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# Weak Convergence of Random Processes from Spaces $F_\psi(\Omega)$

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**Abstract** This paper is devoted to the investigation of conditions for the weak convergence in the space $C(T)$ of the stochastic processes from the space $\mathbf{F}_\psi(\Omega)$. Using these conditions the limit theorem for stochastic processes from the space $\mathbf{F}_\psi(\Omega)$ is obtained. This theorem can be utilized for achieving the given approximation accuracy and reliability of integrals depending on parameter by Monte Carlo method.

**Keywords** weak convergence, pseudometric space, $K_\sigma$-space, $\mathbf{F}_\psi(\Omega)$ space, majorant characteristic, metric massiveness, condition $\mathbf{H}$, stochastic processes, Monte Carlo method

**AMS 2010 subject classifications** 65C05, 60G07

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## 1. Introduction

In this paper we describe conditions for the weak convergence in the space $C(T)$ of the stochastic processes from the space $\mathbf{F}_\psi(\Omega)$. The limit theorem for stochastic processes from this space is proved based on the conditions obtained. One can use this theorem to find approximation accuracy in $C(T)$ and reliability of integrals depending on parameter by Monte Carlo method.

The space $\mathbf{F}_\psi(\Omega)$ was introduced by Yermakov and Ostrovsky in the paper [5]. The paper [8] is devoted to studying properties of such spaces and there were found rules of fulfilling the condition $\mathbf{H}$ in this spaces.

Limits theorems for different classes of processes are investigated, for example, in books [3, 12, 16]. Estimates for the distribution of suprema for the Gaussian stochastic processes can be found in books [3, 16] and papers [2, 6]. Some applications of these results are presented in [7, 13]. The probabilities of large deviations for the sums of independent stochastic processes from the space $\mathbf{F}_\psi(\Omega)$ are considered in the paper [9]. Estimates for the distribution of suprema on $R$ for the stochastic processes from such spaces are describe.

This paper is organized as follows. In Section 2 we introduce the basic definitions related to the weak convergence of random elements. $K_\sigma$-space of random variable and $K_\sigma$-processes are introduced and discussed in Section 3. In the next Section 4, we deal with properties of $\mathbf{F}_\psi(\Omega)$ random variables and processes. Section 5 contains conditions of the weak convergence in the space $\mathbf{C}(T)$ for stochastic processes from the space $\mathbf{F}_\psi(\Omega)$ defined on the compact set. In Section 6 we apply the results obtained to the stochastic processes from $\mathbf{F}_\psi(\Omega)$ and derive the limit theorem for such processes. The accuracy and reliability of estimates of the integrals depending on the parameter evaluated by the Monte Carlo methods are describe in Section 7. In Section 8 we give estimates

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for the distribution of the supremum for Gaussian stochastic processes and examples. Conclusions are presented in Section 9.

## 2. Weak convergence of random elements

Suppose that $(S,\rho)$ is a metric space, $\mathcal{B}(S)$ is the Borel $\sigma$-algebra in $(S,\rho)$ and $(S,\mathcal{B}(S))$ is a mesurable space. A measure $\mu(\cdot)$ on $(S,\mathcal{B}(S))$ is called a probability measure if $\mu(S)=1$.

**Definition 2.1** ([3]). A family $\mu_n$, $n=\overline{1,\infty}$ of probability measures on $(S,\mathcal{B}(S))$ is called weakly convergent as $n\to\infty$ if there exists a probability measure $\mu_\infty$ such that we have

$$\lim_{n\to\infty}\int_S f(x)d\mu_n(x)=\int_S f(x)d\mu_\infty(x) \tag{1}$$

for any bounded continuous real-valued function $f=\{f(x),x\in S\}$. We denote the weak convergence of measures as $\mu_n\Rightarrow\mu_\infty$.

*Remark 2.1.* It is well known that $\mu_n\Rightarrow\mu_\infty$ if and only if $\lim_{n\to\infty}\mu_n(B)=\mu_\infty(B)$ for any $B\in\mathcal{B}(S)$ such that $\mu_\infty(\partial B)=0$, where $\partial B$ denotes the boundary of the set $B$.

Consider the condition for the weak convergence in the space of continuous functions. Suppose that $(T,\rho)$ is a compact metric space and $\mathbf{C}(T)$ is the space of continuous real-valued functions on $T$.

Let $X_n=\{X_n(t),t\in T\}$, $n=\overline{1,\infty}$ be a family of random elements in $\mathbf{C}(T)$ (stochastic processes).

**Definition 2.2** ( [3]). We say, that all finite-dimensional distributions of the stochastic processes $X_n$ converge weakly as $n\to\infty$ to the corresponding finite-dimensional distributions of the process $X=\{X(t),t\in T\}$ if the random vectors $(X_n(t_1),X_n(t_2),\ldots,X_n(t_k))$ converge weakly as $n\to\infty$ to the random vector $(X(t_1),X(t_2),\ldots,X(t_k))$ for any $k\ge 1$ and all $t_1,t_2,\ldots,t_k\in T$.

*Theorem 2.1* ( [3], p. 231)
Let $(T,\rho)$ be a metric space. One has $X_n\Rightarrow X_\infty$ in $\mathbf{C}(T)$ as $n\to\infty$, where $X_\infty=\{X_\infty(t),t\in T\}$ is a random element in $\mathbf{C}(T)$ (stochastic process), if and only if the following conditions hold:

1) all finite-dimensional distributions of the processes $X_n=\{X_n(t),t\in T\}$ converge weakly as $n\to\infty$ to the corresponding finite dimensional distributions of the process $X_\infty=\{X_\infty(t),t\in T\}$;
2) for any $\varepsilon>0$ we have

$$\lim_{h\downarrow 0}\sup_{n=\overline{1,\infty}} P\left\{\sup_{\substack{t,s\in T\\ \rho(t,s)<h}}|X_n(t)-X_n(s)|>\varepsilon\right\}=0. \tag{2}$$

## 3. $K_\sigma$-space of random variable and $K_\sigma$-processes

**Definition 3.1.** [3] A linear subspace $K(\Omega)$ of the space of all random variables $L_0(\Omega)$ is called a lattice if $\max\{\xi,\eta\}\in K(\Omega)$, $\min\{\xi,\eta\}\in K(\Omega)$ for all $\xi,\eta\in K(\Omega)$.

If $K(\Omega)$ is a lattice and $\xi\in K(\Omega)$ then $|\xi|\in K(\Omega)$ since $|\xi|=\max(-\xi,\xi)$.

**Definition 3.2.** [3] A lattice $K(\Omega)$ is called a $K_\sigma$-space of random variable if $K(\Omega)$ is a Banach space equipped with a norm $\|\cdot\|_K$ and the following conditions hold:

a) if $\xi,\eta\in K(\Omega)$ and $|\xi|\le|\eta|$ almost surely, then $\|\xi\|_K\le\|\eta\|_K$;
b) if for a sequence $\{\xi_n,n\ge 1\}$ belonging to $K(\Omega)$, one can find a random variable $\eta\in K(\Omega)$ such that $\sup\limits_{n\ge 1}|\xi_n|<\eta$ almost surely, then $\sup\limits_{n\ge 1}|\xi_n|\in K(\Omega)$.

**Definition 3.3** ([1],[3]). A monotonically nondecreasing sequence of positive numbers $(\varkappa(n), n \geq 1)$ is called an $M$-characteristic (majorant characteristic) of $K_\sigma$-space $K(\Omega)$ if for any $n \geq 1$ and $\xi_k \in K(\Omega)$, $k = \overline{1,n}$ the folowing inequality holds

$$\left\| \max_{1\leq k\leq n} |\xi_k| \right\|_K \leq \varkappa(n) \max_{1\leq k\leq n} \|\xi_k\|_K . \tag{3}$$

Let $K = K(\Omega)$ be a $K_\sigma$-space of random variables and let $\varkappa(n), n \geq 1$ be the $M$-characteristic of the space $K(\Omega)$.

**Definition 3.4.** [3] The stochastic process $X = \{X(t), t \in T\}$ is called $K_\sigma$-stochastic process if random variables $X(t) \in K(\Omega)$ for all $t \in T$.

Consider $X_n = \{X_n(t), t \in T\}$, $n = \overline{1,\infty}$ as a family of stochastic processes from $K(\Omega)$. Suppose that the following conditions hold:

$B_1$) $\varepsilon_0 = \sup\limits_{n=\overline{1,\infty}} \sup\limits_{t,s\in T} \|X_n(t) - X_n(s)\|_K < \infty.$

$B_2$) Let $\rho_n(t,s) = \|X_n(t) - X_n(s)\|_K$, $\rho(t,s) = \sup\limits_{n=\overline{1,\infty}} \rho_n(t,s)$. The pseudometric space $(T,\rho)$ is separable and each process $X_n$ is separable on $(T,\rho)$.

It is clear that $\varepsilon_0 = \sup\limits_{t,s\in T} \rho(t,s)$.

*Remark 3.1.* [3] A pseudometric satisfies all the assumptions of a metric except for the condition: if $\rho(t,s) = 0$ then $t = s$; that is, the set $\{(t,s) : \rho(t,s) = 0\}$ for a pseudometric may be larger than the diagonal $\{(t,s) : t = s\}$. The pair $(T,\rho)$ is called a pseudometric space.

*Theorem 3.1* ([3], p. 109)
Assume that a family of stochastic processes $X_n$, $n = \overline{1,\infty}$ satisfies conditions $B_1$ and $B_2$, and suppose that for all $\tau > 0$

$$\int_0^\tau \varkappa(N(u))du < \infty,$$

where $N(u)$, $u > 0$ is the metric massiveness of the space $(T,\rho)$ (that is least number of close circles with radius less or equal to $u$ and covering the set T), $\varkappa(n)$ is the $M$-characteristic of $K(\Omega)$. Then

a) $$\lim_{h\downarrow 0} \sup_{n=\overline{1,\infty}} \left\| \sup_{\substack{t,s\in T\\ \rho(t,s)<h}} |X_n(t) - X_n(s)| \right\|_K = 0;$$

b) for each $\varepsilon > 0$

$$\lim_{h\downarrow 0} \sup_{n=\overline{1,\infty}} P\left\{ \sup_{\substack{t,s\in T\\ \rho(t,s)<h}} |X_n(t) - X_n(s)| > \varepsilon \right\} = 0;$$

c) the processes $X_n$ are almost surely sample uniformly continuous on $(T,\rho)$ for any $n$.

*Theorem 3.2*
Assume that a family of stochastic processes $X_n$, $n = \overline{1,\infty}$ satisfies conditions of Theorem 3.1 and all finite-dimensional distributions of the processes $X_n(t), t \in T$ converge weakly to the corresponding finite-dimensional distributions of the processes $X_\infty(t), t \in T$ as $n \to \infty$. Then $X_n(t)$ converge weakly in $\mathbf{C}(T)$ to the stochastic processes $X_\infty(t)$ as $n \to \infty$.

*Proof*
This theorem follows from Theorem 2.1 and Theorem 3.1. □

## 4. The space of random variables $\mathbf{F}_\psi(\Omega)$ and stochastic processes from this space

**Definition 4.1.** [8, 9, 10] Let $\psi(u) > 0,\ u \geq 1$ be monotonically increasing, continuous function for which $\psi(u) \to \infty$ as $u \to \infty$. A random variable $\xi$ belongs to the space $\mathbf{F}_\psi(\Omega)$ if

$$\sup_{u\geq 1} \frac{\left(E\left|\xi\right|^u\right)^{1/u}}{\psi(u)} < \infty.$$

Similar definition was formulated in the paper by S. M. Yermakov & Ye. I. Ostrovskii [5]. But there was required that $E\xi = 0$ as $\xi \in \mathbf{F}_\psi(\Omega)$. Moreover, there were considered the random variables for which $E\left|\xi\right|^u = \infty$ for some $u > 0$.

It is proved in [5, 11] that $\mathbf{F}_\psi(\Omega)$ is a Banach space with the norm

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{\left(E\left|\xi\right|^u\right)^{1/u}}{\psi(u)}.$$

The space $\mathbf{F}_\psi(\Omega)$ is a $K_\sigma$-space with the norm $\|\xi\|_\psi$.

Let us provide some examples of random variables from the spaces $\mathbf{F}_\psi(\Omega)$.

*Example 4.1.* [10] The random variable $\xi$ satisfying the condition $|\xi| < C$ with probability one, where $C > 0$ is a constant, belongs to the space $\mathbf{F}_\psi(\Omega)$. Herewith

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{\left(E\left|\xi\right|^u\right)^{1/u}}{\psi(u)} \leq \sup_{u\geq 1} \frac{\left(C^u\right)^{1/u}}{\psi(u)} = \sup_{u\geq 1} \frac{C}{\psi(u)} = \frac{C}{\psi(1)}.$$

*Example 4.2.* [10] The random variable with Laplace distribution (its density function is $p(x) = \frac{1}{2}e^{-|x|}$) belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u$. This follows from the equivalence $\sqrt[k]{E\left|\xi\right|^k} = \sqrt[k]{k!} \sim k$ for $k \geq 1$.

*Example 4.3.* [10] The normally distributed random variable $\xi \sim N(0,1)$ belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^{1/2}$ since $\sqrt[2l]{E\left|\xi\right|^{2l}} = \sqrt[2l]{\frac{(2l)!}{2^l l!}} \sim l^{1/2}$ for $l \geq 1$.

**Definition 4.2.** [8] We say that the condition **H** is fulfilled for the Banach space of random variables $B(\Omega)$, if there exists an absolute constant $C_B$ such that for any centered and independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ from $B(\Omega)$, the following is true:

$$\left\|\sum_{i=1}^n \xi_i\right\|^2 \leq C_B \sum_{i=1}^n \|\xi_i\|^2.$$

The constant $C_B$ is called a scale constant for the space $B(\Omega)$. For space $\mathbf{F}_\psi(\Omega)$ we shall denote the constants $C_{\mathbf{F}_\psi(\Omega)}$ as $C_\psi$.

*Theorem 4.1* ([15])
For the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha \geq \frac{1}{2}$, the condition **H** is fulfilled and the following inequality is true:

$$\left\|\sum_{i=1}^n \xi_i\right\|_\psi^2 \leq 4\cdot 9^\alpha \sum_{i=1}^n \|\xi_i\|_\psi^2.$$

Note, that when $\alpha < \frac{1}{2}$, then the condition **H** is not fulfilled for such a space.

*Theorem 4.2* ([14])
Let $\mathbf{F}_\psi(\Omega)$ be the space defined by the function $\psi(u) = e^{au^\beta}$, where $a > 0$, $0 < \beta < 1$. If $\frac{1}{(2a\beta)^{1/\beta}} = 1$, then the condition **H** is fulfilled for the space $\mathbf{F}_\psi(\Omega)$ with the constant $C_\psi = 4e^{2^\beta a}$. And if $\frac{1}{(2a\beta)^{1/\beta}} > 1$, then for $\mathbf{F}_\psi(\Omega)$

the condition $\mathbf{H}$ is true with the constant

$$C_\psi = \frac{4e^{a\left(2^\beta+1\right)-\frac{1}{2\beta}}}{(2a\beta)^{1/2\beta}}.$$

**Definition 4.3.** [9] It is said that a stochastic process $X = \{X(t),\ t \in T\}$, belongs to the space $\mathbf{F}_\psi(\Omega)$ if for any $t \in T$ the random variable $X(t)$ belongs to the space $\mathbf{F}_\psi(\Omega)$.

*Theorem 4.3*
Assume that a family of stochastic processes $X_n = \{X_n(t), t \in T\}$, $n = \overline{1,\infty}$ from the space $\mathbf{F}_\psi(\Omega)$ is such that following conditions hold:

$\tilde{B}_1)$ $\hat{\varepsilon}_0 = \sup\limits_{n=\overline{1,\infty}} \sup\limits_{t,s\in T} \|X_n(t) - X_n(s)\|_\psi < \infty.$

$\tilde{B}_2)$ The space $(T, \hat{\rho}_x)$ $\left(\hat{\rho}_x(t,s) = \sup\limits_{n=\overline{1,\infty}} \rho_n(t,s)\right)$ is separable and each process $X_n$ is separable on $(T, \hat{\rho}_x)$ and suppose that for any $\tau > 0$

$$\int_0^\tau \varkappa_\psi(N(u))du < \infty, \tag{4}$$

where $N(u)$ is the metric massiveness of the space $(T, \hat{\rho}_x)$, $\varkappa_\psi(n)$ is the $M$-characteristic of $\mathbf{F}_\psi(\Omega)$.

If all finite-dimensional distributions of the processes $X_n(t), t \in T$ converge weakly to the corresponding finite-dimensional distribution of the process $X_\infty(t), t \in T$ as $n \to \infty$, then $X_n(t)$ converge weakly in $\mathbf{C}(T)$ to the process $X_\infty(t)$ as $n \to \infty$.

*Proof*
Theorem 4.3 follows from Theorem 3.2 since the space $\mathbf{F}_\psi(\Omega)$ is $K_\sigma$-space. □

## 5. Limit theorem for stochastic processes from a space $\mathbf{F}_\psi(\Omega)$

Let $X = \{X(t),\ t \in T\}$ be a stochastic process from the space $\mathbf{F}_\psi(\Omega)$, $EX(t) = 0$. Let the condition $\mathbf{H}$ is fulfilled for this space.

Assume that compact pseudometric space $(T, \rho_\psi)$, $\rho_\psi(t,s) = \|X(t) - X(s)\|_\psi$ is separable and the process $X = \{X(t),\ t \in T\}$ is separable as well. Let $X_k(t)$, $k = 1, 2, \ldots, n$ be independent copies of $X(t)$. Consider a stochastic process

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k(t).$$

By Definition (4.2) we have

$$\|Y_n(t) - Y_n(s)\|_\psi^2 \le C_\psi \frac{1}{n} \sum_{k=1}^n \|X_k(t) - X_k(s)\|_\psi^2 = C_\psi \rho_\psi^2(t,s).$$

The pseudometric space $(T, \rho_\psi)$ is separable and the processes $Y_n(t)$ are separable in this space.

*Theorem 5.1*
If the following condition holds

$$\hat{\varepsilon}_0 = \sup_{t,s\in T} \|X(t) - X(s)\|_\psi < \infty,$$

and for any $\tau > 0$

$$\int_0^\tau \varkappa_\psi(\tilde{N}(u))du < \infty, \tag{5}$$

where $\varkappa_\psi(n)$ is the $M$-characteristic of the space $\mathbf{F}_\psi(\Omega)$, $\tilde{N}(\varepsilon)$ is the metric massiveness of the space $(T,\rho_\psi)$, then $Y_n(t)$ converge weakly in $\mathbf{C}(T,\rho_\psi)$ to the Gaussian process $X_\infty(t)$ such that $EX_\infty(t)=0$, $EX_\infty(t)X_\infty(s)=EX(t)X(s)$.

*Proof*
The Central Limit Theorem for random vectors implies that all finite-dimensional distributions of $Y_n(t)$ converge to the ones of the process $X_\infty(t)$.

So, Theorem 5.1 follows from Theorem 4.3. □

## 6. Stochastic processes defined on a metric spaces

Let $(T,m)$ be a compact metric space and let $N_m(u)$ be a metric massiveness of this space. Let $X_n(t), t\in T$ be stochastic processes such that $X_n(t)$ is separable on $(T,m)$ and $X_n(t)\in\mathbf{F}_\psi(\Omega)$.

*Theorem 6.1*
Assume that there exists such a continuous and monotonically increasing function $\sigma(h)$, $h>0$, $\sigma(0)=0$ that the following inequality holds

$$\sup_{n=\overline{1,\infty}}\ \sup_{m(t,s)\le h}\|X_n(t)-X_n(s)\|_\psi\le\sigma(h) \tag{6}$$

and $\sigma(h)<\hat{\varepsilon}_0<\infty$. Suppose that for any $\tau>0$

$$\int_0^\tau \varkappa_\psi(N_m(\sigma^{(-1)}(u)))du<\infty,$$

where $\varkappa_\psi(n)$ is the $M$-characteristic of $\mathbf{F}_\psi(\Omega)$, $N_m(u)$ is metric massiveness of the space $(T,m)$.

If all finite-dimensional distributions of the processes $X_n(t), t\in(T,m)$ converge weakly to the corresponding all finite-dimensional distributions of a process $X_\infty(t), t\in(T,m)$ as $n\to\infty$, then $X_n(t)$ converge weakly in $\mathbf{C}(T,m)$ to the process $X_\infty(t)$.

*Proof*
This Theorem 6.1 follows from Theorem 4.3. Indeed, the condition $\tilde{B}_1$ follows from the condition (6). The condition $\tilde{B}_2$ is fulfilled, because the process $X(t)$ is separable on $(T,m)$, that is separable on $(T,\hat{\rho}_x)$. It is evident that for metric massiveness of the space $(T,\hat{\rho}_x)$ we have then inequality $N(u)\le N_m(\sigma^{(-1)}(u))$. Therefore condition (4) is satisfied. If a function $f(t)$ is continuous on $(T,\hat{\rho}_x)$ it is continuous on $(T,m)$. □

*Example 6.1* ([3], p. 90). Let $T\in R^d$, $d>1$ and $\rho(t,s)=\|t-s\|_d$, where $\|\cdot\|$ is a norm in $R^d$. It is bounded, then there exist such constants $\beta_1$ and $\beta_2$ that

$$\beta_1 u^{-d}\le N_m(T,u)\le\beta_2 u^{-d}.$$

Other examples we can found in the book ([3], p. 90-91).

The next Theorem can be easily proved.

*Theorem 6.2*
Let $(T,m)$ be a compact metric space, let $X=\{X(t),\ t\in(T,m)\}$ be a stochastic process from the space $\mathbf{F}_\psi(\Omega)$, $EX(t)=0$. Assume that the condition $\mathbf{H}$ is satisfied for the space $\mathbf{F}_\psi(\Omega)$ and process $X$ is separable. Let $X_k(t)$, $k=1,2,\dots,n$ be independent copies of $X(t)$. Let

$$\tilde{Y}_n(t)=\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k(t),$$

$$\|X_k(t)-X_k(s)\|_\psi^2=\rho_\psi(t,s).$$

Then

$$\left\|\tilde{Y}_n(t)-\tilde{Y}_n(s)\right\|_\psi^2\le C_\psi\rho_\psi^2(t,s).$$

*Theorem 6.3*
Let the assumptions of the Theorem 6.2 be satisfied and the following conditions hold:

a) $\hat{\varepsilon}_0 = \sup\limits_{t,s\in T} \|X(t)-X(s)\|_\psi < \infty$;
b) $\sup\limits_{m(t,s)\le h} \|X(t)-X(s)\|_\psi \le \sigma(h)$, where $\sigma(h)$, $h>0$ is a continuous, monotonically increasing function and $\sigma(0)=0$;
c) for any $\tau > 0$
$$\int_0^\tau \varkappa_\psi(N_m(\sigma^{(-1)}(u)))du < \infty,$$
where $\varkappa_\psi(n)$ is the $M$-characteristic of $\mathbf{F}_\psi(\Omega)$, $N_m(u)$ is metric massiveness of the space $(T,m)$.

Then $\tilde{Y}_n(t)$ converge weakly in $\mathbf{C}(T,m)$ to the Gaussian process $X_\infty(t)$ such that
$$EX_\infty(t)=0, EX_\infty(t)X_\infty(s)=EX(t)X(s).$$

*Proof*
This Theorem 6.3 follows from Theorem 6.1. □

## 7. On calculation of the integrals depending on a parameter by Monte-Carlo method

Let $\{\mathcal{S},\mathcal{A},\mu\}$ be a measurable space, $\mu$ be a $\sigma$-finite measure and $p(s)\ge 0, s\in\mathcal{S}$ be a measurable function such that $\int\limits_{\mathcal{S}} p(s)d\mu(s)=1$. Let $P(A)$, $A\in\mathcal{A}$ be the measure $P(A)=\int\limits_{A} p(s)d\mu(s)$. The measure $P(A)$ is a probability measure and the space $\{\mathcal{S},\mathcal{A},P\}$ is a probability space.

Let $f(s)$ be a measurable function on $\{\mathcal{S},\mathcal{A},\mu\}$. Suppose, that this integral $\int\limits_{\mathcal{S}} f(s)p(s)d\mu(s)=I$ exists.

*Remark 7.1.* [10] We can consider the integral of the form $\int\limits_{\mathcal{S}}\varphi(s)d\mu(s)$. If $p(s)>0$ is a probability density function in the space $\{\mathcal{S},\mathcal{A},\mu\}$, then
$$\int\limits_{\mathcal{S}}\varphi(s)d\mu(s)=\int\limits_{\mathcal{S}}\frac{\varphi(s)}{p(s)}p(s)d\mu(s)=\int\limits_{\mathcal{S}}f(s)p(s)d\mu(s),$$
where $f(s)=\varphi(s)/p(s)$.

We can consider $f(s)=\xi$ as random variables on $\{\mathcal{S},\mathcal{A},P\}$ and $\int\limits_{\mathcal{S}} f(s)p(s)d\mu(s)=\int\limits_{\mathcal{S}} f(s)dm(s)=E\xi$.

Let $\xi_k$, $k=1,2,\ldots,n$, be the independent copies of random variable $\xi$ and $Z_n=\frac{1}{n}\sum\limits_{k=1}^n\xi_k$. Then according to the strong law of large numbers $Z_n\to E\xi_1=I$ with probability one. We consider $Z_n$ as an estimate for $I$.

Let us consider the integral $\int\limits_{\mathcal{S}} f(s,t)p(s)d\mu(s)=I(t)$ assuming that it exists. Let the function $f(s,t)$ depend on the parameter $t\in T$, where $(T,\rho)$ is some compact set and the function $f(s,t)$ is continuous with regard to $t$.

Suppose $f(s,t)$ is a stochastic process on $\{\mathcal{S},\mathcal{A},P\}$ and which we denote as $\xi(s,t)=\xi(t)$ and
$$I(t)=\int\limits_{\mathcal{S}} f(s,t)p(s)d\mu(s)=\int\limits_{\mathcal{S}} f(s,t)dm(s)=E\xi(t).$$

Let $\xi_k(t)$, $k=1,2,\ldots,n$, be the independent copies of the stochastic process $\xi(t)$ and $Z_n(t)=\frac{1}{n}\sum\limits_{k=1}^n\xi_k(t)$. So, according to the strong law of large numbers $Z_n(t)\to E\xi(t)=I(t)$ with probability one for any $t\in T$.

Let $\xi(t)\in\mathbf{F}_\psi(\Omega)$ and the condition $\mathbf{H}$ is fulfilled for the space $\mathbf{F}_\psi(\Omega)$.

*Theorem 7.1*
Let $\sigma(h)$, $h > 0$ be continuous, monotonically increasing function such that $\sigma(0) = 0$, $\sigma(h) < \hat{\varepsilon}_0 < \infty$. Let the following condition hold

$$\sup_{\rho(t,s)\le h} \|Y(t) - Y(s)\|_{\psi} \le \sigma(h),$$

where $Y(t) = \xi(t) - I(t)$ and for any $\tau > 0$

$$\int_0^{\tau} \varkappa_{\psi}(N_{\rho}(\sigma^{(-1)}(u)))du < \infty, \tag{7}$$

where $\varkappa_{\psi}(n)$ is the $M$-characteristic of $\mathbf{F}_{\psi}(\Omega)$, $N_{\rho}(u)$ is the metric massiveness of the space $(T, \rho)$.

Then $Y_n(t) = \sqrt{n}Z_n(t) = \frac{1}{\sqrt{n}} \sum\limits_{k=1}^{n} \xi_k(t)$ converge weakly in $\mathbf{C}(T)$ to the Gaussian process $X_{\infty}(t)$ such that

$$EX_{\infty}(t) = 0, EX_{\infty}(t)X_{\infty}(s) = E\xi_k(t)\xi_k(s) - I(t)I(s).$$

*Proof*
This Theorem 7.1 follows from Theorem 6.3. □

*Remark 7.2.* In paper [9] it is shown that

$$\|Y(t) - Y(s)\|_{\psi} \le 2C_{\psi} \|\xi(t) - \xi(s)\|_{\psi},$$

where $C_{\psi}$ is the constant from Defnition 4.2. Therefore, the Theorem 7.1 will be fulfilled when there exists continuous, monotonically increasing function $\hat{\sigma}(h) = 2C_{\psi}\sigma(h)$ such that

$$\|\xi(t) - \xi(s)\|_{\psi} \le \hat{\sigma}(h)$$

and

$$\int_0^{\tau} \varkappa_{\psi}(N_{\rho}(\hat{\sigma}^{(-1)}(u)))du < \infty.$$

*Corollary 7.1*
It follows from Remark 2.1 that

$$P\left\{\sup_{t\in T}\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k(t) - I(t)\right| > \varepsilon\right\} = P\left\{\sup_{t\in T}\left|\frac{1}{n}\sum_{k=1}^{n}(\xi_k(t) - I(t))\right| > \varepsilon\right\} =$$

$$P\left\{\sup_{t\in T}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\xi_k(t) - I(t))\right| > \sqrt{n}\varepsilon\right\} = P\left\{\sup_{t\in T}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k(t)\right| > \sqrt{n}\varepsilon\right\} \cong P\left\{\sup_{t\in T}|X_{\infty}(t)| > \sqrt{n}\varepsilon\right\},$$

where $X_k(t) = \xi_k(t) - I(t)$ and $X_{\infty}(t)$ is defined in the Theorem 2.1.

Estimating the last inequality for given $n$ and $\varepsilon$ one can find the approximate reliability for the integral estimator with accuracy $\varepsilon$.

## 8. Theorems about distribution of supremum of Gaussian stochastic processes and examples

Description of distribution of Gaussian stochastic processes we can find in the books [3, 16] and paper [2].

Now we will consider one of the proved theorems.

*Theorem 8.1*
Let $T = [a,b]$, $X = \{X(t), t \in [a,b]\}$ be a centered separable Gaussian process and $D([a,b]) = \sup_{t\in[a,b]} \left(E\left|X(t)\right|^2\right)^{1/2} < \infty$. Assume that the following inequality holds true

$$\sup_{\substack{|t-s|\le h\\ t,s\in[a,b]}} \left(E\left|X(t)-X(s)\right|^2\right)^{1/2} \le ch^\beta,$$

where $c > 0$, $0 < \beta \le 1$. Then for any $0 < \theta < 1$ and $\lambda > 0$

$$E\exp\left\{\lambda \sup_{t\in[a,b]} |X(t)|\right\} \le R\exp\left\{\frac{\lambda^2 (D([a,b]))^2}{2(1-\theta)^2}\right\}, \tag{8}$$

where $R = 2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}}{(\theta D([a,b]))^{1/\beta}} + 1\right)$.

*Proof*
This Theorem 8.1 follows from Theorem 3.44 ([3], p. 107). □

*Corollary 8.1*
Let the assumptions of Theorem 8.1 be true. Then for any $0 < \beta \le 1$, $0 < \theta < 1$, $\varepsilon > 0$, the following inequality holds true

$$P\left\{\sup_{t\in[a,b]} |X(t)| > \varepsilon\right\} \le R\exp\left\{-\frac{\varepsilon^2(1-\theta)^2}{2(D([a,b]))^2}\right\}. \tag{9}$$

*Proof*
By the Chebyshev inequality we have

$$P\left\{\sup_{t\in[a,b]} |X(t)| > \varepsilon\right\} \le \frac{E\exp\left\{\lambda \sup_{t\in[a,b]} |X(t)|\right\}}{\exp\{\lambda\varepsilon\}} \le R\exp\left\{\frac{\lambda^2(D([a,b]))^2}{2(1-\theta)^2}\right\}\exp\{-\lambda\varepsilon\}. \tag{10}$$

The inequality (9) follows from (10) if one takes $\lambda = \frac{\varepsilon(1-\theta)^2}{(D([a,b]))^2}$. □

*Theorem 8.2*
Let assumption of Theorem 8.1 be true. Then for any $0 < \beta \le 1$, $\varepsilon > \sqrt{2}D([a,b])$ the following inequality holds true

$$P\left\{\sup_{t\in[a,b]} |X(t)| > \varepsilon\right\} \le 2^{2/\beta-1}e\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}\varepsilon^{2/\beta}}{2^{1/\beta}(D([a,b]))^{3/\beta}} + 1\right)\exp\left\{-\frac{\varepsilon^2}{2(D([a,b]))^2}\right\}. \tag{11}$$

*Proof*
The inequality (11) follows from (9) putting $(1-\theta)^2 = 1 - \frac{2(D([a,b]))^2}{\varepsilon^2}$ if $1 - \frac{2(D([a,b]))^2}{\varepsilon^2} > 0$. In this case

$$R = 2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}}{\left(\left(1-\sqrt{1-\frac{2(D([a,b]))^2}{\varepsilon^2}}\right)D([a,b])\right)^{1/\beta}} + 1\right) =$$

$$2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}}{(D([a,b]))^{1/\beta}}\left(\frac{\left(1+\sqrt{1-\frac{2(D([a,b]))^2}{\varepsilon^2}}\right)}{\left(1-\sqrt{1-\frac{2(D([a,b]))^2}{\varepsilon^2}}\right)\left(1-\sqrt{1-\frac{2(D([a,b]))^2}{\varepsilon^2}}\right)}\right)^{1/\beta} + 1\right) =$$

$$2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}}{(D([a,b]))^{1/\beta}}\left(\frac{1+\sqrt{1-\frac{2(D([a,b]))^2}{\varepsilon^2}}}{\frac{2(D([a,b]))^2}{\varepsilon^2}}\right)^{1/\beta}+1\right)\leq$$

$$2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}}{(D([a,b]))^{1/\beta}}\left(\frac{\varepsilon^2}{2(D([a,b]))^2}\right)^{1/\beta}+1\right)=$$

$$2^{2/\beta-1}\left(\frac{2^{2/\beta-1}(b-a)c^{1/\beta}\varepsilon^{2/\beta}}{2^{1/\beta}(D([a,b]))^{3/\beta}}+1\right).$$

□

*Example 8.1.* Consider the integral

$$I(t)=rq\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\sqrt{xy}}e^{-rx-qy}\sin\left(\sqrt{txy}\right)dxdy,$$

where $0\leq t\leq l$, $r>0$ and $q>0$.

Let $\xi$ and $\eta$ be the independent random variables exponentially distributed with parameters $r$ and $q$ respectively. Then

$$I(t)=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\sqrt{xy}}re^{-rx}qe^{-qy}\sin\left(\sqrt{txy}\right)dxdy=E\left(\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}\right).$$

Consider the space $\mathbf{F}_\psi(\Omega)$ with $\psi(u)=u^{\frac{1}{2}}$. The condition **H** is fulfilled for this space with the constant $C_\psi=12$ (See [15]). Let

$$\xi(t)=\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}$$

and let $\xi_k(t),\ k=1,2,\ldots,n$ be independent copies of the stochastic process $\xi(t)$. Then $Z_n(t)=\frac{1}{n}\sum_{k=1}^{n}\xi_k(t)$ approximates the integral $I(t)$.

Next we estimate the norm of the process

$$\|\xi(t)\|_\psi=\left\|\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}\right\|_\psi\leq\sqrt{t}$$

and $\inf_{0\leq t\leq T}\|\xi(t)\|_\psi=0$.

An estimate for the norm of increments for this process is given by

$$\|\xi(t)-\xi(s)\|_\psi=\left\|\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}-\frac{\sin\left(\sqrt{s\xi\eta}\right)}{\sqrt{\xi\eta}}\right\|_\psi\leq$$

$$\leq 2\left\|\frac{\sin\left(\sqrt{\xi\eta}\left(\sqrt{t}-\sqrt{s}\right)\right)}{2\sqrt{\xi\eta}}\right\|_\psi\leq\left\|\sqrt{t}-\sqrt{s}\right\|_\psi\leq|t-s|^{\frac{1}{2}}.$$

Therefore

$$\sigma(h)=Ch^{\frac{1}{2}},$$

with $C=1$.

Let us check the condition (7). For the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^{\frac{1}{2}}$ the majorant characteristic has the form (see [8, 9, 10])

$$\varkappa(u) = \left(\frac{e}{\alpha}\right)^{\alpha} (\ln u)^{\alpha}.$$

The condition (7) is fulfilled, when the integral is convergent for any $\tau > 0$

$$I(\tau) = \int_0^{\tau} (\ln(N_\rho(\sigma^{(-1)}(u))))^{1/2} du.$$

This integral is convergent, when, for example, it is convergent for $\tau = 1/\sqrt{2}$

$$I(1/\sqrt{2}) = \int_0^{1/\sqrt{2}} (\ln(N_\rho(\sigma^{(-1)}(u))))^{1/2} du.$$

It's easy to see that $N_\rho(\sigma^{(-1)}(u)) \leq \frac{l}{2\sigma^{(-1)}(u)} + 1$ and for $r \leq 1$, $x > 1$

$$\ln(1+x) = \frac{1}{r}\ln(1+x)^r \leq \frac{1}{r}\ln(1+x^r) \leq \frac{x^r}{r}.$$

Therefore $I(1/\sqrt{2}) \leq \frac{1}{r^{1/2}} \int_0^{1/\sqrt{2}} \left(\frac{l}{2u^2}\right)^{r/2} du$. If $r = \frac{1}{3}$ then $I(1/\sqrt{2}) < \infty$. So, for this example, conditions of Theorem 7.1 are fulfilled. It follows from Corollary 7.1 that for $\varepsilon > 0$

$$P\left\{\sup_{t\in[0,l]} \left|\frac{1}{n}\sum_{k=1}^{n} \xi_k(t) - I(t)\right| > \varepsilon\right\} \cong P\left\{\sup_{t\in[0,l]} |X_\infty(t)| > \sqrt{n}\varepsilon\right\},$$

where $X_\infty(t)$ is a Gaussian process such that $EX_\infty(t) = E\xi(t) - I(t) = 0$, $EX_\infty(t)X_\infty(s) = E(\xi(t) - I(s))(\xi(t) - I(s)) = E\xi(t)\xi(s) - I(t)I(s)$. Therefore

$$E(X_\infty(t) - X_\infty(s))^2 = E(\xi(t) - \xi(s) + (I(t) - I(s)))^2 \leq 2(E(\xi(t) - \xi(s))^2 + (I(t) - I(s))^2).$$

So

$$(I(t) - I(s))^2 = (E(\xi(t) - \xi(s)))^2 \leq E(\xi(t) - \xi(s))^2.$$

Then

$$E(X_\infty(t) - X_\infty(s))^2 \leq 4(E(\xi(t) - \xi(s))^2) \leq 4E\left(\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}} - \frac{\sin\left(\sqrt{s\xi\eta}\right)}{\sqrt{\xi\eta}}\right)^2 \leq$$

$$\leq 4E\left(\frac{\sin\left((\sqrt{t}-\sqrt{s})\sqrt{\xi\eta}\right)}{2\sqrt{\xi\eta}}\right)^2 \leq |t-s|.$$

Therefore, from Theorems 8.1 and 8.2 it follows that $c = 1$, $\beta = 1/2$. Moreover

$$D([0,l]) = \sup_{0\leq t\leq l}\left(E(\xi(t) - I(t))^2\right)^{1/2} = \sup_{0\leq t\leq l}\left((E\xi(t))^2 - I^2(t)\right)^{1/2} \leq \sup_{0\leq t\leq l}\left(E\xi^2(t)\right)^{1/2} \leq 1.$$

The Theorem 8.2 implies for $\varepsilon > \sqrt{2}$ that

$$P\left\{\sup_{t\in[0,l]} |X_\infty(t)| > \varepsilon\right\} \leq 2^3 e(2l\varepsilon^4 + 1)\exp\left\{-\frac{\varepsilon^2}{2}\right\}.$$

Then

$$P\left\{\sup_{t\in[0,l]} \left|\frac{1}{n}\sum_{k=1}^{n} \xi_k(t) - I(t)\right| > v\right\} \leq 2^3 e(2ln^2v^4 + 1)\exp\left\{-\frac{nv^2}{2}\right\}.$$

## 9. Conclusions

In this paper we describe the conditions of the weak convergence in the space $C(T)$ of the stochastic processes from the space $\mathbf{F}_\psi(\Omega)$. Using these conditions the limit theorem for stochastic processes from the space $\mathbf{F}_\psi(\Omega)$ is obtained. Application of this theorem can be used for achieving the approximation accuracy in $C(T)$ and reliability of integrals depending on parameter by Monte Carlo method.

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