right)\alpha E\left(X_i\right)-\left(1-\alpha\right)E\left(X_i\right)\right] \\ &= P-N\left(1+\eta^r\alpha\right)\mu \end{aligned} \tag{17}$$

Finally, the optimization program (9) is reformulated in this case as follows:

$$\begin{cases} \max_{(\alpha,\rho)}\left\langle E\left(B\left(X,\alpha\right)\right)=P-N\left(1+\eta^r\alpha\right)\mu\right\rangle \\ sc \quad \left\langle \begin{array}{l} \left(M_{X_i^A}\left(\rho\right)-1\right)-\mu\left(\left(1+\eta\right)-\alpha\left(1+\eta^r\right)\right)\frac{\rho}{1-\alpha}=0 \qquad , \quad \forall i\in\left[1,...,N\right] \\ \alpha\in\left]0,\frac{\eta}{\eta^r}\right[\\ \rho>0 \end{array}\right. \end{cases} \tag{18}$$

3.2. Case of treaties in "excess of loss"

In this case, the insurer covers each individual claim up to a certain level of retention $L>0$ (the retention limit applies to each individual claim).
Let S be a random variable denoting the total amount of claims; $X^A=\sum_{i=1}^{N}X_i^A$ is the insurer's claims burden and $X^R=\sum_{i=1}^{N}X_i^R$ is the claims burden transferred to the reinsurer.
Then, the total charges are shared as follows:

$$X^A=\sum_{i=1}^{N}X_i^A=\sum_{i=1}^{N}\min\left(X_i,L\right) \text{ and } X^R=\sum_{i=1}^{N}X_i^R=\sum_{i=1}^{N}\left(\left(X_i-L\right)_+\right).\text{ With } S=X^A+X^R \tag{19}$$

The premium charged for reinsurance for a period i is given by:

$$C_r\left(L\right)=\Pi\left(X_i^R\right)=\left(1+\eta^r\right)E\left(\left(X_i-L\right)_+\right) \quad , \quad \forall i\in\left[1,...,N\right] \tag{20}$$

Then the premium charged to the insurer after the reinsurance for a period i is given by:

$$C\left(L\right)=\left(1+\eta\right)E\left(X_i\right)-C_r\left(L\right)=\left(1+\eta\right)\mu-\left(1+\eta^r\right)E\left(\left(X_i-L\right)_+\right) \quad , \quad \forall i\in\left[1,...,N\right] \tag{21}$$

We know that

$$E\left(X_i^A\right)=E\left[\min\left(X_i,L\right)\right]=\int_0^L xdF_{X_i}\left(x\right)+LS_{X_i}\left(\beta\right)=\int_0^L S_{X_i}\left(x\right)dx \quad , \quad \forall i\in\left[1,...,N\right] \tag{22}$$

and

$$E\left(X_i^R\right)=E\left(\left(X_i-L\right)_+\right)=E\left(X_i\right)-E\left(X_i^A\right)=\mu-\int_0^L S_{X_i}\left(x\right)dx \quad , \quad \forall i\in\left[1,...,N\right] \tag{23}$$

Then, the recipe of the premium for each period is given by:

$$\begin{aligned} C(L) &= (1+\eta)\mu - (1+\eta^r) E\left((X_i - L)_+\right) \\ &= (1+\eta)\mu - (1+\eta^r)\left(\mu - \int_0^L S_{X_i}(x)\,dx\right) \quad ,\forall\, i \in [1,\cdots,N] \\ &= (\eta - \eta^r)\mu + (1+\eta^r)\int_0^L S_{X_i}(x)\,dx \end{aligned} \tag{24}$$

The surplus process is given by the following formula:

$$\begin{aligned} R_N(u,L) &= u + C(L)N - X^A \\ &= u + N\left[(\eta-\eta^r)\mu + (1+\eta^r)\int_0^L S_{X_i}(x)\,dx\right] - \sum_{i=1}^{N} X_i^A \end{aligned} \tag{25}$$

Then, the expected reserve is given by:

$$\begin{aligned} E(R_N(u,L)) &= u + N\left[(\eta-\eta^r)\mu + (1+\eta^r)\int_0^L S_{X_i}(x)\,dx\right] - \sum_{i=1}^{N} X_i^A \\ &= u + N\left[(\eta-\eta^r)\mu + (1+\eta^r)\int_0^L S_{X_i}(x)\,dx\right] - N\int_0^L S_{X_i}(x)\,dx \\ &= u + N\left[(\eta-\eta^r)\mu + \eta^r\int_0^L S_{X_i}(x)\,dx\right] \end{aligned} \tag{26}$$

Let us now calculate the mathematical expectation of the technical benefit:
The expected technical benefit is given by:

$$\begin{aligned} E(B(X,L)) &= E\left(\sum_{i=1}^{N}\left[P_i - \prod\left(X_i^R\right) - X_i^A\right]\right) \\ &= \sum_{i=1}^{N}\left[P_i - (1+\eta^r)E\left((X_i-L)_+\right) - E\left(X_i^A\right)\right] \\ &= P - \sum_{i=1}^{N}\left[(1+\eta^r)\left(\mu - \int_0^L S_{X_i}(x)\,dx\right) + \int_0^L S_{X_i}(x)\,dx\right] \end{aligned} \tag{27}$$

Hence, the expected technical benefit is given by:

$$E(B(X,L)) = P - \sum_{i=1}^{N}\left[(1+\eta^r)\mu - \eta^r\int_0^L S_{X_i}(x)\,dx\right] \tag{28}$$

On the other hand, the Lundberg fitting coefficient ρ is always the only root for the following equation:

$$e^{-C(L)\rho}E\left[e^{\rho X_i^A}\right] = 1, \forall i \in [1,\cdots,N] \tag{29}$$

Then the optimization program (9) is reformulated as follows:

$$\begin{cases} \max_{(L,\rho)}\left\langle E(B(X,L)) = P - \sum_{i=1}^{N}\left[(1+\eta^r)\mu - \eta^r\int_0^L S_{X_i}(x)\,dx\right]\right\rangle \\ sc \quad \left\langle \begin{array}{l} e^{-\left((\eta-\eta^r)\mu + (1+\eta^r)\int_0^L S_{X_i}(x)dx\right)\rho} M_{X_i^A}(\rho) = 1 \\ L \in [L^-, L^+] \\ \rho > 0 \end{array}\right. \end{cases} \quad , \quad \forall i \in [1,\cdots,N] \tag{30}$$

Or $L^- > 0$ and $L^+ > 0$ are two parameters that respectively represent the lower and upper limit of the retention, depending on the market reinsurance (the liquidity interval).

3.3. Case of treaties in "stop loss"

Non-proportional reinsurance of "stop loss" type operates similarly to "excess loss" reinsurance. In this case, the retention limit L is applicable to the total of the claims.

Let S be a random variable denoting the total amount of claims; $X^A = \sum_{i=1}^{N} X_i^A$ is the insurer's claims burden and $X^R = \sum_{i=1}^{N} X_i^R$ is the claims burden transferred to the reinsurer.

In this case, the insurer bears the risk $X^A = \min(X, L)$ and the part $X^R = (X-L)_+$ transferred to the reinsurer. The premium charged by the reinsurer for the period X is given by:

$$C_r(L) = \Pi\left(X^R\right) = (1+\eta^r) E\left((S-L)_+\right) \tag{31}$$

We know that

$$E\left(X^A\right) = E\left[\min(S, L)\right] = \int_0^L S_S(x)\, dx \tag{32}$$

Then

$$C_r(L) = (1+\eta^r)\left[E(S) - \int_0^L S_S(x)\, dx\right] \tag{33}$$

Then the premium charged to the insurer after the reinsurance for a period N is given by:

$$C(L) = (1+\eta) E(S) - C_r(L) = (1+\eta) E(S) - (1+\eta^r)\left[E(S) - \int_0^L S_S(x)\, dx\right]$$

$$C(L) = (\eta - \eta^r) E(S) + (1+\eta^r)\int_0^L S_S(x)\, dx \tag{34}$$

The surplus process in this case is given by:

$$R_N(u, L) = u + C(L) - X^A$$

The surplus process in this case is given by:

$$\begin{aligned} E\left(R_N(u, L)\right) &= u + C(L) - E\left(X^A\right) \\ &= u + (\eta - \eta^r) E(S) + \eta^r \int_0^L S_S(x)\, dx \end{aligned} \tag{35}$$

Let us now calculate the expected technical benefit:
we have

$$\begin{aligned} E\left(B(S, L)\right) &= P - \prod\left(X^R\right) - E\left(X^A\right) \\ &= P - (1+\eta^r)\left(E(S) - \int_0^L S_S(x)\, dx\right) - \int_0^L S_S(x)\, dx \end{aligned}$$

Simplifying, then we find:

$$E\left(B(S, L)\right) = P - \left((1+\eta^r) E(S) - \eta^r \int_0^L S_S(x)\, dx\right) \tag{36}$$

The Lundberg adjustment coefficient ρ is the only root for the following equation:

$$e^{-C(L)\rho} E\left[e^{\rho X^A}\right] \equiv e^{-C(L)\rho} M_{X^A}(\rho) = 1 \quad \Rightarrow \quad e^{-\left((\eta-\eta^r)E(S) + (1+\eta^r)\int_0^L S_S(x)dx\right)\rho} M_{X^A}(\rho) = 1 \tag{37}$$

Then the optimization program (9) is reformulated as follows:

$$\begin{cases} \max_{(L,\rho)} \left\langle E\left(B(S, L)\right) = P - \left((1+\eta^r) E(S) - \eta^r \int_0^L S_S(x)\, dx\right)\right\rangle \\ sc \quad \left\langle \begin{array}{l} e^{-\left((\eta-\eta^r)E(S) + (1+\eta^r)\int_0^L S_S(x)dx\right)\rho} M_{X^A}(\rho) = 1 \\ L \in [L^-, L^+] \\ \rho > 0 \end{array} \right. \end{cases} \tag{38}$$

4. Procedure for Optimization by Genetic algorithms

We rely on genetic algorithms to solve our optimization problem. Nevertheless, the use of the latter is conditioned by certain characteristics. For example: the instructions of the optimization program must be realizable.
To make our program workable, we transformed the constrained optimization problem into an unrestrained optimization problem, using the Augmented Lagrangian.
Indeed, our optimization program is in the following general form:

$$\begin{cases} Min_{\lambda,\rho}\ Z(\lambda) \\ \quad K(\lambda,\rho) = 0 \\ \lambda^{\min} \leq \lambda \leq \lambda^{\max} \\ \rho > 0 \end{cases} \tag{39}$$

Or

- $Z(\lambda)$ is the objective function;
- $K(\lambda,\rho)$ is the function of the equality constraint;
- $\lambda \in \left[\lambda^{\min}, \lambda^{\max}\right]$and $\rho > 0$ are domain constraints that limit variations of unknowns λ and ρ.

The Lagrangian function $G(\lambda,\rho,m)$ is defined as follows:

$$G(\lambda,\rho,m) = Z(\lambda) - mK(\lambda,\rho) \tag{40}$$

Or m is the multiplier of Lagrange.
The principle of the method is to solve iteratively the problem without constraints which minimizes the Lagrangian function $G(\lambda,\rho,m)$.
$G(\lambda,\rho,m)$ must be minimized in relation to λ and ρ .

We have developed an iterative algorithm that combines the Lagrange multiplier and genetic algorithms to solve our optimization problem.

Algorithm 3.1. *The solution algorithm*

1. Create Augmented Lagrangian function$G(\lambda,\rho,m)$;
2. Initialize the Lagrange multiplier $m = m_0$;
3. Initialize the couple $(\lambda,\rho) = (\lambda_0,\rho_0)$;
4. i=0;
5. As long as the stopping criterion is not verified, i.e.$\nabla G_i(\lambda_i,\rho_i,m_i) > \varepsilon$, with $\varepsilon > 0$;

   a. $i \leftarrow i+1$;
   b. Create the objective function $G_i(\lambda_i,\rho_i,m_i)$;

6. Run Classic GA for the objective function $G_i(\lambda_i,\rho_i,m_i)$;
7. Update the triple (λ^*,ρ^*,m^*) which minimizes the function $G_i(\lambda_i,\rho_i,m_i)$;
8. Return the result (λ^*,ρ^*).

5. Application

Suppose individual losses $(X_i)_{i=1,\dots,N}$ follow the Uniform Law $U[0,1]$.
such as $E(X_i)=0,5$, $M_{X_i}(\rho)=E\left(e^{\rho X_i}\right)=\int_0^1 e^{\rho X_i}dx=\frac{e^{\rho}-1}{\rho}$ et $S_{X_i}(x)=1-x, \forall i\in\ [1,\cdots,N]$.
Let the following data be:

- $N=9$ (a horizon of 9 years) ;
- $\eta^r=2$ and $\eta=1$ are respectively the security loading of the reinsurer and the insurer ;
- The initial capital is equal to $u=786$;
- The total premium collected is equal $P=\sum_{i=1}^{N}P_i=20$.

5.1. Case of treaties in "quota share"

In the case of a "quota share" reinsurance treaty where the proportionality factor α, we have:

$$E(B(X,\alpha))=P-N(1+\eta^r\alpha)\mu=20-9\frac{(1+2\alpha)}{2}=15,5-9\alpha$$

and

$$\left(M_{X_i^A}(\rho)-1\right)-\mu\left((1+\eta)-\alpha(1+\eta^r)\right)\frac{\rho}{1-\alpha}=0 \quad\Leftrightarrow\quad \rho\left(\frac{e^{\rho}-1}{\rho}-1\right)-\frac{(2-3\alpha)}{2}\rho=1$$

We obtain the following optimization program:

$$\begin{cases}\max_{(\alpha,\rho)}\left\langle E(B(X,\alpha))=15,5-9\alpha\right\rangle \\ sc\ \left\langle \begin{array}{l}\left(\frac{e^{\rho}-1}{\rho}-1\right)-\frac{(2-3\alpha)}{2}\rho=1\\ \alpha\in\left]0,\frac{1}{2}\right[\\ \rho>0\end{array}\right.\end{cases}$$

To solve the optimization program above, we applied the previous procedure using the Genetic algorithms solver (GA) developed by Matlab software.
The result is as follows:

- The optimal objective function, $E(B(X,\alpha^*))=15,5$;
- The best session rate, $\alpha* = 2,354.10^{-8}$;
- The best solution for the coefficient of adjustment, $\rho* = 23,319$.

5.2. Case of treaties in "excess of loss"

Assume that the insurer selects the "excess of loss" type of retention L to cover the risk of loss. In this case we have:

$$M_{X_i^A}(\rho)=E\left(e^{\rho X_i^A}\right)=\int_0^L e^{\rho X_i}dx+\int_L^1 e^{\rho L}dx=\frac{e^{\rho L}-1}{\rho}+e^{\rho L}(1-L)$$

The adjustment coefficient is the positive solution of the following equation:

$$e^{-\left((\eta-\eta^r)\mu+(1+\eta^r)\int_0^L S_{X_i}(x)dx\right)\rho}M_{X_i^A}(\rho)=1\Leftrightarrow\left(\frac{e^{\rho L}-1}{\rho}+e^{\rho L}(1-L)\right)e^{-\rho\left(1-\frac{3(L-1)^2}{2}\right)}=1$$

The expected technical benefit is given by:

$$E(B(X,L))=P-N\left[(1+\eta^r)\left(\mu-\int_0^L S_{X_i}(x)\,dx\right)+\int_0^L S_{X_i}(x)\,dx\right]=6,5+18\left(L-\frac{L^2}{2}\right)$$

Then, we have the following optimization program:

$$\begin{cases} \max_{(L,\rho)} \left\langle E\left(B\left(X,L\right)\right) = 6,5 + 18\left(L - \frac{L^2}{2}\right)\right\rangle \\ sc \left\langle \begin{array}{l} \left(\frac{e^{\rho L}-1}{\rho} + e^{\rho L}\left(1-L\right)\right) e^{-\rho\left(1-\frac{3(L-1)^2}{2}\right)} = 1 \\ L > 0 \\ \rho > 0 \end{array} \right. \end{cases}$$

The result obtained by the Genetic algorithms is:

- The optimal objective function, $E\left(B\left(X,L^{*}\right)\right) = 15,499$;
- The optimal retention limit, $L^{*} = 1$;
- The best solution for the coefficient of adjustment, $\rho^{*} = 0,0001$.

The results for the technical benefit and the adjustment coefficient for the two forms of reinsurance are presented in the following table:

	quota share	excess of loss
Maximum technical benefit	$15,5$	$15,499$
Parameter of optimal reinsurance treaties	$\alpha* = 2,354.10^{-8}$	$L^{*} = 1$
Maximum adjustment coefficient ρ *	$23,319$	$0,0001$

Table 1: Results relative to technical benefit and adjustment coefficient

From Table 1 it can be seen that Lundberg's adjustment coefficient is higher in the case of proportional reinsurance of the "quote share" type than in the case of non-proportional reinsurance of the "excess loss" type with a relatively higher expected technical profit.
Thus, we can conclude from in this example that for a principle of premium of mathematical expectation, the optimal form of reinsurance adopted is the proportional treaty of the "quota share" type.
However, the choice of an optimal form of effective reinsurance obviously depends on the strategy adopted by the insurance company which differs from one insurer to another and it also has several factors: The nature of the portfolio (distribution of claims amounts, independence of occurrence of claims, etc.), we can also take into account the initial assumptions and the statistics observed by the insurance company.

6. Conclusion

In this chapter, we presented a new strategy for the choice of optimal reinsurance under the "Mean-Ruin probability" criterion, using genetic algorithms. This approach makes it possible both to maximize the technical benefit and to minimize the probability of ruin (by maximizing the Lundberg adjustment coefficient).
This strategy is effective because it affects both technical profit and risk at the same time, including the ease of calculation loads and faster execution.

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