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Strictly φ-sub-Gaussian quasi shot noise processes

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Abstract In the paper, strictly φ-sub-Gaussian quasi shot noise processes are considered. There are obtained estimates for distribution of supremum of such a process defined on a compact set and formulated conditions for its sample functions continuity with probability one.

Keywords Short noise processes, φ-sub-Gaussian processes.

AMS 2010 subject classifications 60G07, 60G17, 60F10, 60H05, 60K40.

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Introduction

Studying properties of distribution of supremum of a stochastic process, problems of existence of moments and exponential moments of suprema, characteristics of sample functions, the behaviour of stochastic processes in different functional spaces are topical subjects in the theory of stochastic processes for many years. Classical results can be found in books by Cramer and Lidbetter (1967) [5], Lidbetter, Lindgren and Rootzen (1983) [28], Adler (1990) [1], Ledoux and Talagrand (1991) [29], Buldygin and Kozachenko (1998) [3], Bulinski and Shiryaev (2005) [4]. A lot of papers contain results on estimation of exponential moments and distribution of suprema of Gaussian processes, in particular, papers by Skorokhod (1970) [32], Landau and Shepp (1970) [27], Ledoux and Talagrand (1991) [29], Liphshits (1995) [30], Yurinsky (1995) [33]. In the end of sixties, there appeared papers, in which wider classes of random variables and processes, than Gaussian one, were studied. In the paper by Kahane (1960) [12] sub-Gaussian random variables were introduced. In 1968 in the paper [13] Kozachenko introduced a notion of sub-Gaussian random process. Properties of such processes were studied, in particular, by Buldygin (1977) [2], Fukuda (1990) [9], Ostrovsky (1991) [31].

In 1985 in the paper by Kozachenko and Ostrovsky [15] there were considered spaces $\mathrm{Sub}_\varphi(\Omega)$ of φ-sub-Gaussian random variables and processes, which appeared to be natural generalization of sub-Gaussian spaces. Buldygin and Kozachenko in [3] presented fundamental properties of random variables and processses from $\mathrm{Sub}_\varphi(\Omega)$, conditions for boundedness and estimation of distribution of suprema of such processes in some special cases. Further development of the theory of φ-sub-Gaussian random processes was presented, in particular, in [11, 16, 21, 22, 24, 25]. Since φ-sub-Gaussian random processes are more general, than Gaussian and sub-Gaussian processes, they can be used for simulation of real random processes in queueing theory and financial mathematics. In particular, fractional Brownian motion belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ with $\varphi(x) = \frac{x^2}{2}$. Examples of application and simulation of φ-sub-Gaussian random processes can be found in [17, 18, 19, 20].

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In this paper, some results received for φ-sub-Gaussian random processes are applied to shot noise processes. Shot noise processes serve as mathematical model of various physical phenomena. They are used in electronics, telecommunications, mesoscopic physics.

In [3, 6] there was considered a real-valued homogeneous zero-mean process $\xi = (\xi(t), t \in \mathbb{R})$ with independent increments, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and a real-valued function $g = (g(t,u), t, u \in \mathbb{R})$ satisfying the condition $\int_{-\infty}^{+\infty} g^2(t,u)\mathrm{d}u < \infty, \quad t \in \mathbb{R}$. Then the process $X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u), \quad t \in \mathbb{R}$, was called a *shot noise process generated by the process* ξ *and the response function* g. In the papers [7, 8] and monographs [3, 6] properties of pre-Gaussian shot noise processes were studied and estimates for distribution of suprema for such processes were received.

I shall consider the case, when $\xi = (\xi(t), t \in \mathbb{R})$ is a real-valued zero-mean random process with *uncorrelated* increments defined on a standard probability space such that

$$E(\xi(t) - \xi(s))^2 = t - s, \quad t > s \in \mathbb{R}.$$

If ξ is a strictly φ-sub-Gaussian random process, we shall call the process $X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u)$ *strictly φ-sub-Gaussian quasi shot noise process*.

Basic facts and some properties of the φ-sub-Gaussian random processes are presented in section 1 of this paper. Section 2 contains lemmas, which enable us to consider strictly φ-sub-Gaussian quasi shot noise processes. In section 3, estimates for distribution of suprema of such a process defined on a compact set are obtained and conditions for its sample functions continuity with probability one are formulated.

1. Some results from the theory of φ-sub-Gaussian random processes

Let's recall some basic facts about the space $\mathrm{Sub}_\varphi(\Omega)$ of generalized sub-Gaussian random variables [3, 11, 25].

1.1. Space $\mathrm{Sub}_\varphi(\Omega)$ of φ-sub-Gaussian random variables

Definition 1.1. A continuous even convex function $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ is an *Orlicz N-function* if it is increasing for $x > 0$, $\frac{\varphi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\varphi(x)}{x} \to \infty$ as $x \to \infty$.

Definition 1.2. The *Young-Fenchel transformation* φ^* of an Orlicz N-function $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ is defined as follows

$$\varphi^*(x) := \sup_{y>0} (xy - \varphi(y)), \quad x \geq 0.$$

Lemma 1.3. *[26] The Young-Fenchel transformation of an Orlicz N-function also is N-function and the following inequality holds:*

$$xy \leq \varphi(x) + \varphi^*(y), \quad \text{where} \quad x > 0, y > 0. \tag{1.1}$$

Example 1.4. If $\varphi(x) = \frac{|x|^p}{p}$, $p > 1$, then $\varphi^*(x) = \frac{|x|^q}{q}$, where q is such number that $\frac{1}{q} + \frac{1}{p} = 1$.

Example 1.5. If $\varphi(x) = \exp\{|x|\} - |x| - 1$, then $\varphi^*(x) = (|x| + 1)\ln(|x| + 1) - |x|$.

For more details on properties of convex functions in Orlicz spaces we refer to the book by Krasnosel'skii and Rutickii [26].

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.

Condition Q. We say that *condition Q* is satisfied, if φ is an Orlicz N-function such that

$$\liminf_{x \to 0} \frac{\varphi(x)}{x^2} = C > 0.$$

The constant C may be equal to $+\infty$.

Definition 1.6. A zero mean random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ (the space of φ-sub-Gaussian random variables) if there exists a positive constant a such that the inequality

$$\mathsf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda))$$

holds for all $\lambda \in \mathbb{R}$.

Example 1.7. The following functions are N-functions satisfying condition Q:

$$\varphi(x) = \frac{|x|^\alpha}{\alpha},\ 1 < \alpha \le 2;$$

$$\varphi(x) = \begin{cases} \frac{|x|^2}{\alpha}, & |x| \le 1, \alpha > 2; \\ \frac{|x|^\alpha}{\alpha}, & |x| > 1. \end{cases}$$

Theorem 1.8. *[3, 25] The space* $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with the norm*

$$\tau_\varphi(\xi) = \sup_{\lambda \ne 0} \frac{\varphi^{-1}(\ln \mathsf{E}\exp(\lambda\xi))}{|\lambda|}$$

and the inequalities

$$\begin{aligned} \mathsf{E}\exp(\lambda\xi) &\le \exp(\varphi(\lambda\tau_\varphi(\xi))), \\ (\mathsf{E}\xi^2)^{\frac{1}{2}} &\le C\tau_\varphi(\xi). \end{aligned} \tag{1.2}$$

hold for all $\lambda \in \mathbb{R}$*, where* $C > 0$ *is some constant.*

Remark 1.9. If $\varphi(x) = \frac{x^2}{2}$, then the space $\mathrm{Sub}_\varphi(\Omega) = \mathrm{Sub}(\Omega)$ is called space of sub-Gaussian random variables.

Definition 1.10. Let (T, ρ) be a pseudometric space. The *metric entropy* is defined as

$$H(u) := \ln N_{(T,\rho)}(u)$$

where $N_{(T,\rho)}(u)$ is called a *metric massiveness* of the set T and denotes the least number of closed ρ-balls whose diameter do not exceed $2u$ needed to cover T.

Example 1.11. If $T = [a, b]$ and ρ is the Euclidean distance, then

$$\ln\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \le H_{(T,\rho)}(u) \le \ln\left(\frac{b-a}{2u} + 1\right).$$

Definition 1.12. A stochastic process $X = (X(t), t \in T)$ is a φ-sub-Gaussian process (belongs to the space $\mathrm{Sub}_\varphi(\Omega)$) if the random variable $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in T$.

If $\varphi(x) = \frac{x^2}{2}$, then such processes are called sub-Gaussian.

Example 1.13. A centered Gaussian stochastic process is sub-Gaussian process.

1.2. Strictly φ-sub-Gaussian random variables and processes

Definition 1.14. A family Δ of random variables from the space $\mathrm{Sub}_\varphi(\Omega)$ is called strictly φ-sub-Gaussian family ($\Delta \in \mathrm{SSub}_\varphi(\Omega)$), if there exists a contstant $C_\Delta > 0$ such that for any countable set I, $\xi_i \in \Delta$, $i \in I$ and for any $\lambda_i \in \mathbb{R}^1$ the following inequality holds:

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) \le C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{\frac{1}{2}}. \tag{1.3}$$

The constant C_Δ is called identifying constant of the family Δ.

Remark 1.15. If the set I is countable, then the inequality (1.3) holds true if

$$\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2<\infty.$$

Theorem 1.16. *[25] Let Δ be a strictly φ-sub-Gaussian family of random variables. Then the linear closure of the family Δ in the space $\mathrm{Sub}_\varphi(\Omega)$ (or $L_2(\Omega)$) also is a strictly φ-sub-Gaussian family with the same identifying constant.*

Definition 1.17. A random process $X=\{X(t),t\in T\}$ is called strictly φ-sub-Gaussian process (i.e., $X\in \mathrm{SSub}_\varphi(\Omega)$), if the family of random variables $\{X(t),\ t\in T\}$ is strictly φ-sub-Gaussian one. The identifying constant of this family is called identifying constant of the process X and is denoted as C_X.

Example 1.18. [25] Let k be a deterministic kernel and suppose that the process $X=\{X(t),t\in T\}$ can be represented in the form

$$X(t)=\int\limits_T k(t,s)\mathrm{d}\xi(s),$$

where ξ is a strictly φ-sub-Gaussian random process and the integral above is defined in the mean-square sense. Then the process X is strictly φ-sub-Gaussian random process.

1.3. Some properties of φ-sub-Gaussian random processes

In order to get our main results, we shall use some properties of the φ-sub-Gaussian random processes obtained by Kozachenko and Vasylyk in [21, 23, 25].

Let (T,ρ) be a pseudometric (metric) separable space with a pseudometric (metric) ρ, and $X=\{X(t),t\in T\}$ be a φ-sub-Gaussian random process.

Suppose that there exists a continuous monotonically increasing function $\sigma=\{\sigma(h),h>0\}$, such that $\sigma(h)\to 0$ as $h\to 0$ and

$$\sup_{\rho(t,s)\le h}\tau_\varphi(X(t)-X(s))\le\sigma(h). \tag{1.4}$$

In particular, the following function has this property

$$\sigma(h)=\sup_{\rho(t,s)\le h}\tau_\varphi(X(t)-X(s)),$$

if the process X is continuous in the norm $\tau_\varphi(\cdot)$.

Consider a compact set $B\subset T$, which consists of at least two points t and s such that $\rho(t,s)>0$. In what follows, we shall use the next notations:

- $\gamma=\sup\limits_{t\in B}\tau_\varphi(X(t))$,
- $\beta>0$ is a number such that $\beta\le\sigma\left(\inf\limits_{s\in B}\sup\limits_{t\in B}\rho(t,s)\right)$,
- $\zeta_\varphi(v)=\frac{v}{\varphi^{(-1)}(v)}$.
- $H_B(u)=H_{(B,\rho)}(u)$ is metric entropy of the space (B,ρ).

Theorem 1.19. *[21, 25] Let $X=\{X(t),t\in B\}$ be a φ-sub-Gaussian random process satisfying the condition (1.4). If*

$$\int\limits_0^\beta\zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,du<\infty, \tag{1.5}$$

then for all $x > D(p,\gamma,\beta)$ and any $p \in (0,1)$, we have the following estimates:

$$\mathbf{P}\left\{\sup_{t\in B} X(t) > x\right\} \le W(p,x),$$

$$\mathbf{P}\left\{\inf_{t\in B} X(t) < -x\right\} \le W(p,x),$$

$$\mathbf{P}\left\{\sup_{t\in B} |X(t)| > x\right\} \le 2W(p,x),$$

where

$$W(p,x) = \exp\left\{ - Z^*_{p,\gamma,\beta}(x - D(p,\gamma,\beta))\right\},$$

$$D(p,\gamma,\beta) = 2\left(\gamma\zeta_\varphi(H_B(\sigma^{(-1)}(p\beta))) + \frac{1}{(1-p)p}\int\limits_0^{\beta p^2} \zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,du\right),$$

$Z^*_{p,\gamma,\beta}$ is the Young-Fenchel transformation of the function

$$Z_{p,\gamma,\beta}(\lambda) = \varphi\left(\frac{\lambda\gamma}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p, \quad \lambda \in \mathbb{R}.$$

Theorem 1.20. *[23, 25] If for a separable φ-sub-Gaussian random process $X = \{X(t), t \in B\}$ condition (1.4) holds and for arbitrary $\varepsilon > 0$*

$$\int\limits_0^{\sigma(\varepsilon)} \zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,du < \infty, \tag{1.6}$$

then the process X is sample continuous process with probability one and for any $x > B(p,\varepsilon)$ we have the following estimate:

$$\mathbf{P}\left\{\sup_{\rho(t,s)<\varepsilon} |X(t) - X(s)| > x\right\} \le 2\exp\left\{-\varphi^*\left(\frac{x - B(p,\varepsilon)}{A(p,\varepsilon)}\right)\right\}, \tag{1.7}$$

where $p \in (0,1)$, $A(p,\varepsilon) = \frac{\sigma(\varepsilon)(3-p)}{(1-p)^2}$, $B(p,\varepsilon) = \frac{4(3-p)}{3p(1-p)^2}\int\limits_0^{\sigma(\varepsilon)} \zeta_\varphi(H_B(\sigma^{(-1)}(u)))du$.

2. Strictly φ-sub-Gaussian quasi shot noise processes

Let $\xi = (\xi(t), t \in \mathbb{R})$ be a real-valued zero-mean random process with uncorrelated increments defined on a standard probability space such that

$$E(\xi(t) - \xi(s))^2 = t - s, \quad t > s \in \mathbb{R}.$$

Let $g = (g(t,u), t, u \in \mathbb{R})$ be a real-valued function satisfying the following condition:

$$\int_{-\infty}^{+\infty} g^2(t,u)\mathrm{d}u < \infty, \quad t \in \mathbb{R}. \tag{2.1}$$

Definition 2.1. We shall call the process

$$X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u), \quad t \in \mathbb{R}, \tag{2.2}$$

a *quasi shot noise process generated by the process ξ and the response function g*, where the integral in (2.2) is defined in the mean-square sense.

The covariance function of the process X has the following form:

$$\mathsf{E}X(t)X(s) = \int_{-\infty}^{+\infty} g(t,u)g(s,u)\mathrm{d}u, \quad t,s \in \mathbb{R}.$$

The following lemma will allow us to introduce a strictly φ-sub-Gaussian quasi shot noise process.

Lemma 2.2. *Let the process $\xi = (\xi(t), t \in \mathbb{R})$ be strictly φ-sub-Gaussian random process with uncorrelated increments.*

Then the process $X(t) = \int_{-\infty}^{+\infty} g(t,u)d\xi(u)$, $t \in \mathbb{R}$, also is strictly φ-sub-Gaussian process and for any $t,s \in \mathbb{R}$

$$\tau_\varphi(X(t)) \le c_\xi \left(\int_{-\infty}^{+\infty} g^2(t,u)du\right)^{1/2}, \tag{2.3}$$

$$\tau_\varphi(X(t)-X(s)) \le c_\xi \left(\int_{-\infty}^{+\infty} (g(t,u)-g(s,u))^2 du\right)^{1/2}, \tag{2.4}$$

where c_ξ is the identifying constant of the process ξ.

Proof

From theorem 1.16 and example 1.18 follows that the process $X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u)$, $t \in \mathbb{R}$, is strictly φ-sub-Gaussian process with the identifying constant c_ξ. Therefore, from the definition of strictly φ-sub-Gaussian random process follows that for any $t \in \mathbb{R}$

$$\tau_\varphi(X(t)) \le c_\xi(\mathsf{E}X^2(t))^{1/2} = c_\xi \left(\int_{-\infty}^{+\infty} g^2(t,u)\mathrm{d}u\right)^{1/2},$$

$$\tau_\varphi(X(t)-X(s)) \le c_\xi(\mathsf{E}(X(t)-X(s))^2)^{\frac{1}{2}} = c_\xi \left(\int_{-\infty}^{+\infty} (g(t,u)-g(s,u))^2\mathrm{d}u\right)^{\frac{1}{2}}.$$

□

Lemma 2.3. *Let $X(t) = \int_{-\infty}^{+\infty} g(t,u)d\xi(u)$, $t \in \mathbb{R}$, be strictly φ-sub-Gaussian process and $\{g(t,u), t,u \in \mathbb{R}\}$ be a complete system of functions in $L^2(\mathbb{R})$. Then the process $\xi = (\xi(t), t \in \mathbb{R})$ is strictly φ-sub-Gaussian random process.*

Proof

If the process X can be presented in the form (2.2), then its covariance function has the form $\mathsf{E}X(t)X(s) = \int_{-\infty}^{+\infty} g(t,u)g(s,u)\mathrm{d}u$, $t,s \in \mathbb{R}$. According to the Karhunen theorem [4, 10], the process $\xi = (\xi(t), t \in \mathbb{R})$ belongs to the linear span H_X of the values $X(t), t \in \mathbb{R}$ if and only if the system of functions $\{g(t,u), t,u \in \mathbb{R}\}$ is a complete system of functions in $L^2(\mathbb{R})$. In our case, linear span H_X is a space of strictly φ-sub-Gaussian random variables. Thus, if $\{g(t,u), t,u \in \mathbb{R}\}$ is complete system of functions in $L^2(\mathbb{R})$, then the process $\xi = (\xi(t), t \in \mathbb{R})$ is a strictly φ-sub-Gaussian random process. □

Definition 2.4. We shall call the process $X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u)$ a *strictly φ-sub-Gaussian quasi shot noise process*, if ξ is strictly φ-sub-Gaussian random process.

3. Main results

Suppose that there exist such functions $r = (r(h), h \ge 0)$ and $k = (k(u), u \in \mathbb{R})$ that

$$|g(t,u) - g(s,u)| \le r(t-s)k(u), \quad t,s,u \in \mathbb{R}, \tag{3.1}$$

the function r is nonnegative continuous monotonically increasing function, such that $r(h) \to 0$ as $h \to 0$, and the function k is nonnegative continuous function that satisfies the condition $\int_{\mathbb{R}} k^2(u)\mathrm{d}u < \infty$.

Theorem 3.1. *Let X be a strictly φ-sub-Gaussian quasi shot noise process on the interval $[a,b]$, $a,b \in \mathbb{R}$, with the response function g satisfying condition (3.1). If*

$$\int_0^{\beta} \zeta_\varphi \left(\ln \left(\frac{b-a}{2r^{(-1)}(cu)} + 1 \right) \right) du < \infty, \tag{3.2}$$

then for all $x > \bar{D}(p,\gamma,\beta)$ and any $p \in (0,1)$, we have the following estimates:

$$\mathbf{P}\left\{ \sup_{t\in[a,b]} X(t) > x \right\} \le \bar{W}(p,x),$$

$$\mathbf{P}\left\{ \inf_{t\in[a,b]} X(t) < -x \right\} \le \bar{W}(p,x),$$

$$\mathbf{P}\left\{ \sup_{t\in[a,b]} |X(t)| > x \right\} \le 2\bar{W}(p,x),$$

where

$$\bar{W}(p,x) = \exp\left\{ -Z^*_{p,\gamma,\beta}(x - \bar{D}(p,\gamma,\beta)) \right\},$$

$$\bar{D}(p,\gamma,\beta) = 2\left(\gamma\zeta_\varphi \left(\ln \left(\frac{b-a}{2r^{(-1)}(cp\beta)} + 1 \right) \right) + \right.$$

$$\left. \frac{1}{(1-p)p} \int_0^{\beta p^2} \zeta_\varphi \left(\ln \left(\frac{b-a}{2r^{(-1)}(cu)} + 1 \right) \right) du \right),$$

$Z^*_{p,\gamma,\beta}$ is the Young-Fenchel transformation of the function

$$Z_{p,\gamma,\beta}(\lambda) = \varphi\left(\frac{\lambda\gamma}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p, \quad \lambda \in \mathbb{R},$$

$$\gamma = \sup_{t\in[a,b]} c_\xi \left(\int_{-\infty}^{+\infty} g^2(t,u)du \right)^{1/2},\ \beta \in \left(0, \tfrac{1}{c} r\left(\tfrac{b-a}{2}\right)\right],\ c = \left(c_\xi \left(\int_{-\infty}^{+\infty} k^2(u)du \right)^{1/2} \right)^{-1}.$$

Proof

If condition (3.1) holds for the response function g, then from lemma 2.2 we have that for $t,s \in [a,b]$

$$\tau_\varphi(X(t)-X(s)) \le c_\xi \left(\int_{-\infty}^{+\infty} (g(t,u)-g(s,u))^2 \mathrm{d}u \right)^{1/2} \le$$

$$c_\xi \left(\int_{-\infty}^{+\infty} (r(t-s)k(u))^2 \mathrm{d}u \right)^{1/2} = c_\xi r(t-s) \left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u \right)^{1/2},$$

where the function $r = (r(h), h \ge 0)$ is nonnegative continuous monotonically increasing function, such that $r(h) \to 0$ as $h \to 0$, and the function $k = (k(u), u \in \mathbb{R})$ is nonnegative and satisfies the condition $\int_{\mathbb{R}} k^2(u)\mathrm{d}u < \infty$.

This means that there exists a continuous monotonically increasing function

$$\sigma(h) = c_\xi r(h) \left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u \right)^{1/2}, \quad h \ge 0,$$

such that $\sigma(h) \to 0$ as $h \to 0$ and for which the condition (1.4) holds true:

$$\sup_{|t-s|\le h} \tau_\varphi(X(t)-X(s)) \le \sigma(h).$$

Under condition (1.5) of theorem 1.19, for $B=[a,b]$ we get that $\sigma^{(-1)}(u)=r^{(-1)}(cu)$,

$$H_B(\sigma^{(-1)}(u))\le \ln\left(\frac{b-a}{2\sigma^{(-1)}(u)}+1\right)=\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right),$$

where $c=\left(c_\xi\left(\int_{-\infty}^{+\infty}k^2(u)\mathrm{d}u\right)^{1/2}\right)^{-1}$.

Since $\zeta_\varphi(v)=\frac{v}{\varphi^{(-1)}(v)}$ is continuous monotonically increasing function, then

$$\int_0^\beta \zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,\mathrm{d}u\le\int_0^\beta\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)\mathrm{d}u,$$

$$D(p,\gamma,\beta)=2\Bigg(\gamma\zeta_\varphi(H_B(\sigma^{(-1)}(p\beta)))+\frac{1}{(1-p)p}\int_0^{\beta p^2}\zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,du\Bigg)\le$$

$$2\Bigg(\gamma\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cp\beta)}+1\right)\right)+$$

$$\frac{1}{(1-p)p}\int_0^{\beta p^2}\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)du\Bigg)=:\bar D(p,\gamma,\beta),$$

where $\gamma=\sup\limits_{t\in[a,b]}c_\xi\left(\int_{-\infty}^{+\infty}g^2(t,u)\mathrm{d}u\right)^{1/2}$, $\beta\in\left(0,c_\xi r\left(\frac{b-a}{2}\right)\left(\int_{-\infty}^{+\infty}k^2(u)\mathrm{d}u\right)^{1/2}\right]=\left(0,\frac{1}{c}r\left(\frac{b-a}{2}\right)\right]$.

Thus, if condition (3.2) holds, then the assertion of this theorem follows from theorem 1.19. $\square$

Theorem 3.2. *If for a separable strictly φ-sub-Gaussian quasi shot noise process $X=\{X(t),t\in[a,b]\}$ condition (3.1) holds and for arbitrary $\varepsilon>0$*

$$\int_0^{\sigma(\varepsilon)}\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)du<\infty,\tag{3.3}$$

then the process X is a sample continuous process with probability one and for any $p\in(0,1)$ and $x>\bar B(p,\varepsilon)$ we have the following estimate:

$$\mathbf{P}\left\{\sup_{|t-s|<\varepsilon}|X(t)-X(s)|>x\right\}\le 2\exp\left\{-\varphi^*\left(\frac{x-\bar B(p,\varepsilon)}{\bar A(p,\varepsilon)}\right)\right\},\tag{3.4}$$

where $\sigma(\varepsilon)=c_\xi r(\varepsilon)\left(\int_{-\infty}^{+\infty}k^2(u)du\right)^{1/2}$, $\bar A(p,\varepsilon)=\frac{\sigma(\varepsilon)(3-p)}{(1-p)^2}$,
$\bar B(p,\varepsilon)=\frac{4(3-p)}{3p(1-p)^2}\int_0^{\sigma(\varepsilon)}\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)du$, $c=c_\xi^{-1}\left(\int_{-\infty}^{+\infty}k^2(u)du\right)^{-1/2}$.

Proof
Consider a separable strictly φ-sub-Gaussian quasi shot noise process $X=\{X(t),t\in[a,b]\}$ and let condition (3.1) hold. Then there exists the function

$$\sigma(h)=c_\xi r(h)\left(\int_{-\infty}^{+\infty}k^2(u)\mathrm{d}u\right)^{1/2},\quad h\ge 0,$$

for which condition (1.4) holds true.

From (3.3) we get fulfilment of condition (1.6) of theorem 1.20:

$$H_B(\sigma^{(-1)}(u)) \le \ln\left(\frac{b-a}{2\sigma^{(-1)}(u)}+1\right) = \ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right),$$

$$\int\limits_0^{\sigma(\varepsilon)} \zeta_\varphi(H_B(\sigma^{(-1)}(u)))\,du \le \int\limits_0^{\sigma(\varepsilon)} \zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)du < \infty,$$

where $\sigma(\varepsilon) = c_\xi r(\varepsilon)\left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u\right)^{1/2}$, and $c = c_\xi^{-1}\left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u\right)^{-1/2}$

This implies that the considered separable strictly φ-sub-Gaussian quasi shot noise process X satisfies conditions of theorem 1.20 and from (1.7) estimate (3.4) follows. For $\bar{B}(p,\varepsilon)$ from estimate (3.4) in this particular case we get the following expression:

$$\bar{B}(p,\varepsilon) = \frac{4(3-p)}{3p(1-p)^2}\int\limits_0^{\sigma(\varepsilon)} \zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)\mathrm{d}u.$$

□

Example 3.3. Let's illustrate theorem 3.1 by the following example.

Consider a zero-mean random process $\xi = (\xi(t), t\in\mathbb{R}) \in \mathrm{SSub}_\varphi(\Omega)$ with uncorrelated increments, such that $\varphi(x) = \frac{x^2}{2}$, $x\in\mathbb{R}$. That is, ξ is a strictly sub-Gaussian process and $\mathsf{E}\xi^2(t) = \tau^2(\xi(t))$ for all $t\in\mathbb{R}$, where $\tau(\cdot) = \tau_\varphi(\cdot)$ is a norm in the space of sub-Gaussian variables $\mathrm{Sub}(\Omega)$ (see [3]).

In this case we have the identifying constant $c_\xi = 1$, the inverse function $\varphi^{(-1)}(v) = \sqrt{2v}$, $v\ge 0$, and $\zeta_\varphi(v) = \frac{v}{\varphi^{(-1)}(v)} = \frac{v}{\sqrt{2v}} = \sqrt{\frac{v}{2}}$.

Let a real-valued function $g = (g(t,u), t, u \in\mathbb{R})$ have the following form:

$$g(t,u) = \frac{\sin(tu)}{|u|^\delta+1}, \quad t,u\in\mathbb{R}, \quad \delta > 1.$$

This function satisfies the condition:

$$\int_{-\infty}^{+\infty} g^2(t,u)\mathrm{d}u = \int_{-\infty}^{+\infty} \frac{\sin^2(tu)}{(|u|^\delta+1)^2}\mathrm{d}u < \infty, \quad t\in\mathbb{R}.$$

Now, we can consider a strictly sub-Gaussian quasi shot noise process $X(t) = \int_{-\infty}^{+\infty} g(t,u)\mathrm{d}\xi(u)$, $t\in\mathbb{R}$, generated by the process ξ and the response function g.

For the function g we get the following estimate:

$$|g(t,u)-g(s,u)| = \frac{|\sin(tu)-\sin(su)|}{|u|^\delta+1} = \frac{\left|2\sin\frac{u(t-s)}{2}\cos\frac{u(t-s)}{2}\right|}{|u|^\delta+1} \le$$

$$2\left|\frac{u(t-s)}{2}\right|^\alpha \frac{1}{|u|^\delta+1} = \frac{2^{1-\alpha}|u|^\alpha|t-s|^\alpha}{|u|^\delta+1} = |t-s|^\alpha\cdot\frac{2^{1-\alpha}|u|^\alpha}{|u|^\delta+1} = r(t-s)k(u),$$

where

$$r(t-s) = |t-s|^\alpha, \quad k(u) = \frac{2^{1-\alpha}|u|^\alpha}{|u|^\delta+1}, \quad t,s,u\in\mathbb{R},\ \delta>1,\ \alpha = \min(1,\delta-1)\in(0,1].$$

The function $r(h) = |h|^\alpha$, $h\in\mathbb{R}$, is a nonnegative continuous monotonically increasing function such that $r(h)\to 0$ as $h\to 0$.

The function $k(u) = \frac{2^{1-\alpha}|u|^\alpha}{|u|^\delta+1}$, $u \in \mathbb{R}$, is a nonnegative continuous function and

$$\int_{\mathbb{R}} k^2(u)\mathrm{d}u = \int_{\mathbb{R}} \frac{2^{2-2\alpha}|u|^{2\alpha}}{(|u|^\delta+1)^2}\mathrm{d}u < \infty.$$

This means that condition (3.1) holds for the function g. This implies that in our case there exists a continuous monotonically increasing function

$$\sigma(h) = c_\xi r(h)\left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u\right)^{1/2} = h^\alpha \left(\int_{\mathbb{R}} \frac{2^{2-2\alpha}|u|^{2\alpha}}{(|u|^\delta+1)^2}\mathrm{d}u\right)^{1/2}, \quad h \geq 0,$$

such that $\sigma(h) \to 0$ as $h \to 0$ and for which the condition (1.4) holds true.

Consider the process X on the interval $[a,b] \subset \mathbb{R}$. For functions in condition (3.2) we obtain the following expressions:

$$\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right) = \ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right),$$

$$\zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)\mathrm{d}u = \frac{\ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right)}{\left(2\ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right)\right)^{1/2}} = \left(\frac{1}{2}\ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right)\right)^{1/2},$$

where $c = \left(\int_{-\infty}^{+\infty} k^2(u)\mathrm{d}u\right)^{-\frac{1}{2}}$.

In this case condition (3.2) is of the form

$$\int_0^\beta \zeta_\varphi\left(\ln\left(\frac{b-a}{2r^{(-1)}(cu)}+1\right)\right)\mathrm{d}u = \int_0^\beta \left(\frac{1}{2}\ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right)\right)^{1/2}\mathrm{d}u < \infty,$$

where $\beta \in \left(0, \frac{1}{c}\left(\frac{b-a}{2}\right)^\alpha\right]$. Since the integral above converges, the strictly sub-Gaussian quasi shot noise process X generated by the process ξ and the response function g satisfies conditions of theorem 3.1.

Now consider functions in the expression $\bar{W}(p,x) = \exp\left\{-Z^*_{p,\gamma,\beta}(x-\bar{D}(p,\gamma,\beta))\right\}$.

For $\bar{D}(p,\gamma,\beta)$ we get the expression

$$\bar{D}(p,\gamma,\beta) = 2\left(\gamma\left(\frac{1}{2}\ln\left(\frac{b-a}{2(cp\beta)^{1/\alpha}}+1\right)\right)^{1/2} + \frac{1}{(1-p)p}\int_0^{\beta p^2}\left(\frac{1}{2}\ln\left(\frac{b-a}{2(cu)^{1/\alpha}}+1\right)\right)^{1/2}\mathrm{d}u\right).$$

Recall, that here $p \in (0,1)$ and $\gamma = \sup\limits_{t\in[a,b]} c_\xi\left(\int_{-\infty}^{+\infty} g^2(t,u)\mathrm{d}u\right)^{1/2} = \sup\limits_{t\in[a,b]}\left(\int_{-\infty}^{+\infty}\frac{\sin^2(tu)}{(|u|^\delta+1)^2}\mathrm{d}u\right)^{1/2}$.

The function $Z_{p,\gamma,\beta}(\lambda)$ will take the form

$$Z_{p,\gamma,\beta}(\lambda) = \frac{1}{2}\left(\frac{\lambda\gamma}{1-p}\right)^2(1-p) + \frac{1}{2}\left(\frac{\lambda\beta}{1-p}\right)^2 p = \frac{\lambda^2(\gamma^2(1-p)+\beta^2 p)}{2(1-p)^2}, \quad \lambda \in \mathbb{R},$$

and for its Young-Fenchel transformation we have

$$Z^*_{p,\gamma,\beta}(x-\bar{D}(p,\gamma,\beta)) = \frac{(x-\bar{D}(p,\gamma,\beta))^2(1-p)^2}{2(\gamma^2(1-p)+\beta^2 p)}, \quad x > \bar{D}(p,\gamma,\beta).$$

Thus,

$$\bar{W}(p,x) = \exp\left\{-Z^*_{p,\gamma,\beta}(x-\bar{D}(p,\gamma,\beta))\right\} = \exp\left\{-\frac{(x-\bar{D}(p,\gamma,\beta))^2(1-p)^2}{2(\gamma^2(1-p)+\beta^2 p)}\right\}$$

and, finally, for all $x > \bar{D}(p,\gamma,\beta)$ and any $p \in (0,1)$ we get the following estimate for distribution of supremum of the process X on the interval $[a,b]$:

$$\mathbf{P}\left\{\sup_{t\in[a,b]} X(t) > x\right\} \leq \exp\left\{-\frac{(x-\bar{D}(p,\gamma,\beta))^2(1-p)^2}{2(\gamma^2(1-p)+\beta^2 p)}\right\}.$$

One can see that the right part of the estimate above tends to zero as $x \to \infty$.

4. Conclusions

In this paper, strictly φ-sub-Gaussian quasi shot noise processes are introduced. Some results, obtained for φ-sub-Gaussian stochastics processes, are applied to strictly φ-sub-Gaussian quasi shot noise processes. Estimates for distribution of supremum and conditions for sample functions continuity with probability one for such processes defined on a compact set are presented. As an example, strictly sub-Gaussian quasi shot noise processes are considered.

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