# New Class of duality models in discrete minmax fractional programming based on second-order univexities

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**Abstract**   The main purpose of this paper is first to formulate four new general higher-order parametric duality models for a discrete minmax fractional programming problem, and then prove appropriate duality theorems utilizing various new classes of second-order $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-univex functions.

**Key Words**   Discrete minmax fractional programming; second-order $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-univex functions; Generalized second-order parametric duality models; Duality theorems.

## 1. Introduction and Preliminaries

In this paper, we introduce four generalized second-order parametric duality models and prove a variety of weak, strong, and strict converse duality theorems using the notion of second-order $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-univex functions for the following discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \ \max_{1 \leq i \leq p} \ \frac{f_i(x)}{g_i(x)}$$

subject to    $G_j(x) \leq 0, \ j \in \underline{q}, \quad H_k(x) = 0, \ k \in \underline{r}, \quad x \in X,$
where $X$ is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), $f_i$, $g_i$, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, $G_j$, $j \in \underline{q}$, and $H_k$, $k \in \underline{r}$, are real-valued functions defined on $X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all $x$ satisfying the constraints of $(P)$.

The problems of this nature are more frequently referred to as "generalized fractional programming problems" to the context of mathematical programming. These problems provide realistic models for some significant real-world problems (more notably encountered in multiobjective programming, approximation theory, goal programming, location planning and economics), while their mathematical tractability empowers equivalent parametric nonlinear programming problems with nonfractional objective functions.

Recently, Verma and Zalmai [10] established some second-order parametric necessary optimality conditions as well as numerous second-order sufficient optimality conditions for a discrete minmax fractional programming

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problem applying various second-order $(\phi, \eta, \rho, \theta, m)$-invexity assumptions. This second-order invexity generalizes the concepts of invexity, pseudoinvexity, and quasiinvexity, originally defined by Hanson [3]. The second-order invexity is also referred to as "sonvexity" in the literature.

In the present paper, we plan to introduce new classes of second-order $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-univex functions, which generalize most of the existing notions of second-order univex functions. The second-order univex functions are also referred to as "sounivex functions" in the literature. Here, we shall utilize two partitioning schemes due to Mond and Weir [6] and Yang [13], in conjunction with the generalized versions of the new classes of second-order $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-univex functions to formulate four generalized duality models for $(P)$ and prove appropriate duality theorems. The duality models and the related duality theorems established in this paper generalize those presented in [10 - 12] and others. To the best of our knowledge, all of these duality results are new in the area of discrete minmax fractional programming. In fact, it seems that results of this type, which are based on second-order necessary and sufficient optimality conditions, have not yet appeared in any shape or form for any type of mathematical programming problems. For extensive lists of publications dealing with second-order and higher-order duality results for various categories of nonlinear programming problems, which are essentially based on first-order optimality conditions and different kinds of generalized convexity concepts, we refer the reader ([2]-[5], [7], [9]-[12], [15], [16]).

All the parametric duality results established in this paper can easily be modified (and specialized) for each one of the following three classes of nonlinear programming problems to the context $(P)$:

$(P1)$ $\quad \underset{x \in \mathbb{F}}{\text{Minimize}} \ \dfrac{f_1(x)}{g_1(x)};$

$(P2)$ $\quad \underset{x \in \mathbb{F}}{\text{Minimize}} \ \underset{1 \leq i \leq p}{\max} \ f_i(x);$

$(P3)$ $\quad \underset{x \in \mathbb{F}}{\text{Minimize}} \ f_1(x),$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of $(P)$, that is,

$$\mathbb{F} = \{x \in X : G_j(x) \leq 0, \ j \in \underline{q}, \quad H_k(x) = 0, \ k \in \underline{r}\}.$$

The paper is organized as follows: In Section 1, we introduce a few basic definitions and recall some auxiliary results that will be used in the sequel. In Section 2, we utilize a partitioning scheme due to Mond and Weir [6], and formulate two general second-order parametric duality models for $(P)$ and prove weak, strong, and strict converse duality theorems using various generalized $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-sounivexity assumptions. In Section 3, we shall make use of another partitioning method due to Yang [13] and construct another pair of general second-order parametric duality models with different constraint structures and discuss several second-order duality results under a variety of generalized $(\mathcal{F}, \beta, \phi, \zeta, \rho, \theta, m)$-sounivexity assumptions. Finally, in Section 4 we present some remarks on our main results aiming at some future research endeavors arising from certain modifications of the principal minmax model formulated in the present investigation.

We next introduce some generalized versions of the sounivexity (Zalmai [15]) and others in the literature. Let $f : X \to \mathbb{R}$ be a twice differentiable function. Suppose that $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$, and $\langle a, b \rangle$ is the inner product of the vectors $a$ and $b$.

**Definition 1.1.** *The function $f$ is said to be* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-*sounivex at* $x^* \in X$ *if there exist functions* $\beta : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta, \zeta : X \times X \to \mathbb{R}^n$, *a sublinear function* $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$, *and a positive integer $m$ such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \mathcal{F}\big(x, x^*; \beta(x, x^*)\nabla f(x^*)\big) + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*)z \rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m.$$

*The function $f$ is said to be* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-*sounivex on $X$ if it is* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \theta, m)$-sounivex *at each $x^* \in X$.*

**Definition 1.2.** *The function $f$ is said to be* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \theta, m)$-pseudosounivex *at $x^* \in X$ if there exist functions $\beta : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta, \zeta : X \times X \to \mathbb{R}^n$, a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$, and a positive integer $m$ such that for each $x \in X$ ($x \neq x^*$) and $z \in \mathbb{R}^n$,*

$$\mathcal{F}\big(x, x^*; \beta(x, x^*)\nabla f(x^*)\big) + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*)z \rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^m$$
$$\Rightarrow \quad \phi\big(f(x) - f(x^*)\big)(>) \geq 0.$$

*The function $f$ is said to be* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-pseudosounivex *on $X$ if it is* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-pseudosounivex *at each $x^* \in X$.*

**Definition 1.3.** *The function $f$ is said to be* (prestrictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex *at $x^* \in X$ if there exist functions $\beta : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta, \zeta : X \times X \to \mathbb{R}^n$, a sublinear function $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$, and a positive integer $m$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,*

$$\phi\big(f(x) - f(x^*)\big)(<) \leq 0$$
$$\Rightarrow \quad \mathcal{F}\big(x, x^*; \beta(x, x^*)\nabla f(x^*)\big) + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*)z \rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^m.$$

*The function $f$ is said to be* (prestrictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex *on $X$ if it is* (prestrictly) $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex *at each $x^* \in X$.*

From the above definitions it is clear that if $f$ is $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-sounivex at $x^*$, then it is both $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-pseudosounivex and $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex at $x^* \in X$, if $f$ is $(\mathcal{F}, \beta, \phi, \rho, \theta, m)$-quasisounivex at $x^* \in X$, then it is prestrictly $(\mathcal{F}, \beta, \phi, \rho, \theta, m)$-quasisounivex at $x^* \in X$, and if $f$ is strictly $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-pseudosounivex at $x^* \in X$, then it is $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex at $x^* \in X$.

We observe that in some contexts during proving the duality theorems, it may be easier to apply certain alternative but equivalent forms of the above definitions by considering the contrapositive statements. For example, $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivexity can be defined in the following equivalent way:

The function $f$ is said to be $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-quasisounivex at $x^* \in X$ if there exist functions $\beta : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta, \zeta : X \times X \to \mathbb{R}^n$, a sublinear function $(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$, and a positive integer $m$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\mathcal{F}\big(x, x^*; \beta(x, x^*)\nabla f(x^*)\big) + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*)z \rangle > -\rho(x, x^*)\|\theta(x, x^*)\|^m$$
$$\Rightarrow \quad \phi\big(f(x) - f(x^*)\big) > 0.$$

As the generalized sounivexity of functions generalizes most of the convex functions, including generalized invexity, pseudoinvexity and quasiinvexity by identifying appropriate choices of $\mathcal{F}$, $\beta$, $\phi$, $\rho$, $\zeta$, $\theta$, and $m$. For example, if let $\mathcal{F}\big(x, x^*; \nabla f(x^*)\big) = \langle \nabla f(x^*), \eta(x, x^*) \rangle$ and $\beta(x, x^*) \equiv 1$, where $\eta$ is a given function from $X \times X$ to $\mathbb{R}^n$, then we obtain the definitions of $(\phi, \eta, \rho, \zeta, \theta, m)$-sonvex, $(\phi, \eta, \rho, \zeta, \theta, m)$-pseudosonvex, and $(\phi, \eta, \rho, \zeta, \theta, m)$-quasisonvex functions introduced recently in [10]. For example, let $f : X \to \mathbb{R}$ be a twice differentiable function.

**Definition 1.4.** *The function $f$ is said to be* (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex *at $x^*$ if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $m$ such that for each $x \in X$ ($x \neq x^*$) and $z \in \mathbb{R}^n$,*

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*)z \rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m.$$

**Definition 1.5.** *The function* $f$ *is said to be* (strictly) $(\phi, \eta, \rho, \theta, m)$-*sonvex at* $x^*$ *if there exist functions* $\phi :$ $\mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, *and* $\theta : X \times X \to \mathbb{R}^n$, *and a positive integer* $m$ *such that for each* $x \in X$ $(x \neq x^*)$ *and* $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \langle \nabla f(x^*), \eta(x, x^*) \rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^m.$$

We next present an example on some generalized versions of the souinvexity (Zalmai [15]) and others in the literature.

**Example 1.** *Let* $f : X \to \mathbb{R}$ *be a twice differentiable function with the norm* $\| \cdot \|$ *on* $\mathbb{R}^n$ *and inner product* $\langle a, b \rangle$ *of the vectors* $a$ *and* $b$. *Then the function* $f$ *is said to be* (strictly) $(\mathcal{F}, \beta, \phi, \rho, \xi, \theta, m)$-*sounivex at* $x^*$ *if there exist functions* $\beta : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta, \xi : X \times X \to \mathbb{R}^n$, *a sublinear function* $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$, *and a positive integer* $m$ *such that for each* $x \in X$ $(x \neq x^*)$ *and* $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*) + \frac{1}{2}\langle z, \nabla^2 f(x^*) z \rangle\big)(>) \geq \mathcal{F}\big(x, x^*; \beta(x, x^*)\nabla f(x^*)\big) + \langle \xi(x, x^*), \nabla^2 f(x^*) z \rangle$$
$$+ \rho(x, x^*)\|\theta(x, x^*)\|^m.$$

We conclude this section by recalling a set of second-order necessary optimality conditions for $(P)$.

**Theorem 1.1.** [15] *Let* $x^*$ *be an optimal solution of (P), let* $\lambda^* = \varphi(x^*) \equiv$
$\max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}$, *and assume that the functions* $f_i$, $g_i$, $i \in \underline{p}$, $G_j$, $j \in \underline{q}$, *and* $H_k$, $k \in \underline{r}$, *are twice continuously differentiable at* $x^*$, *and that the second-order Guignard constraint qualification holds at* $x^*$. *Then for each* $z^* \in C(x^*)$, *there exist* $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^{p} u_i = 1\}$, $v^* \in \mathbb{R}_+^q \equiv \{v \in \mathbb{R}^q : v \geq 0\}$, *and* $w^* \in \mathbb{R}^r$ *such that*

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$

$$\left\langle z^*, \left\{ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \right\} z^* \right\rangle \geq 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p},$$

$$v_j^* G_j(x^*) = 0, \quad j \in \underline{q},$$

*where* $C(x^*)$ *is the set of all critical directions of (P) at* $x^*$, *that is,*

$$C(x^*) = \{z \in \mathbb{R}^n : \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z \rangle = 0, \quad i \in A(x^*), \quad \langle \nabla G_j(x^*), z \rangle \leq 0, \quad j \in B(x^*),$$
$$\langle \nabla H_k(x^*), z \rangle = 0, \quad k \in \underline{r}\},$$

$$A(x^*) = \{j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}, \text{ and } B(x^*) = \{j \in \underline{q} : G_j(x^*) = 0\}.$$

For brevity, we shall henceforth refer to $x^*$ as a *normal* optimal solution of $(P)$ if it is an optimal solution and satisfies the second-order Guignard constraint qualification.

In the remainder of this paper, we shall assume that the functions $f_i$, $g_i$, $i \in \underline{p}$, $G_j$, $j \in \underline{q}$, and $H_k$, $k \in \underline{r}$, are twice continuously differentiable on the open set $X$. Moreover, we shall assume, without loss of generality, that $g_i(x) > 0$, $i \in \underline{p}$, and $\varphi(x) \geq 0$ for all $x \in X$.

## 2. Duality Model I

In this section, we discuss several families of duality results under various generalized $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-souinvexity hypotheses imposed on certain combinations of the functions involved in the considered optimization problem. This is accomplished by employing a certain partitioning scheme which was originally proposed in [6] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0, J_1, \ldots, J_M\}$ and $\{K_0, K_1, \ldots, K_M\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu \subseteq \underline{q}$ for each $\mu \in \underline{M} \cup \{0\}$, $J_\mu \cap J_\nu = \emptyset$ for each $\mu, \nu \in \underline{m} \cup \{0\}$ with $\mu \neq \nu$, and $\cup_{\mu=0}^m J_\mu = \underline{q}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_M\}$. Moreover, if $M_1$ and $M_2$ are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $M = \max\{M_1, M_2\}$ and $J_\mu = \emptyset$ or $K_\mu = \emptyset$ for $\mu > \min\{M_1, M_2\}$.

In addition, we use the real-valued functions $\xi \to \Phi_i(\xi, v, w, \lambda)$, $i \in \underline{p}$, $\xi \to \Phi(\xi, u, v, w, \lambda)$, and $\xi \to \Lambda_t(\xi, v, w)$ defined, for fixed $\lambda, u, v,$ and $w$, on $X$ as follows:

$$\Phi_i(\xi, v, w, \lambda) = f_i(\xi) - \lambda g_i(\xi) + \sum_{j \in J_0} v_j G_j(\xi) + \sum_{k \in K_0} w_k H_k(\xi), \quad i \in \underline{p},$$

$$\Phi(\xi, u, v, w, \lambda) = \sum_{i=1}^p u_i[f_i(\xi) - \lambda g_i(\xi)] + \sum_{j \in J_0} v_j G_j(\xi) + \sum_{k \in K_0} w_k H_k(\xi),$$

$$\Lambda_t(\xi, v, w,) = \sum_{j \in J_t} v_j G_j(\xi) + \sum_{k \in K_t} w_k H_k(\xi), \quad t \in \underline{M}.$$

Making use of the sets and functions defined above, we can now formulate our first pair of second-order parametric duality models for $(P)$.

Consider the following two problems:

$(DI)$ Maximize $\lambda$
subject to

$$\sum_{i=1}^p u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^q v_j \nabla G_j(y) + \sum_{k=1}^r w_k \nabla H_k(y) = 0, \tag{2.1}$$

$$\left\langle z, \left\{ \sum_{i=1}^p u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^q v_j \nabla^2 G_j(y) + \sum_{k=1}^r w_k \nabla^2 H_k(y) \right\} z \right\rangle \geq 0, \tag{2.2}$$

$$f_i(y) - \lambda g_i(y) + \sum_{j \in J_0} v_j G_j(y) + \sum_{k \in K_0} w_k H_k(y) \geq 0, \quad i \in \underline{p}, \tag{2.3}$$

$$\sum_{j \in J_t} v_j G_j(y) + \sum_{k \in K_t} w_k H_k(y) \geq 0, \quad t \in \underline{M}, \tag{2.4}$$

$$y \in X, \ z \in C(y), \ u \in \mathbb{R}^p, \ u \geq 0, \sum_{i=1}^p u_i = 1, \ v \in \mathbb{R}^q_+, \ w \in \mathbb{R}^r, \ \lambda \in \mathbb{R}_+; \tag{2.5}$$

$(\tilde{D}I)$ Maximize $\lambda$

subject to (2.2) - (2.5) and

$$\mathcal{F}\left(x, y; \sum_{i=1}^p u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^q v_j \nabla G_j(y) + \sum_{k=1}^r w_k \nabla H_k(y)\right) \geq 0 \ \text{for all} \ x \in \mathbb{F}, \tag{2.6}$$

where $\mathcal{F}(x, y; \cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$.

The feasible set $\mathbb{F}_{(DI)}$ (assumed to be nonempty) of $(DI)$ is defined as:

$$\mathbb{F}_{(DI)} = \left\{ y \in X : z \in \mathbb{R}^n;\ u \in \mathbb{R}^p,\ u \geq 0, \sum_{i=1}^{p} u_i = 1; v \in \mathbb{R}_+^q, v_i \geq 0; w \in \mathbb{R}^r;\ \lambda \in \mathbb{R}_+ \right\}.$$

Comparing $(DI)$ and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than $(DI)$ in the sense that any feasible solution of $(DI)$ is also feasible for $(\tilde{D}I)$, but the converse is not necessarily true. Furthermore, we observe that (2.1) is a system of $n$ equations, whereas (2.6) is a single inequality. Clearly, from a computational point of view, $(DI)$ is preferable to $(\tilde{D}I)$ because of the dependence of (2.6) on the feasible set of $(P)$.

Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for $(P) - (DI)$ and $(P) - (\tilde{D}I)$ are almost identical and, therefore, we shall consider only the pair $(P) - (DI)$.

In the proofs of our duality theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of $(P)$.

**Lemma 1.** [10] *For each $x \in X$,*

$$\varphi(x) = \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

The next two theorems show that $(DI)$ is a dual problem for $(P)$.

**Theorem 2.1.** *(Weak Duality) Let x and y be arbitrary feasible solutions of (P) and (DI), respectively. Furthermore, assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *$\xi \to \Phi(\xi, u, v, w, \lambda)$ is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-pseudosounivex at y and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

   (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-quasisounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

   (iii) *$\bar{\rho}(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;*

(b) (i) *$\xi \to \Phi(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at y and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

   (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \rho_t, \zeta, \theta, m)$-quasisounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

   (iii) *$\bar{\rho}(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) > 0$;*

(c) (i) *$\xi \to \Phi(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at y and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

   (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-pseudosounivex at y, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

   (iii) *$\bar{\rho}(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;*

(d) (i) *$\xi \to \Phi(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at y and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

   (ii) *for each $t \in \underline{M_1}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-quasisounivex at y, for each $t \in \underline{M_2} \neq \emptyset$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-pseudosounivex at y, and for each $t \in \underline{M}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{\underline{M_1}, \underline{M_2}\}$ is a partition of $\underline{M}$;*

   (iii) *$\bar{\rho}(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$.*

*Then $\varphi(x) \geq \lambda$.*

*Proof*
(a): Since $\mathcal{F}(x, y; \cdot)$ is sublinear, it is clear that (2.1) and (2.2) can be expressed as follows:

$$\mathcal{F}\Big(x, y; \beta(x,y)\Big\{\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} v_j \nabla G_j(y) +$$

$$\sum_{k \in K_0} w_k \nabla H_k(y)\Big\}\Big) + \mathcal{F}\Big(x, y; \beta(x,y) \sum_{t=1}^{M} \Big[\sum_{j \in J_t} v_j \nabla G_j(y) + \sum_{k \in K_t} w_k \nabla H_k(y)\Big]\Big) \geq 0. \quad (2.7)$$

$$\frac{1}{2}\Big\langle \zeta(x,y), \Big\{\sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} v_j \nabla^2 G_j(y) +$$

$$\sum_{k \in K_0} w_k \nabla^2 H_k(y)\Big\} z\Big\rangle + \frac{1}{2}\Big\langle \zeta(x,y), \sum_{t=1}^{M} \Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y) + \sum_{k \in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \geq 0. \quad (2.8)$$

Since for each $t \in \underline{M}$,

$$\Lambda_t(x, v, w) = \sum_{j \in J_t} v_j G_j(x) + \sum_{k \in K_t} w_k H_k(x)$$

$$\leq 0 \quad \text{(by the primal feasibility of } x)$$

$$\leq \sum_{j \in J_t} v_j G_j(y) + \sum_{k \in K_t} w_k H_k(y)$$

$$\text{(by (2.4) and the dual feasibility of } y)$$

$$= \Lambda_t(y, v, w),$$

it implies

$$\tilde{\phi}_t\big(\Lambda_t(x, v, w) - \Lambda_t(y, v, w)\big) \leq 0.$$

Thus, it follows from (ii) that

$$\mathcal{F}\Big(x, y; \beta(x,y)\Big[\sum_{j \in J_t} v_j \nabla G_j(y) + \sum_{k \in K_t} w_k \nabla H_k(y)\Big]\Big) + \frac{1}{2}\Big\langle \zeta(x,y), \Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y)$$

$$+ \sum_{k \in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \leq -\tilde{\rho}_t(x,y)\|\theta(x,y)\|^m.$$

Summing over $t \in \underline{M}$ and using the sublinearity of $\mathcal{F}(x, y; \cdot)$, we obtain

$$\mathcal{F}\Big(x, y; \beta(x,y)\sum_{t=1}^{M}\Big[\sum_{j \in J_t} v_j \nabla G_j(y) + \sum_{k \in K_t} w_k \nabla H_k(y)\Big]\Big) + \frac{1}{2}\Big\langle \zeta(x,y), \sum_{t=1}^{M}\Big[\sum_{j \in J_t} v_j \nabla^2 G_j(y)$$

$$+ \sum_{k \in K_t} w_k \nabla^2 H_k(y)\Big] z\Big\rangle \leq -\sum_{t=1}^{M} \tilde{\rho}_t(x,y)\|\theta(x,y)\|^m. \quad (2.9)$$

Combining (2.7) - (2.9), and using (iii) we get

$$\mathcal{F}\Big(x, y; \beta(x,y)\Big\{ \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} v_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(y), \eta(x,y) \Big\}\Big)$$

$$+ \frac{1}{2}\Big\langle \zeta(x,y), \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} v_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle$$

$$\geq \sum_{t=1}^{M} \tilde{\rho}_t(x,y)\|\theta(x,y)\|^m \geq -\bar{\rho}(x,y)\|\theta(x,y)\|^m. \quad (2.10)$$

By (i), the above inequality implies that

$$\bar{\phi}\big(\Phi(x, u, v, w, \lambda) - \Phi(y, u, v, w, \lambda)\big) \geq 0.$$

But $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$, and hence we get

$$\Phi(x, u, v, w, \lambda) \geq \Phi(y, u, v, w, \lambda) \geq 0,$$

where the second inequality follows from (2.3) and the dual feasibility of $y$. Since $x \in \mathbb{F}$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0. \quad (2.11)$$

Now using (2.11) and Lemma 2.1, we see that

$$\varphi(x) = \max_{a \in U} \frac{\sum_{i=1}^{p} a_i f_i(x)}{\sum_{i=1}^{p} a_i g_i(x)} \geq \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} \geq \lambda.$$

(b): The proof is similar to that of part (a).

(c): Suppose to the contrary that $\varphi(x) < \lambda$. This implies that

$$f_i(x) - \lambda g_i(x) < 0, \ i \in \underline{p}. \quad (2.12)$$

Since

$$\Phi(x, u, v, w, \lambda) = \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] + \sum_{j \in J_0} u_j G_j(x) + \sum_{k \in K_0} w_k H_k(x)$$

$$\leq \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \ \text{(by the primal feasibility of } x)$$

$$< 0 \ \ \text{(by (2.12))}$$

$$\leq \Phi(y, u, v, w, \lambda) \ \text{(by (2.3))},$$

it follows from the properties of $\bar{\phi}_i$ that

$$\bar{\phi}_i\big(\Phi(x, u, v, w, \lambda) - \Phi(y, u, v, w, \lambda)\big) < 0.$$

Applying (i), the above inequality implies that

$$\mathcal{F}\Big(x, y; \beta(x,y)\Big\{ \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} u_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(y), \eta(x,y) \Big\}\Big)$$

$$+ \frac{1}{2}\Big\langle \zeta(x,y), \Big[ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} u_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big] z \Big\rangle$$

$$\leq -\bar{\rho}(x,y)\|\theta(x,y)\|^m. \quad (2.13)$$

As seen in the proof of Theorem 2.1, our assumptions in (ii) lead to

$$
\mathcal{F}\Big(x, y; \beta(x,y) \sum_{t=1}^{M} \Big[ \sum_{j \in J_t} u_j \nabla G_j(y) + \sum_{k \in K_t} w_k \nabla H_k(y) \Big] \Big) + \frac{1}{2} \Big\langle \zeta(x,y), \sum_{t=1}^{M} \Big[ \sum_{j \in J_t} u_j \nabla^2 G_j(y)
$$
$$
+ \sum_{k \in K_t} w_k \nabla^2 H_k(y) \Big] z \Big\rangle \le - \sum_{t=1}^{M} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^m,
$$

which when combined with (2.7) and (2.8) results in

$$
\mathcal{F}\Big(x, y; \beta(x,y) \Big\{ \sum_{i=1}^{p} u_i [\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} u_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(y), \eta(x,y) \Big\} \Big) +
$$
$$
\frac{1}{2} \Big\langle \zeta(x,y), \Big\{ \sum_{i=1}^{p} u_i [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} u_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle
$$
$$
\ge \sum_{t=1}^{M} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^m.
$$

In view of (iii), this inequality contradicts (2.13). Hence, $\varphi(x) \ge \lambda$.

(d): The proof is similar to that of part (c).                                  □

**Theorem 2.2.** *(Strong Duality) Let $x^*$ be a normal optimal solution of (P), let $\lambda^* = \varphi(x^*)$, and assume that any one of the four sets of conditions specified in Theorem 2.1 is satisfied for all feasible solutions of (DI). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}_+^q$, and $w^* \in \mathbb{R}^r$ such that $x^*$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$.*

*Proof*
Since $x^*$ is a normal optimal solution of $(P)$, by Theorem 2.1, for each $z^* \in C(x^*)$, there exist $u^*$, $v^*, w^*$, and $\lambda^*(= \varphi(x^*))$, as specified above, such that $x^*$ is a feasible solution of $(DI)$. If, for each $z^* \in C(x^*)$, there exist $u^*$, $v^*, w^*$, and $\lambda^*(= \varphi(x^*))$, as specified above, $x^*$ were not optimal, then there would exist a feasible solution $\tilde{x}$ such that $\tilde{\lambda} > \lambda^* = \varphi(x^*)$ which contradicts Theorem 2.1. Therefore, $x^*$ is an optimal solution of $(DI)$ with respect to $z^* \in C(x^*)$, $u^*$, $v^*, w^*$, and $\lambda^*(= \varphi(x^*))$.                    □

**Theorem 2.3.** *(Strict Converse Duality) Let $x^*$ be a normal optimal solution of (P), and let $\tilde{x}$ be an optimal solution of (DI) with respect to $\tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda}$. Furthermore assume that any one of the following four sets of conditions holds:*

  (a) *The assumptions specified in part (a) of Theorem 2.1 are satisfied for the feasible solution $\tilde{x}$ of (DI), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and the function $\xi \to \Phi(\xi, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ is strictly $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-pseudosounivex at $\tilde{x}$.*

  (b) *The assumptions specified in part (b) of Theorem 2.1 are satisfied for the feasible solution $\tilde{x}$ of (DI), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and the function $\xi \to \Phi(\xi, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at $\tilde{x}$.*

  (c) *The assumptions specified in part (c) of Theorem 2.1 are satisfied for the feasible solution $\tilde{x}$ of (DI), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and the function $\xi \to \Phi(\xi, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at $\tilde{x}$.*

  (d) *The assumptions specified in part (d) of Theorem 2.1 are satisfied for the feasible solution $\tilde{x}$ of (DI), $\bar{\phi}(a) > 0 \Rightarrow a > 0$, and the function $\xi \to \Phi(\xi, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}, \zeta, \theta, m)$-quasisounivex at $\tilde{x}$.*

*Then $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.*

*Proof*
Since $x^*$ is a normal optimal solution of $(P)$, by Theorem 2.1, there exist $\tilde{u}, \tilde{v}, \tilde{w}$, and $\tilde{\lambda}$ such that $\tilde{x}$ (with respect to $\tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda}$) is a feasible solution of $(DI)$ and $\varphi(x^*) = \lambda^*$.

(a): Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of part (a) of Theorem 2.1 (with $x$ replaced by $x^*$ and $y$ by $\tilde{x}$, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i[f_i(x) - \tilde{\lambda}g_i(x)] > 0.$$

Using this inequality along with Lemma 2.1, as in the proof of Theorem 2.1, we get $\varphi(x^*) > \tilde{\lambda}$ which contradicts the fact that $\varphi(x^*) = \lambda^* \leq \tilde{\lambda}$. (b) - (d) : The proofs are similar to that of part (a). $\square$

In Theorems 2.1 - 2.3, various generalized $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-sounivexity conditions were imposed on the function $\xi \to \Phi(\xi, u, v, w, \lambda)$, which is the weighted sum of the functions $\xi \to \Phi_i(\xi, v, w, \lambda)$, $i \in \underline{p}$. In the next few theorems, we shall assume that the individual functions $\xi \to \Phi_i(\xi, v, w, \lambda)$, $i \in \underline{p}$, satisfy appropriate generalized $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-sounivexity hypotheses.

**Theorem 2.4.** *(Weak Duality) Let $x$ and $y$ be arbitrary feasible solutions of (P) and (DI), respectively. Furthermore, assume that any one of the following seven sets of hypotheses is satisfied:*

(a)   (i) *for each $i \in I_+ \equiv \{i \in \underline{p} : u_i > 0\}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-pseudosounivex at $y$, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

    (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-quasisounivex at $y$, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(b)   (i) *for each $i \in I_+$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-quasisounivex at $y$, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

    (ii) *for each $t \in \underline{m}$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-pseudosounivex at $y$, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(c)   (i) *for each $i \in I_+$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-quasisounivex at $y$, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

    (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta)$-quasisounivex at $y$, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) > 0$;

(d)   (i) *for each $i \in I_{1+}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-pseudosounivex at $y$, for each $i \in I_{2+}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-quasisounivex at $y$, and for each $i \in I_+$, $\bar{\phi}_i$ is strictly increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of $I_+$;*

    (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-pseudosounivex at $y$, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(e)   (i) *for each $i \in I_{1+} \neq \emptyset$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-pseudosounivex at $y$, for each $i \in I_{2+}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta)$-quasisounivex at $y$, and for each $i \in I_+$, $\bar{\phi}_i$ is strictly increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of $I_+$;*

    (ii) *for each $t \in \underline{M}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-quasisounivex at $y$, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(f)   (i) *for each $i \in I_+$, $\xi \to \Phi_i(\xi, v, \lambda, \bar{t}, \bar{s})$ is prestrictly $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-quasisounivex at $y$, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each* $t \in \underline{M_1} \neq \emptyset$, $\xi \to \Lambda_t(\xi, v, w)$ *is strictly* $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-*pseudosounivex at* $y$, *for each* $t \in \underline{M_2}$, $\xi \to \Lambda_t(\xi, v, w)$ *is* $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-*quasisounivex at* $y$, *and for each* $t \in \underline{M}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{M_1}, \underline{M_2}\}$ *is a partition of* $\underline{M}$;

(iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(g)  (i) *for each* $i \in I_{1+}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ *is* $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-*pseudosounivex at* $y$, *for each* $i \in I_{2+}$, $\xi \to \Phi_i(\xi, v, w, \lambda)$ *is prestrictly* $(\mathcal{F}, \beta, \bar{\phi}_i, \bar{\rho}_i, \zeta, \theta, m)$-*quasisounivex at* $y$, *and for each* $i \in I_+$, $\bar{\phi}_i$ *is strictly increasing and* $\bar{\phi}_i(0) = 0$, *where* $\{I_{1+}, I_{2+}\}$ *is a partition of* $I_+$;

(ii) *for each* $t \in \underline{M_1}$, $\xi \to \Lambda_t(\xi, v, w)$ *is strictly* $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-*pseudosounivex at* $y$, *for each* $t \in \underline{M_2}$, $\xi \to \Lambda_t(\xi, v, w)$ *is* $(\mathcal{F}, \beta, \tilde{\phi}_t, \tilde{\rho}_t, \zeta, \theta, m)$-*quasisounivex at* $y$, *and for* $t \in \underline{M}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{M_1}, \underline{M_2}\}$ *is a partition of* $\underline{M}$;

(iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) \geq 0$;

(iv) $I_{1+} \neq \emptyset$, $\underline{M_1} \neq \emptyset$, *or* $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{t=1}^{M} \tilde{\rho}_t(x, y) > 0$.

*Then* $\varphi(x) \geq \lambda$.

*Proof*

(a) : Suppose to the contrary that $\varphi(x) < \lambda$. This implies that

$$f_i(x) - \lambda g_i(x) < 0, \ i \in \underline{p}. \tag{2.14}$$

Keeping in mind that $u \geq 0$, we see that for each $i \in I_+$,

$$\Phi_i(x, v, \lambda) = f_i(x) - \lambda g_i(x) + \sum_{j \in J_0} u_j G_j(x) + \sum_{k \in K_0} w_k H_k(x)$$

$$\leq f_i(x) - \lambda g_i(x) \ \text{(by the primal feasibility of } x)$$

$$< 0 \ \text{(by (2.14))}$$

$$\leq \Phi_i(y, v, \lambda) \ \text{(by (2.3))},$$

and so it follows from the properties of $\bar{\phi}_i$ that

$$\bar{\phi}_i\big(\Phi_i(x, v, \lambda) - \Phi_i(y, v, \lambda)\big) < 0.$$

Hence, by (i), for each $i \in I_+$, the above inequality implies that

$$\mathcal{F}\Big(x, y; \beta(x, y)\Big\{\nabla f_i(y) - \lambda \nabla g_i(y) + \sum_{j \in J_0} u_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(y)\Big\}\Big)$$

$$+ \frac{1}{2}\Big\langle \zeta(x, y), \Big[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y) + \sum_{j \in J_0} u_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(y)\Big] z \Big\rangle$$

$$< -\bar{\rho}_i(x, y)\|\theta(x, y)\|^m.$$

Since $u \geq 0$, $u_i = 0$ for each $i \in \underline{p} \backslash I_+$, $\sum_{i=1}^{p} u_i = 1$, and $\mathcal{F}(x, y; \cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\Big(x, y; \beta(x, y)\Big\{\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} u_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(y), \eta(x, y)\Big\}\Big) +$$

$$\frac{1}{2}\Big\langle \zeta(x, y), \Big\{\sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} u_j \nabla^2 G_j(y) + \sum_{k \in K_0} w_k \nabla^2 H_k(x^*)\Big\} z \Big\rangle$$

$$< - \sum_{i \in I_+} u_i \bar{\rho}_i(x, y)\|\theta(x, y)\|^m. \tag{2.15}$$

As seen in the proof of Theorem 2.1, our assumptions in (ii) lead to

$$\mathcal{F}\Big(x,y; \beta(x,y) \sum_{t=1}^{M} \Big[ \sum_{j \in J_t} u_j \nabla G_j(y) + \sum_{k \in K_t} w_k \nabla H_k(y) \Big] \Big) + \frac{1}{2} \Big\langle \zeta(x,y), \sum_{t=1}^{M} \Big[ \sum_{j \in J_t} u_j \nabla^2 G_j(y)$$
$$+ \sum_{k \in K_t} w_k \nabla^2 H_k(y) \Big] z \Big\rangle \leq - \sum_{t=1}^{M} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^m,$$

which when combined with (2.7) and (2.8) results in

$$\mathcal{F}\Big(x,y; \beta(x,y) \Big\{ \sum_{i=1}^{p} u_i [\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j \in J_0} u_j \nabla G_j(y) + \sum_{k \in K_0} w_k \nabla H_k(,y), \eta(x,y) \Big\} \Big) +$$
$$\frac{1}{2} \Big\langle \zeta(x,y), \Big\{ \sum_{i=1}^{p} u_i [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j \in J_0} u_j \nabla^2 G_j(y)$$
$$+ \sum_{k \in K_0} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle \geq \sum_{t=1}^{M} \tilde{\rho}_t(x,y) \|\theta(x,y)\|^m.$$

In view of (iii), this inequality contradicts (2.15). Hence, $\varphi(x) \geq \lambda$.

(b) - (g) : The proofs are similar to that of part (a). $\qquad\square$

**Theorem 2.5.** *(Strong Duality) Let $x^*$ be a normal optimal solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem 2.4 is satisfied for all feasible solutions of (DI). Then for each $z^* \in C(x^*)$, there exist $u^*, v^*, w^*$, and $\lambda^*$ such that $x^*$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$.*

*Proof*
The proof is similar to that of Theorem 2.2. $\qquad\square$

## 3. Duality Model II

In this section we discuss two additional duality models for $(P)$. In these duality formulations, we utilize a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$. This partitioning scheme, which is a slightly extended version of the one initially proposed by Mond and Weir [6], was used by Yang [13] for formulating a generalized duality model for a multiobjective fractional programming problem. In our duality theorems, we impose appropriate generalized $(\mathcal{F}, \beta, \phi, \rho, \zeta, \theta, m)$-sounivexity requirements on certain combinations of the problem functions.

Let $\{I_0, I_1, \ldots, I_\ell\}$ be a partition of $\underline{p}$ such that $\mathcal{L} = \{0, 1, 2, \ldots, \ell\} \subset \mathcal{M} = \{0, 1, \ldots, M\}$, and let the real-valued function $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ be defined, for fixed $u, v, w$, and $\lambda$, on $X$ by

$$\Pi_t(\xi, u, v, w, \lambda) = \sum_{i \in I_t} u_i [f_i(x) - \lambda g_i(x)] + \sum_{j \in J_t} v_j G_j(x) + \sum_{k \in K_t} w_k H_k(x), \ \ t \in \mathcal{M}.$$

Consider the following two problems:

$(DII)$ Maximize $\lambda$
subject to

$$\sum_{i=1}^{p} u_i [\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{3.1}$$

$$\left\langle z, \left\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda\nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j\nabla^2 G_j(y) + \sum_{k=1}^{r} w_k\nabla^2 H_k(y) \right\} z \right\rangle \geq 0, \tag{3.2}$$

$$\sum_{t\in I_t} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j\in J_t} v_j G_j(y) + \sum_{k\in K_t} w_k H_k(y) \geq 0, \quad t\in\mathcal{M}, \tag{3.3}$$

$$\sum_{j\in J_t} v_j G_j(y) + \sum_{k\in K_t} w_k H_k(y) \geq 0, \quad t\in\mathcal{L}\backslash M, \tag{3.4}$$

$$y\in X,\ z\in C(y),\ u\in\mathbb{R}^p,\ u\geq 0, \sum_{i=1}^{p} u_i = 1,\ v\in\mathbb{R}_+^q,\ w\in\mathbb{R}^r,\ \lambda\in\mathbb{R}_+; \tag{3.5}$$

$(\tilde{D}II)$     Maximize     $\lambda$

subject to (3.2) - (3.5) and

$$\mathcal{F}\left(x, y; \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda\nabla g_i(y)] + \sum_{j=1}^{q} v_j\nabla G_j(y) + \sum_{k=1}^{r} w_k\nabla H_k(y)\right) \geq 0 \ \text{ for all } x\in\mathbb{F},$$

where $\mathcal{F}(x, y; \cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$.

The feasible set $\mathbb{F}_{(DII)}$ (assumed to be nonempty) of $(DII)$ is defined as:

$$\mathbb{F}_{(DII)} = \left\{ y\in X : z\in\mathbb{R}^n;\ u\in\mathbb{R}^p,\ u\geq 0, \sum_{i=1}^{p} u_i = 1; v\in\mathbb{R}_+^q, v_i\geq 0; w\in\mathbb{R}^r;\ \lambda\in\mathbb{R}_+ \right\}.$$

The comments and observations made earlier about the relationship between $(DI)$ and $(\tilde{D}I)$ are, of course, also valid for $(DII)$ and $(\tilde{D}II)$.

The following two theorems show that $(DII)$ is a dual problem for $(P)$.

**Theorem 3.1.** *(Weak Duality) Let $x$ and $y$ be arbitrary feasible solutions of (P) and (DII), respectively. Furthermore, assume that any one of the following seven sets of hypotheses is satisfied:*

(a)  (i) *for each $t\in\mathcal{L}$, $\xi\to\Pi_t(\xi, u, v, w, \lambda)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (ii) *for each $t\in\mathcal{M}\setminus\mathcal{L}$, $\xi\to\Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (iii) *$\sum_{t\in\mathcal{M}} \rho_t(x, y) \geq 0$ for all $x\in\mathbb{F}$;*

(b)  (i) *for each $t\in\mathcal{L}$, $\xi\to\Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (ii) *for each $t\in\mathcal{M}\setminus\mathcal{L}$, $\xi\to\Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (iii) *$\sum_{t\in\mathcal{M}} \rho_t(x, y) \geq 0$ for all $x\in\mathbb{F}$;*

(c)  (i) *for each $t\in\mathcal{L}$, $\xi\to\Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (ii) *for each $t\in\mathcal{M}\setminus\mathcal{L}$, $\xi\to\Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;*

  (iii) *$\sum_{t\in\mathcal{M}} \rho_t(x, y) > 0$ for all $x\in\mathbb{F}$;*

(d) (i) for each $t \in \mathcal{L}_1$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \theta, m)$-pseudosounivex at $y$, for each $t \in \mathcal{L}_2$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, and for each $t \in \mathcal{L}$, $\phi_t$ is increasing and $\phi_t(0) = 0$, where $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a partition of $\mathcal{L}$;

    (ii) for each $t \in \mathcal{M} \setminus \mathcal{L}$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;

    (iii) $\sum_{t \in \mathcal{M}} \rho_t(x, y) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $t \in \mathcal{L}_1 \neq \emptyset$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, for each $t \in \mathcal{L}_2$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, and for each $t \in \mathcal{L}$, $\phi_t$ is increasing and $\phi_t(0) = 0$, where $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a partition of $\mathcal{L}$;

    (ii) for each $t \in \mathcal{M} \setminus \mathcal{L}$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;

    (iii) $\sum_{t \in \mathcal{M}} \rho_t(x, y) \geq 0$ for all $x \in \mathbb{F}$;

(f) (i) for each $t \in \mathcal{L}$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, $\phi_t$ is increasing, and $\phi_t(0) = 0$;

    (ii) for each $t \in (\mathcal{M} \setminus \mathcal{L})_1 \neq \emptyset$, $\xi \to \Lambda_t(\xi, v, w)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, for each $t \in (\mathcal{M} \setminus \mathcal{L})_2$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, and for each $t \in \mathcal{L}$, $\phi_t$ is increasing and $\phi_t(0) = 0$, where $\{(\mathcal{M} \setminus \mathcal{L})_1, (\mathcal{M} \setminus \mathcal{L})_2\}$ is a partition of $\mathcal{M} \setminus \mathcal{L}$;

    (iii) $\sum_{t \in \mathcal{M}} \rho_t(x, y) \geq 0$ for all $x \in \mathbb{F}$;

(g) (i) for each $t \in \mathcal{L}_1$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, for each $t \in \mathcal{L}_2$, $\xi \to \Pi_t(\xi, u, v, w, \lambda)$ is prestrictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, and for each $t \in \mathcal{L}$, $\phi_t$ is increasing and $\phi_t(0) = 0$, where $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a partition of $\mathcal{L}$;

    (ii) for each $t \in (\mathcal{M} \setminus \mathcal{L})_1$, $\xi \to \Lambda_t(z, v, w)$ is strictly $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-pseudosounivex at $y$, for each $t \in (\mathcal{M} \setminus \mathcal{L})_2$, $\xi \to \Lambda_t(\xi, v, w)$ is $(\mathcal{F}, \beta, \phi_t, \rho_t, \zeta, \theta, m)$-quasisounivex at $y$, and for each $t \in \mathcal{M} \setminus \mathcal{L}$, $\phi_t$ is increasing and $\phi_t(0) = 0$, where $\{(\mathcal{M} \setminus \mathcal{L})_1, (\mathcal{M} \setminus \mathcal{L})_2\}$ is a partition of $\mathcal{M} \setminus \mathcal{L}$;

    (iii) $\sum_{t \in \mathcal{M}} \rho_t(x, y) \geq 0$ for all $x \in \mathbb{F}$;

    (iv) $\mathcal{L}_1 \neq \emptyset$, $(\mathcal{M} \setminus \mathcal{L})_1 \neq \emptyset$, or $\sum_{t \in \mathcal{M}} \rho_t(x, y) > 0$.

*Then $\varphi(x) \geq \lambda$.*

*Proof*

(a) : Suppose to the contrary that $\varphi(x) < \lambda$. This implies that

$$f_i(x) - \lambda g_i(x) < 0, \quad i \in \underline{p}.$$

Since $u \geq 0$ and $u \neq 0$, we see that for each $t \in \mathcal{L}$,

$$\sum_{i \in I_t} u_i[f_i(x) - \lambda g_i(x)] \leq 0. \tag{3.6}$$

Now using this inequality, we see that for each $t \in \mathcal{L}$,

$$\begin{aligned}
\Pi_t(x, u, v, w, \lambda) &= \sum_{i \in I_t} u_i[f_i(x) - \lambda g_i(x)] + \sum_{j \in J_t} v_j G_j(x) + \sum_{k \in K_t} w_k H_k(x) \\
&\leq \sum_{i \in I_t} u_i[f_i(x) - \lambda g_i(x)] \quad \text{(by the primal feasibility of } x) \\
&\leq 0 \quad \text{(by (3.6))} \\
&\leq \sum_{i \in I_t} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j \in J_t} v_j G_j(y) + \sum_{k \in K_t} w_k H_k(y) \\
&\quad \text{(by (3.3))} \\
&= \Pi_t(y, u, v, w, \lambda),
\end{aligned}$$

and hence
$$\phi_t\big(\Pi_t(x,u,v,w,\lambda) - \Pi_t(y,u,v,w,\lambda)\big) \le 0.$$
Hence, by (i), for each $t \in \mathcal{L}$, the above inequality implies that

$$\mathcal{F}\Big(x,y;\beta(x,y)\Big\{\sum_{i\in I_t}u_i[\nabla f_i(y) - \lambda\nabla g_i(y)] + \sum_{j\in J_t}v_j\nabla G_j(y) + \sum_{k\in K_t}w_k\nabla H_k(y)\Big\}\Big)$$
$$+ \frac{1}{2}\Big\langle\zeta(x,y), \Big\{\sum_{i\in I_t}u_i[\nabla^2 f_i(y) - \lambda\nabla^2 g_i(y)] + \sum_{j\in J_t}v_j\nabla^2 G_j(y)$$
$$+ \sum_{k\in K_t}w_k\nabla^2 H_k(y)\Big\}z\Big\rangle < -\rho_t(x,y)\|\theta(x,y)\|^m.$$

Summing over $t \in \mathcal{L}$ and using the sublinearity of $\mathcal{F}(x,y;\cdot)$, we obtain

$$\mathcal{F}\Big(x,y;\beta(x,y)\Big\{\sum_{i=1}^{p}u_i[\nabla f_i(y) - \lambda\nabla g_i(y)] + \sum_{t\in\mathcal{L}}\Big[\sum_{j\in J_t}v_j\nabla G_j(y) + \sum_{k\in K_t}w_k\nabla H_k(y)\Big]\Big\}\Big)$$
$$+ \frac{1}{2}\Big\langle\zeta(x,y), \sum_{t\in\mathcal{L}}\Big\{\sum_{i\in I_t}u_i[\nabla^2 f_i(y) - \lambda\nabla^2 g_i(y)] + \sum_{j\in J_t}v_j\nabla^2 G_j(y)$$
$$+ \sum_{k\in K_t}w_k\nabla^2 H_k(y)\Big\}z\Big\rangle < -\sum_{t\in\mathcal{L}}\rho_t(x,y)\|\theta(x,y)\|^m. \quad (3.7)$$

Proceeding as in the proof of Theorem 2.1, we find that for each $t \in \mathcal{M}\backslash\mathcal{L}$,
$$\Lambda_t(x,v,w) \le \Lambda_t(y,v,w),$$
and so
$$\phi_t\big(\Lambda_t(x,v,w) - \Lambda_t(y,v,w)\big) \le 0,$$
which in view of (ii) implies that

$$\mathcal{F}\Big(x,y;\beta(x,y)\Big[\sum_{j\in J_t}v_j\nabla G_j(y) + \sum_{k\in K_t}w_k\nabla H_k(y)\Big]\Big) + \frac{1}{2}\Big\langle\zeta(x,y), \Big[\sum_{j\in J_t}v_j\nabla^2 G_j(y)$$
$$+ \sum_{k\in K_t}w_k\nabla^2 H_k(y)\Big]z\Big\rangle \le -\rho_t(x,y)\|\theta(x,y)\|^m.$$

Summing over $t \in \mathcal{M}\backslash\mathcal{L}$ and using the sublinearity of $\mathcal{F}(x,y;\cdot)$, we get

$$\mathcal{F}\Big(x,y;\beta(x,y)\sum_{t\in\mathcal{M}\backslash\mathcal{L}}\Big[\sum_{j\in J_t}v_j\nabla G_j(y) + \sum_{k\in K_t}w_k\nabla H_k(y)\Big]\Big) + \frac{1}{2}\Big\langle\zeta(x,y), \sum_{t\in\mathcal{M}\backslash\mathcal{L}}\Big[\sum_{j\in J_t}v_j\nabla^2 G_j(y)$$
$$+ \sum_{k\in K_t}w_k\nabla^2 H_k(y)\Big]z\Big\rangle \le -\sum_{t\in\mathcal{M}\backslash\mathcal{L}}\rho_t(x,y)\|\theta(x,y)\|^m. \quad (3.8)$$

Now combining (3.7) and (3.8) and using (iii), we obtain

$$\mathcal{F}\Big(x,y;\beta(x,y)\Big\{\sum_{i=1}^{p}u_i[\nabla f_i(y) - \lambda\nabla g_i(y)] + \sum_{j=1}^{q}v_j\nabla G_j(y) + \sum_{k=1}^{r}w_k\nabla H_k(y)\Big\}\Big)$$
$$+ \frac{1}{2}\Big\langle\zeta(x,y), \Big\{\sum_{i=1}^{p}u_i[\nabla^2 f_i(y) - \lambda\nabla^2 g_i(y)] + \sum_{j=1}^{q}v_j\nabla^2 G_j(y)$$
$$+ \sum_{k=1}^{r}w_k\nabla^2 H_k(y)\Big\}z\Big\rangle < -\sum_{t\in\mathcal{M}}\rho_t(x,y)\|\theta(x,y)\|^m \le 0. \quad (3.9)$$

Now multiplying (3.1) by $\beta(x, y)$, applying the sublinear function $\mathcal{F}(x, y; \cdot)$ to both sides of the resulting equation, and then adding the equation to (3.2), we get

$$\mathcal{F}\Big(x, y; \beta(x, y)\Big\{ \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y) \Big\}\Big)$$

$$+ \frac{1}{2}\Big\langle \zeta(x, y), \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle \geq 0,$$

which contradicts (3.9). Therefore, we conclude that $\varphi(x) \geq \lambda$.

(b) - (g) : The proofs are similar to that of part (a).  □

**Theorem 3.2.** *(Strong Duality) Let $x^*$ be a normal optimal solution of (P) and assume that any one of the seven sets of conditions set forth in Theorem 3.1 is satisfied for all feasible solutions of (DII). Then for each $z^* \in C(x^*)$, there exist $u^*, v^*, w^*$, and $\lambda^*$ such that $x^*$ is an optimal solution of (DII) and $\varphi(x^*) = \lambda^*$.*

*Proof*
The proof is similar to that of Theorem 2.2.  □

## 4. Concluding Remarks

It seems that the results presented in this paper will prove useful in investigating other related classes on nonlinear programming problems and applying similar generalized convexity concepts to nonlinear fractional programming problems, including finite and semiinfinite aspects, for example, a class of semiinfinite minmax fractional programming problems.

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