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Inference on the parameters and reliability characteristics of generalized inverted scale family of distributions based on records

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Abstract A generalized inverted scale family of distributions is considered. Two measures of reliability are discussed, namely $\rho(t) = P(X > t)$ and $P = P(X > Y)$. Point and interval estimation procedures are developed for the parameters, $\rho(t)$ and P based on records. Two types of point estimators are developed - uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES). A comparative study of different methods of estimation is done through simulation studies and asymptotic confidence intervals of the parameters based on MLE and log transformed MLE are constructed. Testing procedures are also developed for the parametric functions of the distribution and a real life example has been analysed for illustrative purposes.

Keywords Generalized inverted scale family of distributions, Point estimation, Interval estimation, Records, Simulation studies

AMS 2010 subject classifications 62N05, 62F10

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1. Introduction

A scale family of distributions plays an important role in reliability analysis with some of its most common members being exponential distribution, Rayleigh distribution, half-logistic distribution etc. Gupta and Kundu (1999, 2001a, 2001b) introduced the generalized exponential distribution. If Y is an exponential random variable (rv), then $X = \frac{1}{Y}$ has an inverted exponential distribution. Lin et al. (1989) and Dey (2007) discussed inverted exponential distribution (IED) to analyze lifetime data. Abouammoh and Alshingiti (2009) discussed generalized inverted exponential distribution (GIED) by introducing a shape parameter and discussed their statistical and reliability properties. Under Type II censoring, Krishna and Kumar (2012) estimated reliability characteristics of GIED. Potdar and Shirke (2012, 2013) discussed inference on the scale family of lifetime distributions based on progressively censored data and generalized inverted scale family of distributions respectively. In this paper, we develop UMVUES and MLES of the powers of the parameters and reliability functions of the generalized inverted scale family of distributions based on record data. The powers of the parameter are estimated as they appear in expressions for moments and the hazard rate of the distributions.

The reliability function $\rho(t)$ is defined as the probability that a system survives until time t. Thus, if the rv X denotes the lifetime of an item or a system, then $\rho(t) = P(X > t)$. The reliability characteristic under stress-strength setup defined as $P = P(X > Y)$, is another measure of reliability function which represents the reliability of a system (or an item) of random strength (or supply) X subject to random stress (or demand) Y. Thus, P is a measure of system's performance. A lot of work has been done in the literature for the point estimation and testing of $\rho(t)$ and P. For example, Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Kelley et al. (1976), Sathe and Shah (1981), Chao (1982), Chaturvedi and Surinder (1999) developed

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inferential procedures for $\rho(t)$ and P for exponential distribution. Constantine et al. (1986) derived UMVUE and MLE for P associated with gamma distribution. Awad and Gharraf (1986) estimated P for Burr distribution. For estimation of $\rho(t)$ corresponding to Maxwell and generalized Maxwell distributions, one may refer to Tyagi and Bhattacharya (1981) and Chaturvedi and Rani (1998), respectively. Inferences have been drawn for $\rho(t)$ and P for some families of lifetime distributions by Chaturvedi and Rani (1997), Chaturvedi and Tomer (2003), Chaturvedi and Singh (2006, 2008), Chaturvedi and Kumari (2015) and Chaturvedi and Malhotra (2016, 2917). Chaturvedi and Tomer (2002) derived UMVUE for $\rho(t)$ and P for negative binomial distribution. For exponentiated Weibull and Lomax distributions, the inferential procedures are available in Chaturvedi and Pathak (2012, 2013, 2014). Chaturvedi and Vyas (2017) developed estimation and testing procedures for the reliability functions of exponentiated distributions under Type I and Type II censoring.

Chandler (1952) introduced the concept of records as a statistic of successive extremes from a sequence of independent and identically distributed rvs. This theory is largely based on the theory of order statistics and is especially closely related to extreme order statistics. Record values and the associated statistics are of particular interest in the areas of climatology, sports, traffic, medicine, economics etc. A large number of record data saved for a long time motivated the development of several mathematical models reflecting the corresponding record processes and forecasting the future record results. Several inferential procedures for the parameters of different distributions, based on record data, have been developed by Glick (1978), Nagaraja (1988a,1988b), Balakrishan et al. (1995), Arnold et al. (1992), Habibi et al. (2006), Arashi and Emadi (2008), Razmkhah and Ahmadi (2011), Belaghi et al. (2015) and others.

The rest of the paper is organised as follows. In Section 2, we discuss the generalized inverted family of distributions proposed by Potdar and Shirke (2013) who introduced a shape parameter to the scale family of distributions. In Section 3, we develop point estimation procedures based on records when the scale parameter is known and also discuss the case when both the shape and scale parameters are unknown. As far as point estimation is concerned, we derive UMVUES and MLES. A new technique of obtaining these estimators is developed, in which first of all the estimators of powers of parameter are obtained. These estimators are used to obtain estimators of $\rho(t)$. Using the derivatives of the estimators of $\rho(t)$, the estimators of sampled probability density function (pdf), at a specified point, are obtained which are subsequently used to obtain estimators of P. The estimators of P are derived for the cases when X and Y belong to the same and different families of distributions. In Section 4, asymptotic confidence intervals for scale and shape parameters and reliability function are constructed and in Section 5, testing procedures are developed for various parametric functions. In Section 6, we present numerical findings and illustrate a real example.

2. The Generalized Inverted Scale Family of Distributions

Let Y be a rv having distribution belonging to a scale family of distributions with cumulative distribution function (cdf) G, probability density function (pdf) g and scale parameter λ. Potdar and Shirke (2013) generalized this family by introducing a shape parameter α to obtain a generalized scale family of distributions. Let $X = \frac{1}{Y}$, then distribution of X belongs to generalized inverted scale family of distributions. The pdf and cdf of the generalized inverted scale family of distributions are respectively given as:

$$f_X(x;\lambda,\alpha) = \frac{\alpha}{\lambda x^2}\, g\left(\frac{1}{\lambda x}\right)\left[G\left(\frac{1}{\lambda x}\right)\right]^{\alpha-1}; \quad x>0, \lambda>0, \alpha>0 \tag{2.1}$$

$$F_X(x;\lambda,\alpha) = 1-\left[G\left(\frac{1}{\lambda x}\right)\right]^{\alpha}; \quad x>0, \lambda>0, \alpha>0 \tag{2.2}$$

We obtain the model in equation (2.1) by differentiating $F_X(x;\lambda,\alpha)$ in (2.2) with respect to x. Some of the members of the family of distributions in (2.1) are generalized inverted exponential distribution (GIED), generalized inverted half-logistic distribution (GIHD), generalized inverted Rayleigh distribution (GIRD) and so on. The following Figure 1 shows the pdf plot of the generalized inverted scale family of distributions.

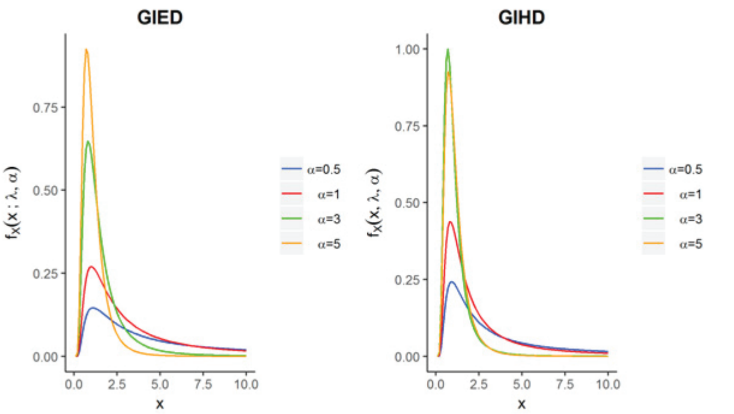

Figure 1. The pdf plot of some members of Generalized Inverted Scale Family of Distributions for different values of the shape parameter α.

3. Point Estimation Procedures

Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed (iid) rvs from (2.1). An observation X_j will be called an upper record value (or simply a record) if its value exceeds than all the previous observations up to time j. Thus X_j is a record if $X_j > X_i$ for every $i < j$.

The record time sequence $\{T_n, n \geq 0\}$ is a sequence of all time points when an observation is marked as a record and is mathematically defined as:

$$\begin{cases} T_0 = 1; & \text{with probability } 1 \\ T_n = \min\{j : X_j > X_{T_{n-1}}\}; & n \geq 1 \end{cases}$$

The record value sequence $\{R_n\}$ is a sequence of all observations marked as records and is mathematically defined as:

$$R_n = X_{T_n}; \quad n = 0, 1, 2, \ldots$$

We can rewrite (2.1) as follows:

$$f_X(x;\lambda,\alpha) = \frac{\alpha g\left(\frac{1}{\lambda x}\right)}{\left(\lambda x^2 G\left(\frac{1}{\lambda x}\right)\right)} \exp\left\{-\alpha \log\left(\frac{1}{G\left(\frac{1}{\lambda x}\right)}\right)\right\}; \quad x > 0, \lambda > 0, \alpha > 0$$

The likelihood function of the first $n+1$ upper record values $R_0, R_1, R_2, \ldots, R_n$ is:

$$L(\alpha|R_0, R_1, R_2, \ldots, R_n) = f_X(R_n;\lambda,\alpha) \prod_{i=0}^{n-1} \frac{f_X(R_i;\lambda,\alpha)}{1 - F_X(R_i;\lambda,\alpha)}$$

It is easy to see that

$$L(\alpha|R_0, R_1, R_2, \ldots, R_n) = \left(\frac{\alpha}{\lambda}\right)^{n+1} \exp\left(-\alpha \log\left(\frac{1}{G\left(\frac{1}{\lambda R_n}\right)}\right)\right) \prod_{i=0}^{n} \frac{g\left(\frac{1}{\lambda R_i}\right)}{R_i^2 G\left(\frac{1}{\lambda R_i}\right)} \tag{3.1}$$

The following theorem provides UMVUE of powers of α. This estimator will be utilized to obtain the UMVUE of reliability functions. For simplicity, we define:

$$U(x) = \log\left(\frac{1}{G\left(\frac{1}{\lambda x}\right)}\right)$$

Theorem 1
For $q \in (-\infty, \infty)$, $q \neq 0$, the UMVUE of α^q is given by:

$$\tilde{\alpha}^q = \begin{cases} \left\{\frac{\Gamma(n+1)}{\Gamma(n-q+1)}\right\}(U(R_n))^{-q}; & n > q-1 \\ 0; & \text{otherwise} \end{cases}$$

Proof
It follows from (3.1) and factorisation theorem [see Rohtagi and Saleh (2012, p.361)] that $U(R_n)$ is a sufficient statistic for α and the *pdf* of $U(R_n)$ is:

$$h(U(R_n)|\alpha) = \frac{\alpha^{n+1}U(R_n)^n}{\Gamma(n+1)}\exp(-\alpha U(R_n));\ U(R_n) \geq 0 \tag{3.2}$$

From (3.2), since the distribution of $U(R_n)$ belongs to exponential family, it is also complete [see Rohtagi and Saleh (2012, p.367)]. The result now follows from (3.2) that

$$E[U(R_n)^{-q}] = \left\{\frac{\Gamma(n-q+1)}{\Gamma(n+1)}\right\}\alpha^q$$

□

In the following theorem, we obtain UMVUE of the reliability function.

Theorem 2
The UMVUE of the reliability function is

$$\tilde{\rho}(t) = \begin{cases} \left[1 - \frac{U(t)}{U(R_n)}\right]^n; & U(t) < U(R_n) \\ 0; & \text{otherwise} \end{cases}$$

Proof
It is easy to see that

$$\begin{aligned}\rho(t) &= \exp\{-\alpha U(t)\} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\{\alpha U(t)\}^i\end{aligned} \tag{3.3}$$

Applying Theorem 1, it follows from (3.3) that

$$\begin{aligned}\tilde{\rho}(t) &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\{U(t)\}^i\tilde{\alpha}^i \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\left\{\frac{U(t)}{U(R_n)}\right\}^i\end{aligned}$$

and the theorem follows. □

The following corollary provides UMVUE of the sampled *pdf*. This estimator is derived with the help of Theorem 2.

Corollary 1
The UMVUE of the sampled *pdf* (2.1) at a specified point x is

$$\tilde{f}_X(x;\lambda,\alpha) = \begin{cases} \frac{ng\left(\frac{1}{\lambda x}\right)}{\lambda x^2 U(R_n) G\left(\frac{1}{\lambda x}\right)}\left[1 - \frac{U(x)}{U(R_n)}\right]^{n-1}; & U(x) < U(R_n) \\ 0; & \text{otherwise} \end{cases}$$

Proof

We note that the expectation of $\int_t^\infty \tilde{f}_X(x;\lambda,\alpha)dx$ with respect to R_n is $\rho(t)$. Hence, $\tilde{\rho}(t) = \displaystyle\int_t^\infty \tilde{f}_X(x;\lambda,\alpha)dx$. The result follows from Theorem 2. $\square$

In the following theorem, we obtain expression for the variance of $\tilde{\rho}(t)$, which will be needed to study its efficiency.

Theorem 3

The variance of $\tilde{\rho}(t)$ is given by:

$$\begin{aligned}\mathrm{Var}\{\tilde{\rho}(t)\} = \frac{1}{n!}\{\alpha U(t)\}^{(n+1)} \exp\{-\alpha U(t)\}\Bigg[&\frac{a_n}{\alpha U(t)} - a_{n-1}\exp\{\alpha U(t)\}E_i(-\alpha U(t))\\ &+ \sum_{i=0}^{n-2} a_i \Bigg\{\sum_{m=1}^{n-i-1}\frac{(m-1)!}{(n-i-1)!}(-\alpha U(t))^{n-i-m-1}\\ &- \frac{1}{(n-i-1)!}(-\alpha U(t))^{n-i-1}\exp(\alpha U(t))E_i(-\alpha U(t))\Bigg\}\\ &+ \sum_{i=n+1}^{2n} a_i (i-n)!\left(\frac{1}{(\alpha U(t))}\right)^{i-n+1}\sum_{r=0}^{i-n}\frac{1}{r!}(\alpha U(t))^r\Bigg]\\ &- \exp\{-2\alpha U(t)\},\end{aligned}\tag{3.4}$$

where $a_i = (-1)^i\binom{2n}{i}$ and $-E_i(-x) = \int_x^\infty \frac{e^{-u}}{u}du$.

Proof

Using (3.2) and Theorem 2,

$$\begin{aligned}E\{\tilde{\rho}(t)^2\} &= \frac{\alpha^{n+1}}{\Gamma(n+1)}\int_{U(t)}^\infty \left[1-\frac{U(t)}{U(R_n)}\right]^{2n}\{U(R_n)\}^n \exp\{-\alpha U(R_n)\}dU(R_n)\\ &= \frac{1}{(\Gamma(n+1))}(\alpha U(t))^{n+1}\exp(-\alpha U(t))\int_0^\infty \frac{z^{2n}}{(1+z)^n}\exp(-z\alpha U(t))dz\\ &= \frac{1}{(\Gamma(n+1))}(\alpha U(t))^{n+1}\exp(-\alpha U(t))I, \quad \text{(say)}\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}I &= \sum_{i=0}^{n} a_i \int_0^\infty \frac{1}{(z+1)^{n-i}}\exp(-z\alpha U(t))dz\\ &+ \sum_{i=n+1}^{2n} a_i \int_0^\infty (z+1)^{i-n}\exp(-z\alpha U(t))dz\end{aligned}\tag{3.6}$$

Using a result of Erdélyi (1954) that

$$\int_0^\infty \frac{(\exp?(-up))}{(u+a)^n}du = \sum_{m=1}^{n-1}\frac{(m-1)!(-p)^{n-m-1}}{(n-1)!a^m} - \frac{(-p)^{n-1}}{(n-1)!}\exp?(ap)E_i(-ap)$$

we have

$$\begin{aligned}
&\int_0^\infty \frac{1}{(z+1)^{n-i}} \exp(-z\alpha U(t))dz \\
&= \sum_{m=1}^{n-i-1} \frac{(m-1)!}{(n-i-1)!}(-\alpha U(t))^{n-i-m-1} \\
&\quad - \frac{1}{(n-i-1)!}(-\alpha U(t))^{n-i-1} \exp(\alpha U(t)) E_i(-\alpha U(t)), \quad i = 0, 1, 2, \ldots, n-2
\end{aligned} \tag{3.7}$$

Furthermore,

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+z)} \exp(-z\alpha U(t))dz &= \exp(\alpha U(t)) \int_0^\infty \frac{1}{(z+1)} \exp(-\alpha U(t)(z+1))dz \\
&= \exp(\alpha U(t)) \int_{(\alpha U(t))}^\infty \frac{e^{-y}}{y} dy = -\exp(\alpha U(t)) E_i(-\alpha U(t)).
\end{aligned} \tag{3.8}$$

We have

$$\int_0^\infty \exp(-z\alpha U(t))du = \left(\frac{1}{\alpha U(t)}\right) \tag{3.9}$$

Finally,

$$\begin{aligned}
\int_0^\infty (1+z)^{i-n} \exp(-z\alpha U(t))dz &= \sum_{r=0}^{i-n} \binom{i-n}{r} \int_0^\infty z^{i-n-r} \exp?(-z\alpha U(t))dz \\
&= \sum_{r=0}^{i-n} \binom{i-n}{r} \left\{\frac{1}{\alpha U(t)}\right\}^{i-n-r+1} \Gamma(i-n-r+1)
\end{aligned} \tag{3.10}$$

The theorem now follows on making substitutions from (3.7), (3.8), (3.9) and (3.10) in (3.6) and then using (3.5). □

Theorem 4
The MLE of $\rho(t)$ is given by:

$$\hat{\rho}(t) = \exp\left\{\frac{-(n+1)U(t)}{U(R_n)}\right\}.$$

Proof
It can be easily seen from (3.1) that the MLE of α is $\hat{\alpha} = \frac{(n+1)}{U(R_n)}$. The theorem now follows from invariance property of MLE. □

In the following corollary, we obtain the MLE of sampled pdf with the help of Theorem 4. This will be used to obtain MLE of P.

Corollary 2
The MLE of $f_X(x; \lambda, \alpha)$ at a specified point x is

$$\hat{f}_X(x; \lambda, \alpha) = \frac{(n+1)g\left(\frac{1}{\lambda x}\right)}{\lambda x^2 U(R_n) G\left(\frac{1}{\lambda x}\right)} \exp\left\{\frac{-(n+1)U(x)}{U(R_n)}\right\}; \quad x > 0,\ \lambda > 0, \alpha > 0$$

Proof
The result follows from Theorem 4 on using the fact that $\hat{f}_X(x; \lambda, \alpha) = -\frac{d}{dt}\hat{\rho}(t)$. □

In the following theorem, we obtain the expression for variance of $\hat{\rho}(t)$.

Theorem 5
The variance of $\hat{\rho}(t)$ is given by:

$$\begin{aligned}\mathrm{Var}\{\hat{\rho}(t)\} &= \frac{2}{n!}\{2(n+1)\alpha U(t)\}^{\frac{n+1}{2}} K_{n+1}(2\sqrt{2(n+1)\alpha U(t)}) \\ &\quad - \left[\frac{2}{n!}\{(n+1)\alpha U(t)\}^{\frac{n+1}{2}} K_{n+1}(2\sqrt{(n+1)\alpha U(t)})\right]^2\end{aligned}$$

where $K_r(\cdot)$ is modified Bessel function of second kind of order r.

Proof
Using (3.2) and Theorem 4, we have

$$\begin{aligned}E\{\hat{\rho}(t)\} &= \frac{\alpha^{n+1}}{\Gamma(n+1)} \int_0^\infty \exp\left[-\left\{\alpha U(R_n) + \frac{(n+1)U(t)}{U(R_n)}\right\}\right] \{U(R_n)\}^n dU(R_n) \\ &= \frac{1}{\Gamma(n+1)} \int_0^\infty \exp\left[-\left\{y + \frac{(n+1)\alpha U(t)}{y}\right\}\right] y^n dy \end{aligned} \tag{3.11}$$

Applying a result of Watson (1952) that

$$\int_0^\infty u^{-m} \exp\left\{-\left(au + \frac{b}{u}\right)\right\} du = 2\left(\frac{a}{b}\right)^{\frac{m-1}{2}} K_{m-1}(2\sqrt{ab})$$

[it is to be noted that $K_{-m}(\cdot) = K_m(\cdot)$ for $m = 0, 1, 2, \ldots$], we obtain from (3.11) that

$$E\{\hat{\rho}(t)\} = \frac{2}{n!}\{(n+1)\alpha U(t)\}^{\frac{n+1}{2}} K_{n+1}(2\sqrt{(n+1)\alpha U(t)})$$

Similarly, we can obtain the expression for $E\{\hat{\rho}(t)^2\}$ and the result follows. □

Let X and Y be two independent rvs following the generalized inverted scale families of distributions $f_X(x; \lambda_1, \alpha_1)$ and $f_Y(y; \lambda_2, \alpha_2)$ respectively. We consider the case when X and Y belong to different families of distributions, i.e.

$$f_X(x; \lambda_1, \alpha_1) = \frac{\alpha_1 g\left(\frac{1}{\lambda_1 x}\right)}{\lambda_1 x^2 G\left(\frac{1}{\lambda_1 x}\right)} \exp\left\{-\alpha_1 \log\left(\frac{1}{G\left(\frac{1}{\lambda_1 x}\right)}\right)\right\}; \quad x > 0,\ \lambda_1 > 0,\ \alpha_1 > 0$$

and

$$f_Y(y; \lambda_2, \alpha_2) = \frac{\alpha_2 h\left(\frac{1}{\lambda_2 y}\right)}{\lambda_2 y^2 H\left(\frac{1}{\lambda_2 y}\right)} \exp\left\{-\alpha_2 \log\left(\frac{1}{H\left(\frac{1}{\lambda_2 y}\right)}\right)\right\}; \quad y > 0,\ \lambda_2 > 0,\ \alpha_2 > 0$$

Let $\{R_n\}$ and $\{R_m^*\}$ be the record value sequences for X's and Y's respectively. For simplicity, we define:

$$U(x) = \log\left(\frac{1}{G\left(\frac{1}{\lambda_1 x}\right)}\right)$$

$$V(x) = \log\left(\frac{1}{H\left(\frac{1}{\lambda_2 y}\right)}\right)$$

The following theorem provides the UMVUE of P when X and Y belong to different families of distributions.

Theorem 6
The UMVUE of P is given by

$$\tilde{P} = \begin{cases} m\displaystyle\int_{\frac{1}{V(R_m^*)\log\left(\frac{1}{H\left(\frac{\lambda_1}{\lambda_2}G^{-1}(e^{-U(R_n)})\right)}\right)}}^{\infty} (1-z)^{m-1}\left[1 - U(R_n)^{-1}\log\left\{\frac{1}{G\left(\frac{\lambda_2}{\lambda_1}H^{-1}(e^{-zV(R_m^*)})\right)}\right\}\right]^n dz; \\ \qquad\qquad \lambda_1 G^{-1}(e^{-U(R_n)}) \le \lambda_2 H^{-1}(e^{-V(R_m^*)}) \\ m\displaystyle\int_1^{\infty}(1-z)^{m-1}\left[1 - U(R_n)^{-1}\log\left\{\frac{1}{G\left(\frac{\lambda_2}{\lambda_1}H^{-1}(e^{-zV(R_m^*)})\right)}\right\}\right]^n dz; \\ \qquad\qquad \lambda_1 G^{-1}(e^{-U(R_n)}) > \lambda_2 H^{-1}(e^{-V(R_m^*)}) \end{cases}$$

Proof
It follows from Corollary 1 that the UMVUES of $f_X(x;\lambda_1,\alpha_1)$ and $f_Y(y;\lambda_2,\alpha_2)$ at specified points x and y are respectively:

$$\tilde{f}_X(x;\lambda_1,\alpha_1) = \begin{cases} \dfrac{ng\left(\frac{1}{\lambda_1 x}\right)}{\lambda_1 x^2 U(R_n) G\left(\frac{1}{\lambda_1 x}\right)}\left[1 - \dfrac{U(x)}{U(R_n)}\right]^{n-1}; & U(x) < U(R_n) \\ 0; & \text{otherwise} \end{cases}$$

and

$$\tilde{f}_Y(y;\lambda_2,\alpha_2) = \begin{cases} \dfrac{mh\left(\frac{1}{\lambda_2 y}\right)}{\lambda_2 y^2 V(R_m^*) H\left(\frac{1}{\lambda_2 y}\right)}\left[1 - \dfrac{V(y)}{V(R_m^*)}\right]^{n-1}; & V(y) < V(R_m^*) \\ 0; & \text{otherwise} \end{cases}$$

From the arguments similar to those used in the proof of Corollary 1,

$$\begin{aligned} \tilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \tilde{f}_X(x;\lambda_1,\alpha_1)\tilde{f}_Y(y;\lambda_2,\alpha_2)dx\,dy \\ &= \int_{y=0}^{\infty} \tilde{\rho}_X(y)\left\{-\frac{d}{dy}\tilde{\rho}_Y(y)\right\}dy \\ &= m\int_{\max[\lambda_1 G^{-1}(e^{-U(R_n)}),\lambda_2 H^{-1}(e^{(}-V(R_m^*)))]}^{\infty}\left[1 - \frac{U(y)}{U(R_n)}\right]^n \left\{\frac{h\left(\frac{1}{\lambda_2 y}\right)}{\lambda_2 y^2 V(R_m^*) H\left(\frac{1}{\lambda_2 y}\right)}\right\}\left[1 - \frac{V(y)}{V(R_m^*)}\right]^{m-1} dy \end{aligned}$$

The theorem now follows on considering the two cases and putting $\frac{V(y)}{V(R_m^*)} = z$.

In the following theorem, we obtain the UMVUE of P when X and Y belong to same families of distributions.

Theorem 7
When X and Y belong to same families of distributions and $\lambda_1 = \lambda_2$

$$\tilde{P} = \begin{cases} 1 - m\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{i+1} B(i+1, n+1); & U(R_n) < U(R_m^*) \\ 1 - m\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left\{\frac{U(R_m^*)}{U(R_n)}\right\}^{i} B(i+1, m); & U(R_m^*) < U(R_n) \end{cases}$$

Proof
Taking $G(\cdot) = H(\cdot)$ in Theorem 6, then for $U(R_n) < U(R_m^*)$

$$\begin{aligned} \tilde{P} &= m\int_{\frac{U(R_n)}{U(R_m^*)}}^{\infty}(1-z)^{m-1}\left\{1 - \frac{zU(R_m^*)}{U(R_n)}\right\}^n dz \\ &= 1 - m\left\{\frac{U(R_n)}{U(R_m^*)}\right\}\int_0^1\left\{1 - \frac{wU(R_n)}{U(R_m^*)}\right\}^{m-1}(1-w)^n dw \\ &= 1 - m\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{i+1}\int_0^1 w^i(1-w)^n dw \end{aligned}$$

and the first assertion follows. Similarly, we can prove the second assertion. □

The following theorem provides the MLE of P when X and Y belong to different families of distributions.

Theorem 8
The MLE of P when X and Y belong to different families of distributions, is

$$\hat{P} = \int_0^{\infty} e^{-z}\exp\left\{\frac{-(n+1)}{U(R_n)}\log\left(\frac{1}{G\left(\frac{\lambda_2}{\lambda_1}H^{-1}(e^{\frac{-zV(R_m^*)}{m+1}})\right)}\right)\right\}dz$$

Proof
We have,

$$\begin{aligned} \hat{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty}\hat{f}_X(x;\lambda_1,\alpha_1)\hat{f}_Y(y;\lambda_2,\alpha_2)dx\,dy \\ &= \int_{y=0}^{\infty}\hat{\rho}_X(y)\hat{f}_Y(y;\lambda_2,\hat{\alpha}_2)dy \\ &= \int_{y=0}^{\infty}\exp\left\{\frac{-(n+1)U(y)}{U(R_n)}\right\}\left\{\frac{(m+1)h\left(\frac{1}{\lambda_2 y}\right)}{\lambda_2 y^2 V(R_m^*)H\left(\frac{1}{\lambda_2 y}\right)}\right\}\exp\left\{\frac{-(m+1)V(y)}{V(R_m^*)}\right\}dy \end{aligned}$$

The result now follows on putting $\left\{\frac{(m+1)V(y)}{V(R_m^*)}\right\} = z$. □

The following theorem provides MLE of P when X and Y belong to same families of distributions. The result follows from Theorem 8.

Theorem 9
When X and Y belong to same families of distributions and $\lambda_1 = \lambda_2$, the MLE of P is given by

$$\hat{P} = \frac{(m+1)U(R_n)}{(m+1)U(R_n) + (n+1)U(R_m^*)}$$

Now we consider the case when both the parameters α and λ are unknown. From (3.1), the log-likelihood function is given as:

$$
\begin{aligned}
l(\alpha,\lambda) &= L(\alpha,\lambda|R_0,R_1,R_2,\ldots,R_n)\\
&= (n+1)\log(\alpha)-(n+1)log(\lambda)-\alpha U(R_n)+\sum_{i=0}^{n}\log\left(g\left(\frac{1}{\lambda R_i}\right)\right)-2\log(R_i)-\log\left(G\left(\frac{1}{(\lambda R_i)}\right)\right)
\end{aligned}
\tag{3.12}
$$

where

$$
U(x)=\log\left(\frac{1}{G\left(\frac{1}{\lambda x}\right)}\right)
$$

The MLES of α and λ are the solutions of the two simultaneous equations given below:

$$
\frac{n+1}{\alpha}-U(R_n)=0 \tag{3.13}
$$

and

$$
\frac{-(n+1)}{\lambda}-\frac{1}{\lambda^2}\sum_{i=0}^{n}\frac{g'\left(\frac{1}{\lambda R_i}\right)}{R_i g\left(\frac{1}{\lambda R_i}\right)}+\frac{1}{\lambda^2}\sum_{i=0}^{n}\frac{g\left(\frac{1}{\lambda R_i}\right)}{R_i G\left(\frac{1}{\lambda R_i}\right)}-\alpha\frac{g\left(\frac{1}{\lambda R_n}\right)}{\lambda^2 R_n G\left(\frac{1}{\lambda R_n}\right)}=0 \tag{3.14}
$$

From (3.13), we get

$$
\hat{\alpha}=\frac{n+1}{\log\left(\frac{1}{G\left(\frac{1}{\hat{\lambda}R_n}\right)}\right)} \tag{3.15}
$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are the MLES of α and λ respectively.

Since these non-linear equation does not have a closed form solution, therefore we apply Newton Raphson algorithm to compute MLE of λ. Using this values of $\hat{\lambda}$, we can compute $\hat{\alpha}$ from (3.15).

It is to be noted that from Theorem 4, Theorem 8 and invariance property of MLE, the MLE of $\rho(t)$ is given as:

$$
\hat{\rho}(t)=\exp\left\{\frac{-(n+1)U(t)}{U(R_n)}\right\}
$$

where $U(x)=\log\left(\frac{1}{G\left(\frac{1}{\hat{\lambda}x}\right)}\right)$, $\hat{\lambda}$ is the MLE of λ. Whereas the MLE of P when X and Y belong to different family of distribution is given by:

$$
\hat{P}=\int_0^\infty e^{-z}\exp\left\{\frac{-(n+1)}{U(R_n)}\log\left(\frac{1}{G\left(\frac{\hat{\lambda}_2}{\hat{\lambda}_1}H^{-1}(e^{-zV(R^*_m)}m+1)\right)}\right)\right\}dz
$$

where $U(x)=\log\left(\frac{1}{G\left(\frac{1}{\hat{\lambda}_1 x}\right)}\right)$, $V(x)=\log\left(\frac{1}{H\left(\frac{1}{\hat{\lambda}_2 x}\right)}\right)$ and $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the MLES of λ_1 and λ_2 respectively. Similarly, the MLE of P when X and Y belong to same family of distribution and $\lambda_1=\lambda_2$ can be derived from Theorem 9.

4. Confidence Intervals

Now, Fisher information matrix of $\theta = (\alpha, \lambda)^T$ is:

$$I(\theta) = -E\begin{bmatrix} \dfrac{\partial^2 l}{\partial \alpha^2} & \dfrac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \dfrac{\partial^2 l}{\partial \lambda \partial \alpha} & \dfrac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}$$

where $\frac{\partial^2 l}{\partial \alpha^2} = \frac{-(n+1)}{\alpha^2}$, $\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \frac{\partial^2 l}{\partial \lambda \partial \alpha} = \frac{-g\left(\frac{1}{\lambda R_n}\right)}{\lambda^2 R_n G\left(\frac{1}{\lambda R_n}\right)}$

$$\begin{aligned}\frac{\partial^2 l}{\partial \lambda^2} = \frac{n+1}{\lambda^2} &+ \frac{1}{\lambda^4}\sum_{i=0}^{n} \frac{\left\{g\left(\frac{1}{\lambda R_i}\right)g''\left(\frac{1}{\lambda R_i}\right) - (g'\left(\frac{1}{\lambda R_i}\right))^2 + 2\lambda R_i g\left(\frac{1}{\lambda R_i}\right)g'\left(\frac{1}{\lambda R_i}\right)\right\}}{\left\{R_i g\left(\frac{1}{\lambda R_i}\right)\right\}^2} \\ &- \frac{1}{\lambda^4}\sum_{i=0}^{n} \frac{\left\{G\left(\frac{1}{\lambda R_i}\right)g'\left(\frac{1}{\lambda R_i}\right) - (g\left(\frac{1}{\lambda R_i}\right))^2 + 2\lambda R_i g\left(\frac{1}{\lambda R_i}\right)G\left(\frac{1}{\lambda R_i}\right)\right\}}{\left\{R_i G\left(\frac{1}{\lambda R_i}\right)\right\}^2} \\ &+ \frac{\alpha}{\lambda^4} \frac{\left\{G\left(\frac{1}{\lambda R_n}\right)g'\left(\frac{1}{\lambda R_n}\right) - \left(g\left(\frac{1}{\lambda R_n}\right)\right)^2 + 2\lambda R_n g\left(\frac{1}{\lambda R_n}\right)G\left(\frac{1}{\lambda R_n}\right)\right\}}{\left\{R_n G\left(\frac{1}{\lambda R_n}\right)\right\}^2}\end{aligned}$$

where $g'(\cdot) = \frac{d}{d\lambda}g(\cdot)$ and $g''(\cdot) = \frac{d}{\lambda}g'(\cdot)$.

Since it is a complicated task to obtain the expectation of the above expressions, therefore we use observed Fisher information matrix which is obtained by dropping the expectation sign. The asymptotic variance-covariance matrix of the MLES is the inverse of $I(\hat{\theta})$. After obtaining the inverse matrix, we get variance of $\hat{\alpha}$ and $\hat{\lambda}$. We use these values to construct confidence intervals of α and λ respectively.

Assuming asymptotic normality of the MLES, CIs for α and λ are constructed. Let $\hat{\sigma}^2(\hat{\alpha})$ and $\hat{\sigma}^2(\hat{\lambda})$ be the estimated variances of $\hat{\alpha}$ and $\hat{\lambda}$ respectively. Then $100(1-\varepsilon)\%$ asymptotic CIs for α and λ are respectively given by:

$$\left(\hat{\alpha} - Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha}), \hat{\alpha} + Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha})\right) \text{ and } \left(\hat{\lambda} - Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\lambda}), \hat{\lambda} + Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\lambda})\right)$$

where $Z_{\frac{\varepsilon}{2}}$ is the upper $100(1-\varepsilon)$ percentile point of standard normal distribution. Using this CI for α and λ, one can easily obtain the $100(1-\varepsilon)\%$ asymptotic CI for $\rho(t)$ as follows:

$$\left(\left(G\left(\frac{1}{t(\hat{\lambda} + Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\lambda}))}\right)\right)^{\hat{\alpha}+Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha})}, \left(G\left(\frac{1}{t(\hat{\lambda} - Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\lambda}))}\right)\right)^{\hat{\alpha}-Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\hat{\alpha})}\right)$$

Meeker and Escober (1998) reported that the asymptotic CI based on log(MLE) has better coverage probability. An approximate $100(1-\varepsilon)\%$ CI for $\log(\alpha)$ and $\log(\lambda)$ are:

$$(\log(\hat{\alpha}) - Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\alpha})), \log(\hat{\alpha}) + Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\alpha})))$$

and

$$(\log(\hat{\lambda}) - Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\lambda})), \log(\hat{\lambda}) + Z_{\frac{\varepsilon}{2}}\hat{\sigma}(\log(\hat{\lambda})))$$

where $\hat{\sigma}^2(\log(\hat{\alpha}))$ is the estimated variance of $\log(\alpha)$ and is approximated by $\hat{\sigma}^2(\log(\hat{\alpha})) = \frac{\hat{\sigma}^2(\hat{\alpha})}{\hat{\alpha}^2}$. Similarly, $\hat{\sigma}^2(\log(\hat{\lambda}))$ is the estimated variance of $\log(\lambda)$ and is approximated by $\hat{\sigma}^2(\log(\hat{\lambda})) = \frac{\hat{\sigma}^2(\hat{\lambda})}{\hat{\lambda}^2}$. Hence, approximate

$100(1-\varepsilon)\%$ CI for α and λ are:

$$\left(\hat{\alpha}e^{-Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\alpha})}{\hat{\alpha}}}, \hat{\alpha}e^{Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\alpha})}{\hat{\alpha}}}\right) \quad \text{and} \quad \left(\hat{\lambda}e^{-Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\lambda})}{\hat{\lambda}}}, \hat{\lambda}e^{Z_{\frac{\varepsilon}{2}}\frac{\hat{\sigma}(\hat{\lambda})}{\hat{\lambda}}}\right)$$

5. Testing of Hypotheses

Suppose, for known value of λ, we have to test the hypothesis $H_0 : \alpha = \alpha_0$ against $H_1 : \alpha \neq \alpha_0$. It follows from (3.1) that, under H_0,

$$\sup_{\Theta_0}?L(\alpha|R_0, R_1, \ldots, R_n) = \left(\frac{\alpha_0}{\lambda}\right)^{n+1} \exp\left(-\alpha_0 \log\left(1/G\left(\frac{1}{\lambda R_n}\right)\right)\right) \prod_{i=0}^{n} \frac{g\left(\frac{1}{\lambda R_i}\right)}{R_i^2 G\left(\frac{1}{\lambda R_i}\right)}; \quad \Theta_0 = \{\alpha : \alpha = \alpha_0\}$$

and

$$\sup_{\Theta} L(\alpha|R_0, R_1, \ldots, R_n) = \left(\frac{n+1}{\lambda \log\left(\frac{1}{G\left(\frac{1}{\lambda R_n}\right)}\right)}\right)^{n+1} \exp(-(n+1)) \prod_{i=0}^{n} \frac{g\left(\frac{1}{\lambda R_i}\right)}{R_i^2 G\left(\frac{1}{\lambda R_i}\right)}; \quad \Theta = \{\alpha : \alpha > 0\}$$

Denoting $\log\left(\frac{1}{G\left(\frac{1}{\lambda x}\right)}\right)$ by $U(x)$. The likelihood ratio (LR) is given by:

$$\begin{aligned}\Phi(R_0, R_1, \ldots, R_n) &= \frac{\sup_{\Theta_0}?L(\alpha|R_0, R_1, \ldots, R_n)}{\sup_{\Theta} L(\alpha|R_0, R_1, \ldots, R_n)} \\ &= \left\{\frac{\alpha_0 U(R_n)}{(n+1)}\right\}^{n+1} \exp\{-\alpha_0 U(R_n) + (n+1)\} \end{aligned} \tag{5.1}$$

We note that the first term on the right hand side of (5.1) is monotonically increasing and the second term is monotonically decreasing in $U(R_n)$. It follows from (3.2) that $2\alpha_0 U(R_n) \sim \chi^2_{2(n+1)}$. Thus, the critical region is given by:

$$\{0 < U(R_n) < k_0\} \cup \{k_0' < U(R_n) < \infty\}$$

where k_0 and k_0' are obtained such that $k_0 = \frac{\chi^2_{2(n+1)}\left(\frac{\varepsilon}{2}\right)}{2\alpha_0}$ and $k_0' = \frac{\chi^2_{2(n+1)}\left(1-\frac{\varepsilon}{2}\right)}{2\alpha_0}$ where ε is the level of significance.

An important hypothesis in life-testing experiments is $H_0 : \alpha \leq \alpha_0$ against $H_1 : \alpha > \alpha_0$. It follows from (3.1) that for $\alpha_1 > \alpha_2$,

$$\frac{L(\alpha_1|R_0, R_1, \ldots, R_n)}{L(\alpha_2|R_0, R_1, \ldots, R_n)} = \left(\frac{\alpha_1}{\alpha_2}\right)^{n+1} \exp\{(\alpha_2 - \alpha_1)U(R_n)\} \tag{5.2}$$

It follows from (5.2) that the family of distributions $f_X(x; \lambda, \alpha)$ has monotone likelihood ratio in $U(R_n)$. Thus, the uniformly most powerful critical region for testing H_0 against H_1 is given by [see Lehmann (1959, p.88)]

$$\Phi(R_0, R_1, \ldots, R_n) = \begin{cases} 1; & U(R_n) \leq k_0'' \\ 0; & \text{otherwise} \end{cases}$$

where $k_0'' = \frac{\chi^2_{2(n+1)}(\varepsilon)}{2\alpha_0}$.

It can be seen that when X and Y belong to same families of distributions and $\lambda_1 = \lambda_2 = \lambda$, $P = \frac{\alpha_2}{\alpha_1+\alpha_2}$.

Suppose we want to test $H_0 : P = P_0$ against $H_1 : P \neq P_0$. It follows that H_0 is equivalent to $\alpha_2 = k\alpha_1$ where $k = \frac{P_0}{1-P_0}$. Thus, $H_0 : \alpha_2 = k\alpha_1$ and $H_1 : \alpha_2 \neq k\alpha_1$.

It can be shown that, under H_0,

$$\hat{\alpha}_1 = \frac{n+m+2}{U(R_n) + kU(R_m^*)}$$

and

$$\hat{\alpha}_2 = \frac{k(n+m+2)}{U(R_n) + kU(R_m^*)}$$

The likelihood for observing α_1 and α_2 is

$$\begin{aligned} &L(\alpha_1, \alpha_2 | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*) \\ &= \left(\frac{\alpha_1}{\lambda}\right)^{n+1} \left(\frac{\alpha_2}{\lambda}\right)^{m+1} \exp(-\{\alpha_1 U(R_n) + \alpha_2 U(R_m^*)\}) \prod_{i=0}^{n} \frac{g\left(\frac{1}{\lambda R_i}\right)}{R_i^2 G\left(\frac{1}{\lambda R_i}\right)} \prod_{j=0}^{m} \frac{g\left(\frac{1}{\lambda R_j^*}\right)}{(R_j^*)^2 G\left(\frac{1}{\lambda R_j^*}\right)} \end{aligned}$$

Thus, for a generic constant C,

$$\begin{aligned} &\sup_{\Theta_0} ? L(\alpha_1, \alpha_2 | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*) \\ &= \frac{C}{\{U(R_n) + kU(R_m^*)\}^{n+m+2}} \exp\{-(n+m+2)\}; \quad \Theta_0 = \{\alpha_1, \alpha_2 : \alpha_2 = k\alpha_1\} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\sup_{\Theta} L(\alpha_1, \alpha_2 | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*) \\ &= \frac{C}{\{U(R_n)\}^{n+1} \{U(R_m^*)\}^{m+1}} \exp\{-(n+m+2)\}; \quad \Theta = \{\alpha_1, \alpha_2 : \alpha_1 > 0, \alpha_2 > 0\} \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), the LR is:

$$\Theta(R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*) = \frac{C \left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{m+1}}{\{1 + U(R_n)/kU(R_m^*)\}^{n+m+2}}$$

Denoting by $F_{a,b}(\cdot)$, the F-Statistic with (a, b) degrees of freedom and using the fact that $\frac{U(R_n)}{U(R_m^*)} \sim \frac{(n+1)\alpha_2}{(m+1)\alpha_1} F_{2(n+1),2(m+1)}$, the critical region is given by

$$\left\{\frac{U(R_n)}{U(R_m^*)} < k_2\right\} \cup \left\{\frac{U(R_n)}{U(R_m^*)} > k_2'\right\}$$

where $k_2 = \frac{k(n+1)}{(m+1)} F_{2(n+1),2(m+1)} \left(\frac{\varepsilon}{2}\right)$ and $k_2' = \frac{k(n+1)}{(m+1)} F_{2(n+1),2(m+1)} \left(1 - \frac{\varepsilon}{2}\right)$.

6. Numerical Findings

A simulation study is carried out to study the performance of MLES of α and λ and compare the performance of UMVUE and MLE of α where we consider Generalized Inverted Exponential distribution (GIED). We compute bias and mean square errors of the estimators for comparison. Also, the length of asymptotic confidence intervals based on MLE and log-transformed MLE of α and λ are compared.

Simulation is carried out for $(\alpha, \lambda) = (0.5, 0.5), (0.5, 1), (1, 0.5)$ and $(1, 2)$ for $n = 5, 8, 10$ and 12. For each n, 1000 observations from $gamma(n+1, \alpha)$ were generated. Let us denote these observations by Y_i; $i = 1, 2, \ldots, 1000$. Thus the average estimate of complete and sufficient statistic $U(R_n)$ is given by $U(R_n) = \frac{1}{1000}\sum_{i=1}^{1000} Y_i$. Tables 1 to 4 show the bias and mean square errors of the MLES of α and λ and UMVUE of α. In Tables 5 to 8, the length of asymptotic confidence intervals based on MLE and log-transformed MLE of α and λ at 95% and 90% level of significance are compared for different sample sizes n.

Table 1. When $\alpha = 0.5$ and $\lambda = 0.5$

	$\tilde{\alpha}$		$\hat{\alpha}$			$\hat{\lambda}$		
n	$\tilde{\alpha}$	$MSE(\tilde{\alpha})$	$\hat{\alpha}$	$Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$\hat{\lambda}$	$Bias(\hat{\lambda})$	$MSE(\hat{\lambda})$
5	0.5642	0.0625	0.6770	0.1000	0.1000	0.4871	-0.0128	0.5291
8	0.6033	0.0357	0.6787	0.0625	0.0491	0.5112	0.0112	0.2185
10	0.5971	0.0277	0.6568	0.0500	0.0361	1.0561	0.5561	0.3361
12	0.6667	0.0227	0.7223	0.0416	0.0284	0.5216	0.0216	0.1232

Table 2. When $\alpha = 0.5$ and $\lambda = 1$

	$\tilde{\alpha}$		$\hat{\alpha}$			$\hat{\lambda}$		
n	$\tilde{\alpha}$	$MSE(\tilde{\alpha})$	$\hat{\alpha}$	$Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$\hat{\lambda}$	$Bias(\hat{\lambda})$	$MSE(\hat{\lambda})$
5	0.4634	0.0625	0.5561	0.1000	0.1000	1.1571	0.1571	0.8439
8	0.6205	0.0357	0.6980	0.0625	0.0491	2.4667	1.4667	2.2360
10	0.7070	0.0277	0.7777	0.0500	0.0361	1.4080	0.4080	0.3954
12	0.7729	0.0208	0.8323	0.0384	0.0256	2.4620	1.4620	2.1895

Table 3. When $\alpha = 1$ and $\lambda = 0.5$

	$\tilde{\alpha}$		$\hat{\alpha}$			$\hat{\lambda}$		
n	$\tilde{\alpha}$	$MSE(\tilde{\alpha})$	$\hat{\alpha}$	$Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$\hat{\lambda}$	$Bias(\hat{\lambda})$	$MSE(\hat{\lambda})$
5	0.9667	0.2000	1.1279	0.1666	0.3000	0.8953	0.3953	0.1707
8	1.2006	0.1428	1.3507	0.1250	0.1964	0.4682	0.0317	0.0586
10	1.6277	0.1111	1.7905	0.1000	0.1444	0.4682	0.0317	0.0520
12	1.5924	0.0909	1.7251	0.0833	0.1136	1.2191	0.7191	0.5211

Table 4. When $\alpha = 1$ and $\lambda = 2$

	$\tilde{\alpha}$		$\hat{\alpha}$			$\hat{\lambda}$		
n	$\tilde{\alpha}$	$MSE(\tilde{\alpha})$	$\hat{\alpha}$	$Bias(\hat{\alpha})$	$MSE(\hat{\alpha})$	$\hat{\lambda}$	$Bias(\hat{\lambda})$	$MSE(\hat{\lambda})$
5	1.2415	0.2500	1.4898	0.2000	0.4000	2.7793	0.7793	1.2073
8	1.0903	0.1428	1.2266	0.1250	0.1964	3.0306	1.0306	1.3169
10	0.9326	0.1111	1.0259	0.1000	0.1444	2.0310	0.0310	0.4920
12	1.0497	0.0909	1.1372	0.0833	0.1136	1.4164	0.5835	2.0961

From the above tables we observe that for all values of n and α, the mean square error of UMVUE of α is less than that of MLE of α. Also, as sample size n increases, these mean square errors decrease.

Table 5. Length of CI of α, $\log(\alpha)$, λ and $\log(\lambda)$ when $\alpha = 0.5$, $\lambda = 0.5$ and significance level 95% and 90%

	α		$\log(\alpha)$		λ		$\log(\lambda)$	
n	95%	90%	95%	90%	95%	90%	95%	90%
5	1.1759	0.98869	1.3294	1.0766	2.8511	2.3927	9.065	5.6372
8	0.8333	0.6994	0.8867	0.7307	1.8322	1.5376	2.9829	2.1864
10	0.7186	0.6031	0.755	0.6245	0.6431	0.5397	0.653	0.5456
12	0.6401	0.5497	0.6613	0.5497	1.3737	1.1529	1.8065	1.4022

Table 6. Length of CI of α, $\log(\alpha)$, λ and $\log(\lambda)$ when $\alpha = 0.5$, $\lambda = 1$ and significance level 95% and 90%

	α		$\log(\alpha)$		λ		$\log(\lambda)$	
n	95%	90%	95%	90%	95%	90%	95%	90%
5	1.1759	0.9869	1.4076	1.1215	3.5481	2.9776	5.1109	3.87
8	0.8333	0.6994	0.8837	0.729	1.1413	0.9578	1.1515	0.9639
10	0.7186	0.6031	0.7444	0.6183	1.8756	1.5741	2.0174	1.6573
12	0.6093	0.5113	0.623	0.5194	0.8932	0.7496	0.8981	0.7525

Table 7. Length of CI of α, $\log(\alpha)$, λ and $\log(\lambda)$ when $\alpha = 1$, $\lambda = 2$ and significance level 95% and 90%

	α		$\log(\alpha)$		λ		$\log(\lambda)$	
n	95%	90%	95%	90%	95%	90%	95%	90%
5	2.3519	1.9738	2.6038	2.1213	3.0361	2.5479	3.1893	2.6381
8	1.6667	1.3988	71.7980	1.4758	1.9783	1.6602	2.0136	1.6810
10	1.4373	1.2062	1.5577	1.2769	2.7470	2.3054	2.9612	2.4311
12	1.2803	1.0745	1.3490	1.1149	5.1938	4.3588	8.6336	6.2941

Table 8. Length of CI of α, $\log(\alpha)$, λ and $\log(\lambda)$ when $\alpha = 1$, $\lambda = 0.5$ and significance level 95% and 90%

	α		$\log(\alpha)$		λ		$\log(\lambda)$	
n	95%	90%	95%	90%	95%	90%	95%	90%
5	2.0452	1.7164	2.3371	1.8868	0.4719	0.396	0.4774	0.3992
8	1.6667	1.3988	1.7745	1.4621	0.9411	0.7898	1.1077	0.8868
10	1.4373	1.2062	1.4762	1.2291	0.8855	0.7431	1.0235	0.8236
12	1.2803	1.0745	1.3099	1.0919	0.2469	0.2072	0.2473	0.2075

From Tables 4 to 8 we observe that as sample size n increases, the length of CIs based on MLE and log-transformed MLE decreases. As reported by Meeker and Escober (1998), we too observe that asymptotic CIs based on log-transformed MLE have better coverage probability.

Table 9. Mean square error of MLE and UMVUE of $\rho(t)$ and length of CI of $\rho(t)$ when $\alpha = 2$ and $\lambda = 0.5$ at significance level 95% and 90%

t	$\rho(t)$	$\tilde{\rho}(t)$	$\hat{\rho}(t)$	$Var(\tilde{\rho}(t))$	$MSE(\hat{\rho}(t))$	95%	90%
1	0.7758	0.8062	0.7764	0.0084	0.0105	0.4142	0.3562
2	0.5433	0.5772	0.5355	0.4409	0.0211	0.2079	0.1731
3	0.4365	0.4768	0.4378	0.6729	0.0216	0.0952	0.0794
4	0.3874	0.4244	0.3886	1.6828	0.0209	0.0498	0.0416
5	0.3584	0.3928	0.3596	2.8916	0.0202	0.0289	0.0242

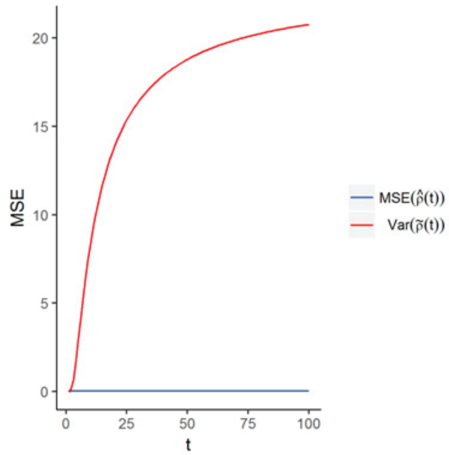

Figure 2. Mean Square Error of MLE and UMVUE of $\rho(t)$.

From Table 9 we observe that as time t increases, the length of CI of $\rho(t)$ based on MLE of α and λ decreases. Figure 2 compares the variance of UMVUE of reliability function with the MSE of MLE of reliability function calculated in Table 9 as time t increases. Since the variance of UMVUE of $\rho(t)$ is always greater than the MSE of MLE of $\rho(t)$, thus the MLE of $\rho(t)$ is a more efficient estimator of $\rho(t)$.

In the theory developed in Section 5, for testing the hypothesis $H_0 : \alpha = \alpha_0 = 2$ against $H_1 : \alpha \neq \alpha_0 = 2$ under this scheme, we have considered the following sample.

0.1431 0.7565 0.8903 1.5914 1.6962 2.88554.7279 9.6573 14.4171

Now with the help of Chi-Square tables at 5% level of significance, we obtained $k_0 = 2.0576$ and $k_0' = 7.8815$. Hence, in this case we may accept H_0 at 5% level of significance since $U(R_8) = 3.7785$.

Again, for testing $H_0 : \alpha \leq \alpha_0 = 2$ against $H_1 : \alpha > \alpha_0 = 2$, we have considered the above sample. Now at 5% level of significance we obtained $k_0'' = 2.3476$ and hence, in this case we may accept H_0 at 5% level of significance since $U(R_8) = 3.7785$.

In order to test $H_0 : P = P_0 = 0.6666$ against $H_1 : P \neq P_0 = 0.6666$ under this scheme, we have considered the following Sample X and Sample Y.

Sample X: 0.1260 0.2755 0.3638 0.5159 0.5316 1.0305 1.9092

Sample Y: 0.1535 0.1653 0.2414 0.2604 0.3426 0.4431 0.5511 0.5709

For these two samples we obtained $U(R_n)/U(R_m^*) = 2.2445$. Now, with the help of F-tables at 5% level of significance, we obtained $k_2 = 0.5986$ and $k_2' = 4.9297$. Hence, in this case we may accept H_0 at 5% level of significance.

An Example on Real Data

Lawless (1982) provided real data which represents the number of million revolutions before failure for each of 23 ball bearings in a life test:

17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

Potdar and Shirke (2013) showed that according to Kolmogorov-Smirnov test, this data set best fits generalized inverted half logistic distribution (GIHD). This is also confirmed in the following Figure 3.

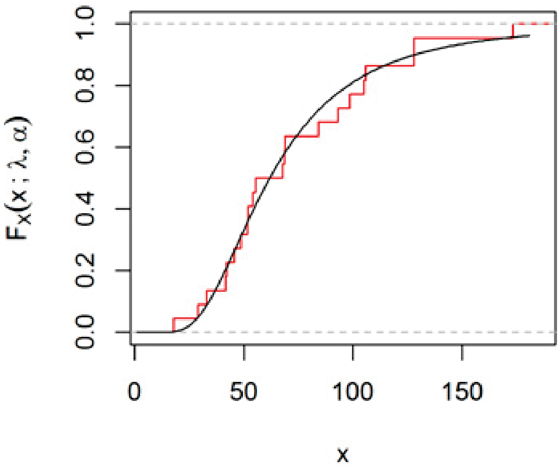

Figure 3. Empirical and Theoretical cdf of GIHD.

In such a case, since both the parameters α and λ are unknown, thus by applying Newton Raphson algorithm we obtain the MLES of these parameters. The computed observed Fisher Information matrix is used to obtain the confidence intervals of the parameters and hence the reliability function $\rho(t)$. Table 10 shows the MLE of the parameters and length of CIs based on MLE and log-transformed MLE of α and λ. We can see that CIs based on log transformation of MLE of the parameters α and λ have a higher coverage probability. In Table 11, MLE and UMVUE of reliability function along with their respective variances and the confidence interval of $\rho(t)$ are computed. We also see that the variance of UMVUE of $\rho(t)$ is smaller than that of its MLE. Hence UMVUE of $\rho(t)$ is more efficient estimator of $\rho(t)$.

Table 10. MLE of α and λ and length of CI of α, $\log(\alpha)$, λ and $\log(\lambda)$ at significance level 95% and 90%

	α		$\log(\alpha)$		λ		$\log(\lambda)$		
n	95%	90%	95%	90%	95%	90%	95%	90%	
3.2023	0.0073	4.2370	3.5558	4.5529	3.7413	0.00477	0.00400	0.00485	0.00405

Table 11. MLE and UMVUE of $\rho(t)$ and CI of $\rho(t)$ when $t = 20$ at significance level 95% and 90%

$\tilde{\rho}(t)$	$Var(\hat{\rho}(t))$	$\hat{\rho}(t)$	$Var(\hat{\rho}(t))$	95%	90%
0.9384	6.587E-07	0.9357	7.224E-07	[0.9863,0.9984]	[0.9874,0.9978]

7. Discussion

This article proposes results on generalized inverted family of distributions having scale and shape parameters. Point and interval estimation procedures for the parameters and reliability characteristics of the family have been developed. As a member of this family, generalized inverted exponential distribution is considered and through simulation techniques, performance of the estimators and confidence intervals are studied. Testing procedures for various parametric functions have been developed. A real life example on generalized inverted half logistic distribution has also been analysed.

Tables 1 to 4 show that for all values of n and α, the mean square error of UMVUE of α is less than that of MLE of α. Also, as sample size n increases, these mean square errors decrease. Tables 5 to 8 show that as sample size n increases, we obtain better interval estimates of the parameters of the model under study. As reported by Meeker and Escober (1998), we too observe that asymptotic CIs based on log-transformed MLE have better coverage probability. Table 9 shows that as time t increases, we obtain better interval estimates of $R(t)$ based on MLE of α and λ. Figure 2 compares the mean square error of UMVUE and MLE of reliability function calculated in Table 9 with respect to time t. In all we note that the UMVUE of the shape parameter and the reliability function are better estimators than their respective MLES. In all we note that the UMVUE of the shape parameter and the MLE of the reliability function $\rho(t)$ are better estimators than their respective MLE and UMVUE.

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